Strong homotopy Lie algebras, homotopy Poisson manifolds and Courant algebroids

Yunhe Sheng (Jilin University)

Geometry of Jets and Fields, Banach Center, Bedlewo
In honer of Prof. Janusz Grabowski
May 12, 2015
Joint work with Honglei Lang and Xiaomeng Xu

Outline

Lie 2-algebras are the categorification of Lie algebras. They are the infinitesimal of Lie 2-groups. Lie 2-groups are the categorification of Lie groups, which describe symmetries between symmetries.

The category of Lie 2-algebras and the category of 2-term L_∞-algebras (also called strong homotopy Lie algebras) are equivalent.

J. C. Baez and A. S. Crans, Higher-Dimensional Algebra VI: Lie 2-Algebras, *Theory and Appl. Categ.* 12 (2004), 492-528.

The notion of a Courant algebroid was introduced in

Z. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Diff. Geom.* 45 (1997), 547-574.

The notion of a Courant algebroid was introduced in

Z. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Diff. Geom.* 45 (1997), 547-574.

See the following paper for its history:

Y. Kosmann-Schwarzbach, Courant algebroids. A short history. *SIGMA Symmetry Integrability Geom. Methods Appl.* 9 (2013), Paper 014, 8 pp.

Many people contribute on this theory. In particular, Prof. Grabowski and his collaborators give the deformation and contraction theory of Courant algebroids.

- J. Grabowski, Courant-Nijenhuis tensors and generalized geometries. *Groups, geometry and physics*, 101-112, 2006.
- J. Carinena, J. Grabowski and G. Marmo, Courant algebroid and Lie bialgebroid contractions. *J. Phys. A* 37 (2004), no. 19, 5189-5202.

One can obtain a Courant algebroid from a degree 2 symplectic NQ manifold.

D. Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson Brackets and Beyond*, 169ĺC185, *Contemp. Math.*, 315, Amer. Math. Soc., Providence, RI, 2002.

One can obtain a Courant algebroid from a degree 2 symplectic NQ manifold.

D. Roytenberg, On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson Brackets and Beyond*, 169ĺC185, *Contemp. Math.*, 315, Amer. Math. Soc., Providence, RI, 2002.

A Courant algebroid could give rise to a Lie 2-algebra according to Roytenberg-Weinstein construction.

D. Roytenberg and A. Weinstein, Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.*, 46(1) (1998), 81-93.

The notion of a homotopy Poisson manifold of degree n was introduced in

R. A. Mehta, On homotopy Poisson actions and reduction of symplectic Q-manifolds, *Diff. Geom. Appl.* 29(3) (2011), 319-328.

The notion of a homotopy Poisson manifold of degree n was introduced in

R. A. Mehta, On homotopy Poisson actions and reduction of symplectic Q-manifolds, *Diff. Geom. Appl.* 29(3) (2011), 319-328.

There is a linear Poisson structure on the dual space of a Lie algebra. It is natural to ask what is the structure on the "dual" of a Lie 2-algebra. Motivated by this question, we find some relations between Lie 2-algebras, homotopy Poisson manifolds and Courant algebroids. This is the content of this talk.

Outline

Definition

A **homotopy Poisson algebra** *of degree n is a graded commutative algebra $\mathfrak{a}$ with an L_∞-algebra structure $\{l_m\}_{m\geq 1}$ on $\mathfrak{a}[n]$, such that the map*

$$x \longrightarrow l_m(x_1, \cdots, x_{m-1}, x), \qquad x_1, \cdots, x_{m-1}, x \in \mathfrak{a}$$

is a derivation of degree $2 - m - n(m-1) + \sum_{i=1}^{m-1} |x_i|$. Here, $|x|$ denotes the degree of $x \in \mathfrak{a}$.

A homotopy Poisson algebra of degree n is of **finite type** *if there exists a q such that $l_m = 0$ for all $m > q$.*

A **homotopy Poisson manifold** *of degree n is a graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is equipped with a degree n homotopy Poisson algebra structure of finite type.*

Several related structures:

(i) A P_∞-**algebra** is a graded commutative algebra $\mathfrak{a}$ over a field of characteristic zero such that there is an L_∞-algebra structure $\{l_m\}_{m\geq 1}$ on $\mathfrak{a}$, and the map

$$x \longrightarrow l_m(x_1, \cdots, x_{m-1}, x),$$

is a derivation of degree $2 - m - (|x_1| + \cdots + |x_{m-1}|)$. Their P_∞-algebra is a homotopy Poisson algebra of degree 0.

A. S. Cattaneo and G. Felder, Relative formality theorem and quantisation of coisotropic submanifolds. *Adv. Math.*, 2007, 208(2): 521-548.

(ii) The notion of **a higher Poisson structure** was introduced in

- T. Voronov, Higher derived brackets and homotopy algebras. *J. Pure Appl. Algebra* 202 (2005), no. 1-3, 133-153.

and further studied in

- A. Bruce, From L_∞-algebroids to higher Schouten Poisson structures. *Rep. Math. Phys.* 67 (2011), no. 2, 157-177.
- H. M. Khudaverdian and Th. Th. Voronov, Higher Poisson brackets and differential forms. Geometric methods in physics, 203-215, *AIP Conf. Proc.*, 1079, Amer. Inst. Phys., Melville, NY, 2008.

where the authors used the superized version of an L_∞-algebra.

(iii) A **graded Poisson algebra of degree** k is a graded commutative algebra $\mathfrak{a}$ with a degree $-k$ Lie bracket, such that the bracket is a biderivation of the product, namely

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{|y|(|x|+k)} y \cdot [x, z].$$

Thus, a graded Poisson algebra of degree k is a homotopy Poisson algebra of degree k. In particular, the associated L_∞-algebra has only one non-zero map l_2.

A. S. Cattaneo, D. Fiorenza and R. Longoni, Graded Poisson Algebras. *Encyclopedia of Math. Phys.*, 2006, 2: 560-567.

Example

Let A be a Lie algebroid. Consider its dual vector bundle, $A^*[1]$, which is an N-manifold of degree 1. Its algebra of polynomial functions is

$$\cdots \oplus \Gamma(A) \oplus C^\infty(M),$$

where $\Gamma(A)$ is of degree 1, and $C^\infty(M)$ is of degree 0. The Poisson bracket is in fact the Schouten bracket $[\cdot,\cdot]_S$ on $\Gamma(\wedge^\bullet A)$. It is straightforward to see that $A^*[1]$ is a homotopy Poisson manifold of degree 1.

Example

Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2, l_3)$, its graded dual space $\mathfrak{g}^*[1] = \mathfrak{g}_0^*[1] \oplus \mathfrak{g}_{-1}^*[1]$ is an N-manifold of degree 1 with the base manifold $\mathfrak{g}_{-1}^*$. Its algebra of polynomial functions is

$$\cdots \oplus \left(C^\infty(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0\right) \oplus C^\infty(\mathfrak{g}_{-1}^*).$$

There is a degree 1 homotopy Poisson algebra structure on it obtained by extending the original 2-term L_∞-algebra structure using the Leibniz rule. Thus, the dual of a 2-term L_∞-algebra is a homotopy Poisson manifold of degree 1. This generalize the fact that the dual of a Lie algebra is a linear Poisson manifold. Similarly, the dual of an n-term L_∞-algebra is a homotopy Poisson manifold of degree $n-1$.

(M,π) is Poisson manifold if and only if $(T^*[1]M, Q)$ is a symplectic NQ-manifold of degree 1, where the homological vector field Q is given by $Q=\{\pi,\cdot\}$. More generally, the cotangent bundle of a homotopy Poisson manifold of degree n gives rise to a symplectic NQ-manifold of degree $n+1$.

Theorem

Given a degree n homotopy Poisson manifold $(\mathcal{M},\{l_i\}_{1\leq i<\infty})$, the cotangent bundle $T^*[n+1]\mathcal{M}$ is a symplectic NQ-manifold of degree $n+1$, where $Q=\{\sum l_i,\cdot\}$, and $\{\cdot,\cdot\}$ is the canonical Poisson structure on $T^*[n+1]\mathcal{M}$. Moreover, for any $a_1,\cdots,a_k\in C^\infty(\mathcal{M})$, we have

$$l_k(a_1,\cdots,a_k)=\{a_k,\cdots,\{a_2,\{a_1,\sum l_i\}\}\cdots\}|_{\mathcal{M}}. \qquad (1)$$

Corollary

Given an n-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{-n+1}, \{l_i\}_{1\leq i\leq n+1})$, the cotangent bundle $T^*[n]\mathfrak{g}^*[n-1]$ is a symplectic NQ-manifold of degree n, where the degree 1 homological vector field Q is given by

$$Q = \{\sum l_i, \cdot\}, \tag{2}$$

in which $\{\cdot,\cdot\}$ is the canonical Poisson structure, and $\sum l_i \in Sym(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}[1-n]$ is viewed as a polynomial function of degree $n+1$ on $T^*[n]\mathfrak{g}^*[n-1]$.

Definition

A **Maurer-Cartan element** α *on a degree n homotopy Poisson manifold* $\mathcal{M}$ *is a function on* $\mathcal{M}$ *satisfying the Maurer-Cartan equation*

$$\sum_i \frac{(-1)^i}{i!} l_i(\alpha, \cdots, \alpha) = 0. \tag{3}$$

Example

(**quasi-Poisson $\mathfrak{g}$-manifolds**) *Let M be a manifold and $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}}, K)$ a quadratic Lie algebra. We define $R \in \wedge^3\mathfrak{k}^*$ by*

$$R(u,v,w) = K([u,v]_{\mathfrak{k}}, w), \quad \forall\ u,v,w \in \mathfrak{k}.$$

Then $\mathcal{M} := T^*[1]M \times \mathfrak{k}[1]$ is a homotopy Poisson manifold of degree 1:

$$\begin{aligned} l_1(\xi) &= \delta(\xi), \\ l_2(X,Y) &= [X,Y]_S, \quad l_2(X,f) = Xf, \\ l_3(\xi,\eta,\gamma) &= K(R)(\xi,\eta,\gamma), \end{aligned}$$

where $f \in C^\infty(M), X, Y \in \mathfrak{X}(M)$, $\xi,\eta,\gamma \in \mathfrak{k}^*$, and $\delta : \wedge^\bullet\mathfrak{k}^* \longrightarrow \wedge^{\bullet+1}\mathfrak{k}^*$ is the coboundary operator associated to the Lie algebra $\mathfrak{k}$.

Example

(continue) A degree 2 function $\alpha = \pi + \rho$, where $\pi \in \wedge^2 \mathfrak{X}(M)$ and $\rho \in \mathfrak{k}^* \otimes \mathfrak{X}(M)$, is a Maurer-Cartan element, i.e.

$$-l_1(\alpha) + \frac{1}{2} l_2(\alpha, \alpha) - \frac{1}{3!} l_3(\alpha, \alpha, \alpha) = 0,$$

iff the following three conditions hold:

$$l_1(\rho) = \frac{1}{2}[\rho, \rho]_S, \quad [\pi, \rho]_S = 0, \quad \frac{1}{2}[\pi, \pi]_S = \frac{1}{6} K(R)(\rho, \rho, \rho).$$

These conditions are equivalent to that $\rho : \mathfrak{k} \longrightarrow \mathfrak{X}(M)$ is a Lie algebra morphism, π is $\mathfrak{k}$-invariant and $\frac{1}{2}[\pi, \pi]_S = \wedge^3 \rho(K(R))$ respectively. Therefore, a quasi-Poisson $\mathfrak{g}$-manifold gives rise to a Maurer-Cartan element on $\mathcal{M}$.

Example

(**twisted Poisson structures**) The shifted cotangent bundle $T^*[1]M$ of a manifold M is canonically a symplectic N-manifold of degree 1. For a choice of a closed 3-form H, we define l_3 by $l_3(X,Y,Z) = H(X,Y,Z)$. The compatibility of l_2 and l_3 is due to the fact that H is closed. Thus, $(T^*[1]M, l_2 = [\cdot,\cdot]_S, l_3 = H)$ is a homotopy symplectic manifold of degree 1.
Choose a local coordinate (x^i, p_i) on $T^*[1]M$. A degree 2 function $\pi = \frac{1}{2}\pi^{ij}(x)p_ip_j$ is a Maurer-Cartan element of $T^*[1]M$ if and only if

$$\frac{1}{2}l_2(\pi,\pi) - \frac{1}{3!}l_3(\pi,\pi,\pi) = 0,$$

which is equivalent to $\frac{1}{2}[\pi,\pi] = \wedge^3\pi^\sharp H$, that is, π is a twisted Poisson structure on M.

Example

(**twisted Courant algebroids**) *Let $E \longrightarrow M$ be a vector bundle with a fiber metric K, and H a closed 4-form on M. Let $\mathcal{M}$ be its minimal symplectic realization. We define a new degree 2 homotopy Poisson algebra structure on the algebra of functions of $\mathcal{M}$ by adding*

$$l_4(\chi_1,\chi_2,\chi_3,\chi_4) = H(a(\chi_1), a(\chi_2), a(\chi_3), a(\chi_4)), \quad \forall\ \chi_i \in \mathcal{A}^2.$$

Choose a local coordinate (x^i, p_i, ξ^a). A degree 3 function $\alpha = \rho^i_a p_i \xi^a - \frac{1}{3!} f_{abc}\xi^a\xi^b\xi^c$ is a Maurer-Cartan element if and only if

$$\frac{1}{2} l_2(\alpha,\alpha) + \frac{1}{24} l_4(\alpha,\alpha,\alpha,\alpha) = 0.$$

Example

(continue) Define an anchor $\rho : E \to TM$ and a derived bracket $\lceil \cdot, \cdot \rceil$ on $\Gamma(E)$ by

$$\rho(e)f = l_2(l_2(e, \alpha), f), \quad \lceil e_1, e_2 \rceil = l_2(l_2(e_1, \alpha), e_2).$$

On the other hand, a straightforward calculation gives that

$$l_4(\alpha, \alpha, \alpha, \alpha) = \rho^* H.$$

Thus, the condition that α is a Maurer-Cartan element is equivalent to the fact that $(E, K, \rho, \lceil \cdot, \cdot \rceil, H)$ is a twisted Courant algebroid, which arises from the study of three dimensional sigma models with Wess-Zumino term.

Theorem

A degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{2\leq i<\infty})$ *with a degree* $n+1$ *Maurer-Cartan element* α *one-to-one corresponds to a twisted symplectic NQ-manifold* $(\mathcal{M}, \{\cdot,\cdot\}_s, \alpha)$ *with* $\Theta|_{\mathcal{M}} = 0$.

Theorem

A degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{2\le i<\infty})$ *with a degree* $n+1$ *Maurer-Cartan element* α *one-to-one corresponds to a twisted symplectic NQ-manifold* $(\mathcal{M}, \{\cdot,\cdot\}_s, \alpha)$ *with* $\Theta|_{\mathcal{M}} = 0$.

Thus, associated to a degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{2\le i<\infty})$ with a degree $n+1$ Maurer-Cartan element α, there is a AKSZ sigma model with boundary.

N. Ikeda and X. Xu, Canonical functions, differential graded symplectic pairs in supergeometry, and AKSZ sigma models with boundaries, *J. Math. Phys*, 55, 113505 (2014).

Outline

Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$, the cotangent bundle $T^*[2]\mathfrak{g}^*[1]$ is a symplectic NQ-manifold of degree 2. Furthermore, symplectic NQ-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids. Thus, from $T^*[2]\mathfrak{g}^*[1]$, we obtain a Courant algebroid E:

$$E = \mathfrak{g}^*_{-1} \times (\mathfrak{g}_0^* \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}^*_{-1}, \tag{4}$$

in which the anchor and the Dorfman bracket are defined by the derived bracket using the degree 3 function $l = -\sum l_i$.

Proposition

Consider the Courant algebroid E given above, for constant sections $x, y \in \mathfrak{g}_0, \xi, \eta \in \mathfrak{g}_0^*$ and linear function $m \in \mathfrak{g}_{-1}$, we have

(i) $\rho(x)(m) = l_2^1(x, m)$, *i.e. the anchor of x is a linear vector field;*

(ii) $\rho(\xi) = -l_1^*(\xi)$, *i.e. the anchor of ξ is a constant vector field;*

(iii) *the image of a linear function under the operator D is not a constant section, we have*

$$Dm = l_1(m) - l_2^1(m, \cdot) \in \mathfrak{g}_0 + \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}; \tag{5}$$

(iv) $[x, y] = l_2^0(x, y) + l_3(x, y, \cdot) \in \mathfrak{g}_0 + \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1};$

Given a Courant algebroid $E \longrightarrow M$, using the skew-symmetric Courant bracket, we get a 2-term L_∞-algebra structure on $C^\infty(M) \oplus \Gamma(E)$. Now consider the Courant algebroid (4) obtained from a 2-term L_∞-algebra. Since it is linear, we pick linear functions on $\mathfrak{g}_{-1}^*$ as the degree -1 part and $\mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0^*)$ as the degree 0 part.

Proposition

For any $x, y \in \mathfrak{g}_0$, and $\xi \otimes m, \eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have

$$\begin{cases} [\![x, y]\!] = x \circ y = l_2^0(x, y) + l_3(x, y, \cdot); \\ [\![x, \xi \otimes m]\!] = \xi \otimes l_2^1(x, m) + (l_2^0(x, \cdot)^* \xi) \otimes m \\ \qquad -\frac{1}{2}\xi(x)(l_1(m) - l_2^1(m, \cdot)); \\ [\![\xi \otimes m, \eta \otimes n]\!] = l_1^* \eta(m) \xi \otimes n - l_1^* \xi(n) \eta \otimes m. \end{cases} \tag{6}$$

Theorem

Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$, we can obtain a new 2-term L_∞-algebra

$$\tilde{\mathfrak{g}} = (\mathfrak{g}_{-1} \xrightarrow{\tilde{l}_1} \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}_0^*), \tilde{l}_2 = \tilde{l}_2^0 + \tilde{l}_2^1, \tilde{l}_3)$$

from the corresponding Courant algebroid (4), in which $\tilde{l}_1 = D$ (given by (5)), $\tilde{l}_2^0$ is given by (6), $\tilde{l}_2^1$ and $\tilde{l}_3$ are given by

$$\begin{aligned} &\tilde{l}_2^1(x + \xi \otimes m, n) = \frac{1}{2} l_2^1(x, n) + \frac{1}{2}\langle \xi, l_1(n)\rangle m, \\ &\tilde{l}_3(x_1 + \xi_1 \otimes m_1, x_2 + \xi_2 \otimes m_2, x_3 + \xi_3 \otimes m_3) \\ = \ &-\frac{1}{2} l_3(x_1, x_2, x_3) - \frac{1}{2}(\langle l_2^0(x_1, x_2), \xi_3\rangle m_3 + c.p.) \\ &+ \textit{some terms} \end{aligned}$$

2-term L_∞-algebra $\mathfrak{g}$ $\longmapsto$ homotopy Poisson manifold $\mathfrak{g}^*[1]$ $\longmapsto$ degree 2 symplectic NQ-manifold $T^*[2]\mathfrak{g}^*[1]$

$\longmapsto$ Courant algebroid $E = \mathfrak{g}^*_{-1} \times (\mathfrak{g}_0 \oplus \mathfrak{g}_0^*)$ $\longmapsto$ 2-term L_∞-algebra $\tilde{\mathfrak{g}}$.

Example

(**omni-Lie algebra**) Let V be a vector space. Consider the abelian 2-term L_∞-algebra $(V \xrightarrow{\mathrm{id}} V, l_2 = 0, l_3 = 0)$, we get a new 2-term L_∞-algebra

$$(V \xrightarrow{\mathbb{i}} V \oplus \mathfrak{gl}(V), \tilde{l}_2, \tilde{l}_3),$$

in which $\mathbb{i}$ is the natural inclusion, and

$$\begin{cases} \tilde{l}_2^1(u + A, m) &= \frac{1}{2}Am, \\ \tilde{l}_2^0(u + A, v + B) &= \frac{1}{2}(Av - Bu) + [A, B], \\ \tilde{l}_3(u + A, v + B, w + C) &= -\frac{1}{4}([A, B]w + [B, C]u + [C, A]v), \end{cases}$$

for all $u, v, w \in V_0 = V$, $m \in V_{-1} = V$ and $A, B, C \in \mathfrak{gl}(V)$. This 2-term L_∞-algebra is the one associated to the omni-Lie algebra $V \oplus \mathfrak{gl}(V)$.

Example

(**2-term L_∞-algebra of string type**) Let $(\mathfrak{k},[\cdot,\cdot]_{\mathfrak{k}})$ be a Lie algebra. Consider the 2-term L_∞-algebra $(\mathbb{R}\stackrel{0}{\longrightarrow}\mathfrak{k}, l_2=[\cdot,\cdot]_{\mathfrak{k}}, l_3=0)$, we get a new 2-term L_∞-algebra

$$(\mathbb{R}\stackrel{0}{\longrightarrow}\mathfrak{k}\oplus\mathfrak{k}^*,\tilde{l}_2,\tilde{l}_3),$$

where $\tilde{l}_2$ and $\tilde{l}_3$ are given by

$$\left\{\begin{array}{rcl}\tilde{l}_2^1(u+\xi,r)&=&0,\\ \tilde{l}_2^0(u+\xi,v+\eta)&=&[u,v]_{\mathfrak{k}}+\mathrm{ad}_u^*\eta-\mathrm{ad}_v^*\xi,\\ \tilde{l}_3(u+\xi,v+\eta,w+\zeta)&=&-\frac{1}{2}(\langle[u,v]_{\mathfrak{k}},\zeta\rangle+\langle[v,w]_{\mathfrak{k}},\xi\rangle+\langle[w,u]_{\mathfrak{k}},\eta\rangle\end{array}\right.$$

This is exactly the 2-term L_∞-algebra of string type.

Example

Consider the 2-term L_∞-algebra $(\mathfrak{k} \xrightarrow{\mathrm{id}} \mathfrak{k}, l_2 = [\cdot,\cdot]_{\mathfrak{k}}, l_3 = 0)$, where $(\mathfrak{k}, [\cdot,\cdot]_{\mathfrak{k}})$ is a Lie algebra. By Theorem 3.3, we obtain a new 2-term L_∞-algebra

$$(\mathfrak{k} \xrightarrow{\tilde{l}_1} \mathfrak{k} \oplus \mathfrak{gl}(\mathfrak{k}), \tilde{l}_2, \tilde{l}_3),$$

$$\begin{aligned}
\tilde{l}_1(m) &= m - \mathrm{ad}_m,\\
\tilde{l}_2^1(u + A, m) &= \frac{1}{2}[u, m]_{\mathfrak{k}} + \frac{1}{2}Am,\\
\tilde{l}_2^0(u + A, v + B) &= [u, v]_{\mathfrak{k}} + \frac{1}{2}(Av - Bu) + [\mathrm{ad}_u, B] + [A, \mathrm{ad}_v]\\
&\quad + \frac{1}{2}(\mathrm{ad}_{Bu} - \mathrm{ad}_{Av}) + [A, B],\\
\tilde{l}_3(u + A, v + B, w + C) &= -\frac{1}{2}C[u, v]_{\mathfrak{k}} - \frac{1}{4}[A, B]w\\
&\quad - \frac{1}{4}([u, Bw]_{\mathfrak{k}} + [Bu, w]_{\mathfrak{k}}) + c.p.
\end{aligned}$$

Definition

A nonabelian omni-Lie algebra associated to a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *is a triple* $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \{\cdot,\cdot\}_{\mathfrak{g}})$*, where* $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ *is the symmetric* $\mathfrak{g}$*-valued pairing given by*

$$\langle A + u, B + v\rangle_{\mathfrak{g}} = Av + Bu,$$

and $\{\cdot,\cdot\}_{\mathfrak{g}}$ *is the bilinear bracket given by*

$$\{A + u, B + v\}_{\mathfrak{g}} = [A, B] + [A, \mathrm{ad}_v] + [\mathrm{ad}_u, B] - \mathrm{ad}_{Av} + Av + [u, v]_{\mathfrak{g}}.$$

Proposition

$(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\}_{\mathfrak{g}})$ *is a Leibniz (Loday) algebra.*

Proposition

$(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \{\cdot,\cdot\}_{\mathfrak{g}})$ *is a Leibniz (Loday) algebra.*

Proposition

$(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \{\cdot,\cdot\}_{\mathfrak{g}})$ *is a trivial deformation of the omni-Lie algebra* $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \langle\cdot,\cdot\rangle, \{\cdot,\cdot\}_0)$ *via the Nijenhuis operator*

$$N = \begin{pmatrix} 0 & \mathrm{ad} \\ 0 & 0 \end{pmatrix} : \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}.$$

The Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ yields a Lie-Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$. Then we obtain a Courant algebroid structure on $T\mathfrak{g}^* \oplus T^*_{\pi_{\mathfrak{g}}}\mathfrak{g}^*$. Denote the sets of linear vector fields and constant 1-forms on $\mathfrak{g}^*$ by $\mathfrak{X}_{lin}(\mathfrak{g}^*)$ and $\Omega^1_{con}(\mathfrak{g}^*)$. For any $x \in \mathfrak{g}$, denote by l_x the corresponding linear function on $\mathfrak{g}^*$. Let $\{x^i\}$ be a basis of the vector space underlying $\mathfrak{g}$. Then $\{l_{x^i}\}$ constitute a local coordinate of $\mathfrak{g}^*$. So $\{\frac{\partial}{\partial l_{x^i}}\}$ constitute a basis of vector fields on $\mathfrak{g}^*$ and $\{dl_{x^i}\}$ constitute a basis of 1-forms on $\mathfrak{g}^*$. For $A \in \mathfrak{gl}(\mathfrak{g})$, we get a linear vector field $\hat{A} = \sum_j l_{A(x^j)} \frac{\partial}{\partial l_{x^j}}$ on $\mathfrak{g}^*$. Also $u = \sum_i u_i x^i \in \mathfrak{g}$ defines a constant 1-form $\hat{u} = \sum_i u_i dl_{x^i}$ on $\mathfrak{g}^*$.

Define $\Phi : \mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g} \longrightarrow \mathfrak{X}_{lin}(\mathfrak{g}^*) \oplus \Omega^1_{con}(\mathfrak{g}^*)$ by

$$\Phi(A+u) = \hat{A} + \hat{u}.$$

Obviously, Φ is an isomorphism between vector spaces.

Theorem

The nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g}) \oplus \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \{\cdot,\cdot\}_{\mathfrak{g}})$ *is induced from the Courant algebroid* $T\mathfrak{g}^* \oplus T^*_{\pi_{\mathfrak{g}}}\mathfrak{g}^*$ *via the restriction on* $\mathfrak{X}_{lin}(\mathfrak{g}^*) \oplus \Omega^1_{con}(\mathfrak{g}^*)$. *More precisely, we have*

$$\begin{cases} \langle \Phi(A+u), \Phi(B+v)\rangle_{CA} & = & I_{\langle A+u, B+v\rangle_{\mathfrak{g}}}, \\ \{\Phi(A+u), \Phi(B+v)\}_{CA} & = & \Phi\{A+u, B+v\}_{\mathfrak{g}}. \end{cases}$$

For any bilinear map $F:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$, define $\mathrm{ad}^F:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g})$ by $\mathrm{ad}^F(u)=F(u,\cdot)$.

Proposition

Given a bilinear map $F:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$, the graph of ad^F, which we denote by $\mathcal{G}_F$, is a Dirac structure of the nonabelian omni-Lie algebra $(\mathfrak{gl}(\mathfrak{g})\oplus\mathfrak{g},\langle\cdot,\cdot\rangle,\{\cdot,\cdot\}_{\mathfrak{g}})$ iff F satisfies the Maurer Cartan equation

$$dF+\frac{1}{2}[F,F]=0,$$

Thus, Dirac structures of the form $\mathcal{G}_F$ characterize deformations of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$.

Outline

Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ be a 2-term L_∞-algebra. The vector bundle $E = \mathfrak{g}_{-1}^* \times (\mathfrak{g}_0^* \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}_{-1}^*$ can be decomposed as $A \oplus A^*$, where $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ is a trivial vector bundle. $\{l, l\} = 0$, where $l = l_1 + (l_2^1 + l_2^0) + l_3$, implies that

$$\{l_1, l_1\} = 0 \quad \{l_1, l_2\} = 0, \quad \frac{1}{2}\{l_2, l_2\} + \{l_1, l_3\} = 0, \quad \{l_2, l_3\} = 0,$$

where $\{\cdot, \cdot\}$ is Kosmann-Schwarzbach's big bracket.

Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ be a 2-term L_∞-algebra. The vector bundle $E = \mathfrak{g}_{-1}^* \times (\mathfrak{g}_0^* \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}_{-1}^*$ can be decomposed as $A \oplus A^*$, where $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ is a trivial vector bundle. $\{l, l\} = 0$, where $l = l_1 + (l_2^1 + l_2^0) + l_3$, implies that

$$\{l_1, l_1\} = 0 \quad \{l_1, l_2\} = 0, \quad \frac{1}{2}\{l_2, l_2\} + \{l_1, l_3\} = 0, \quad \{l_2, l_3\} = 0,$$

where $\{\cdot, \cdot\}$ is Kosmann-Schwarzbach's big bracket.

Theorem

Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ be a 2-term L_∞-algebra, then $(A, -l)$ is a Lie quasi-bialgebroid, and the Courant algebroid E is the double of the Lie quasi-bialgebroid $(A, -l)$.

Corollary

The Lie algebroid structure on A, determined by $-l_1$, is given by

(i) *for any constant section $\xi \in \mathfrak{g}_0^*$, $\rho_A(\xi) = -l_1^*(\xi)$;*

(ii) *for any constant sections $\xi, \eta \in \mathfrak{g}_0^*$, we have $[\xi, \eta]_A = 0$;*

(iii) *for any constant section $\xi \in \mathfrak{g}_0^*$ and linear section $\eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have $[\xi, \eta \otimes n]_A = \langle \xi, l_1(n) \rangle \eta$;*

(iv) *for any linear sections $\xi \otimes m, \eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have*

$$[\xi \otimes m, \eta \otimes n]_A = \langle l_1^* \eta, m \rangle \xi \otimes n - \langle l_1^* \xi, n \rangle \eta \otimes m.$$

Thus, A is an action Lie algebroid of the abelian Lie algebra $\mathfrak{g}_0^*$ acting on $\mathfrak{g}_{-1}^*$ via $-l_1^*$, which sends an element $\xi \in \mathfrak{g}_0^*$ to a constant vector field $-l_1^*(\xi) \in \mathfrak{g}_{-1}^*$.

Corollary

For all constant section $x \in \mathfrak{g}_0$ of A^*, $-l_2^1$ gives rise to an anchor map ρ_{A^*} of A^* via

$$\rho_{A^*}(x) = l_2^1(x, \cdot),$$

which is a linear vector field. For all constant sections $x, y \in \mathfrak{g}_0$, $-l_2^0$ gives rise to the bracket operation on A^*:

$$[x, y]_{A^*} = l_2^0(x, y).$$

The Jacobi identity of $[\cdot, \cdot]_{A^*}$ is controlled by $\phi = -l_3 \in \wedge^3 \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1} \subset \Gamma(\wedge^3 A)$. More precisely, we have

$$\begin{aligned}[[x, y]_{A^*}, z]_{A^*} + c.p. &= d_A\phi(x, y, z) + \phi(d_A x, y, z) - \phi(x, d_A y, z) \\ &\quad + \phi(x, y, d_A z).\end{aligned}$$

Corollary

Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$, we obtain a Lie quasi-bialgebroid (A, δ, ϕ), where the Lie algebroid $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ is determined by $-l_1$, $\delta : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k+1} A)$ is the generalized Chevalley-Eilenberg operator determined by the anchor ρ_{A^*} and the bracket $[\cdot, \cdot]_{A^*}$, and $\phi = -l_3$.

Theorem

The quasi-Poisson groupoid corresponding to (A, δ, ϕ) *is* $(\Gamma, \Pi, \phi = -l_3)$, *where* $\Gamma : \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}_{-1}^*$ *is the action groupoid integrating* A, Π *is characterized by*

$$\Pi(dx, dy) = -l_2^0(x, y), \quad \Pi(dx, dm) = -l_2^1(x, m), \quad \Pi(dm, dn) = -l_2^1(l_1(m), n),$$

where d *is the usual de Rham differential, and* $x, y \in \mathfrak{g}_0, m, n \in \mathfrak{g}_{-1}$ *are linear functions on* $\mathfrak{g}_{-1}^* \times \mathfrak{g}_0^*$.

Theorem

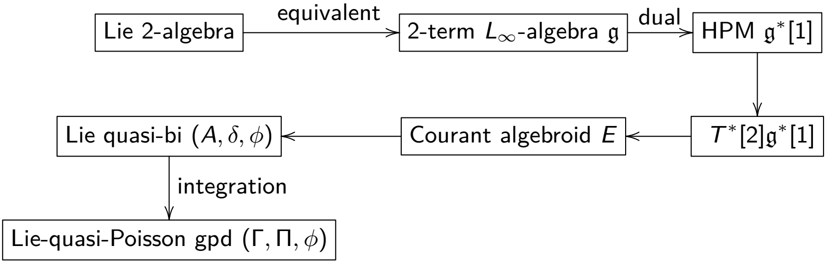
The quasi-Poisson groupoid corresponding to (A, δ, ϕ) *is* $(\Gamma, \Pi, \phi = -l_3)$, *where* $\Gamma : \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}_{-1}^*$ *is the action groupoid integrating* A, Π *is characterized by*

$$\Pi(dx, dy) = -l_2^0(x, y), \quad \Pi(dx, dm) = -l_2^1(x, m), \quad \Pi(dm, dn) = -l_2^1(l_1(m), n),$$

where d *is the usual de Rham differential, and* $x, y \in \mathfrak{g}_0$, $m, n \in \mathfrak{g}_{-1}$ *are linear functions on* $\mathfrak{g}_{-1}^* \times \mathfrak{g}_0^*$.

One can also obtain a quasi-Poisson Lie 2-group from a Lie 2-algebra via the integration of Lie 2-bialgebras directly, see

Z. Chen, M. Stienon and P. Xu, Poisson 2-groups, *J. Diff. Geom.*, 2013, 94(2): 209-240.


Lie 2-algebra
equivalent
2-term L_∞-algebra $\mathfrak{g}$
dual
HPM $\mathfrak{g}^*[1]$
$T^*[2]\mathfrak{g}^*[1]$
Courant algebroid E
Lie quasi-bi (A, δ, ϕ)
integration
Lie-quasi-Poisson gpd (Γ, Π, ϕ)

Outline

One construct an Ikeda-Uchino algebroid structure on a vector bundle E from degree 3 symplectic NQ-manifold $(T^*[3]E[1], \Theta)$, where the Q-structure is given by Θ, which is a function on $T^*[3]E[1]$ of degree 4:

N. Ikeda and K. Uchino, QP-structures of degree 3 and 4D topological field theory. *Comm. Math. Phys.* 303 (2011), no. 2, 317-330.

Definition

An Ikeda-Uchino algebroid is a vector bundle $E \longrightarrow M$ together with a skew-symmetric bracket operation $[\cdot,\cdot] : \Gamma(\wedge^2 E) \longrightarrow \Gamma(E)$, a bundle map $\rho : E \longrightarrow TM$, a symmetric bundle map $\partial : E^* \longrightarrow E$ which induces a fiber metric $(\cdot,\cdot)_+$ (not necessarily nondegenerate) on E^* via $(\alpha_1,\alpha_2)_+ := \langle \partial\alpha_1, \alpha_2\rangle$, and $\Omega \in \Gamma(\wedge^4 E^*)$, such that for all $e_1, e_2, e_3, e_4 \in \Gamma(E), \alpha_1, \alpha_2 \in \Gamma(E^*)$, and $f \in C^\infty(M)$, the following equalities are satisfied:

(A$_1$) $\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)], \quad [e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2;$

(A$_2$) $[[e_1, e_2], e_3] + c.p. = \partial\Omega(e_1, e_2, e_3);$

(A$_3$) $\rho \circ \partial = 0, \quad \delta\Omega = 0;$

(A$_4$) $\rho(e_1)(\alpha_1, \alpha_2)_+ = (L_{e_1}\alpha_1, \alpha_2)_+ + (\alpha_1, L_{e_1}\alpha_2)_+.$

Let $\mathfrak{g} = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}; l_1, l_2, l_3, l_4)$ be a 3-term L_∞-algebra, then $T^*[3]\mathfrak{g}^*[2]$ is a symplectic NQ-manifold of degree 3, where the Q-structure is given by $Q = \{\sum l_i, \cdot\}$. On the other hand, $T^*[3]\mathfrak{g}^*[2] = T^*[3]E[1]$, where $E = \mathfrak{g}^*_{-2} \times (\mathfrak{g}^*_{-1} \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}^*_{-2}$.

Theorem

From a 3-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}; l_1, l_2, l_3, l_4)$, we can get an Ikeda-Uchino algebroid

$$E = \mathfrak{g}^*_{-2} \times (\mathfrak{g}^*_{-1} \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}^*_{-2}.$$

Theorem

(continue) For all constant sections $x, y, z, w \in \mathfrak{g}_0, \alpha, \beta \in \mathfrak{g}^*_{-1}$ *of* E*, all constant sections* $\xi \in \mathfrak{g}^*_0, m, n \in \mathfrak{g}_{-1}$ *of* E^**, and all linear functions* $f \in \mathfrak{g}_{-2}$ *on the base manifold, we have*

(i) *the anchor* ρ *is given by* $\rho(x+\alpha)(f) = -l^2_2(x, f) - \langle \alpha, l^1_1(f)\rangle$;

(ii) *the skew-symmetric brackets* $[\cdot,\cdot]$ *is given by*

$$[x, y] = -l^0_2(x, y) - l^1_3(x, y, \cdot), \ \ [x, \alpha] = -l^1_2(x, \cdot)^*\alpha, \ \ [\alpha, \beta] = 0;$$

(iii) *the symmetric pairing on* E^* *is given by*

$$(m, n)_+ = l^3_2(m, n), \ \ (m, \xi)_+ = \langle l^0_1(m), \xi\rangle;$$

(iv) *the 4-form* Ω *on* E *is defined by*

$$\Omega(x, y, z, w) = l_4(x, y, z, w), \ \ \Omega(x, y, z, \alpha) = -\langle l^0_3(x, y, z), \alpha\rangle.$$

Thanks for your attention!

Thanks for your attention!

Happy Birthday to Janusz!