Vector Bundle Valued Differential Forms on Non-Negatively Graded DG Manifolds

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Introduction

Graded geometry encodes efficiently (non-graded) geometric structures, e.g. $\mathbb{N}Q$-manifolds encode Lie algebroids and their higher analogues.

Remark

$\mathbb{N}Q$-*manifolds* $(\mathcal{M}, Q)$ + *a compatible geometric structure*
encode
higher Lie algebroids + *a compatible structure*.

Differential forms on $\mathcal{M}$ preserved by Q are of a special interest. Vector bundle (VB) valued forms are even more interesting!

VB valued forms describe several interesting geometries: *foliated*, *(pre)contact*, *(pre)symplectic*, *locally conformal symplectic*, *poly-symplectic*, *cosymplectic*, *multisymplectic*, ...

Introduction

Examples

deg	$\mathbb{N}Q$-manifold	standard geometry	proved in
1	foliated	infinitesimal ideal system	[Zambon & Zhu 2012]
1	contact	Jacobi	[Grabowski 2013] [Mehta 2013]
1	symplectic	Poisson	[Roytenberg 2002]
2	contact	contact-Courant	[Grabowski 2013]
2	symplectic	Courant	[Roytenberg 2002]

Remark

All above cases can be regarded as:

$\mathbb{N}Q$-*manifold + a compatible VB valued form.*

Introduction

The above examples motivate the study of *VB valued differential forms* on $\mathbb{N}Q$-manifolds!

Aims of the Talk

1. describe VB valued differential forms on $\mathbb{N}$-manifolds in terms of non-graded geometric data,
2. use this description as a *unified formalism* for examples above,
3. enlarge the list of examples.

Remark

I work in the simplest case: deg 1, i.e. (non-higher) Lie algebroids.

Outline

Outline

Reminder on Graded Manifolds

Definition: *graded manifold*

A pair $\mathcal{M} = (M, C^\infty(\mathcal{M}))$ consisting of a manifold M and a graded $C^\infty(M)$-algebra $C^\infty(\mathcal{M}) \approx \Gamma(S^\bullet E^\bullet)$ for some graded VB $E^\bullet \to M$.

Remark

Smooth maps, vector fields, differential forms, etc. on $\mathcal{M}$ are defined algebraically via graded differential calculus on $C^\infty(\mathcal{M})$.

Think of $\mathcal{M}$ as a space locally coordinatized by (x^i, z^α) :

- $\deg x^i = 0 \Longrightarrow$ the x^i's *commute*,
- $\deg z^\alpha =: |\alpha| \in \mathbb{Z} \smallsetminus 0 \Longrightarrow$ the z^α's *graded commute*.

The *Euler vector field*

$$\Delta = |\alpha|\, z^\alpha \frac{\partial}{\partial z^\alpha}$$

measures the *internal degree* of geometric objects.

Q-manifolds and Lie algebroids

Remark

I work with $\mathbb{N}$-manifolds $\mathcal{M}$, i.e. $C^\infty(\mathcal{M})$ is non-negatively graded. The *degree* of $\mathcal{M}$ is the *highest* degree of its coordinates.

Definition: $\mathbb{N}Q$-*manifold*

An $\mathbb{N}$-manifold $\mathcal{M}$ + an homological vector field Q, i.e.

$$\deg Q = 1, \quad \text{and} \quad [Q, Q] = 0.$$

Proposition

There is a one-to-one correspondence between deg 1 $\mathbb{N}Q$*-manifolds and Lie algebroids, given by* $(A[1], Q = d_A) \Longleftarrow (A, \rho_A, [-,-]_A)$. *Conversely*

$$[\alpha, \beta]_A^\vee = [[Q, \alpha^\vee], \beta^\vee] \quad \text{and} \quad \rho_A(\alpha) f = [Q, \alpha^\vee] f, \quad \alpha, \beta \in \Gamma(A).$$

where $\alpha^\vee \in \mathfrak{X}(A[1]) :=$ vertical lift of $\alpha \in \Gamma(A)$.

VB Valued Forms on $\mathbb{N}$-manifolds

Let $\mathcal{E} \to \mathcal{M}$ be a VB in the category of graded manifolds. There is a Cartan calculus on $\Omega(\mathcal{M}, \mathcal{E}) := \{\mathcal{E}$-valued differential forms on $\mathcal{M}\}$.

Definition: *derivation of $\mathcal{E}$*

An $\mathbb{R}$-linear, graded operator $\mathbb{X} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\mathbb{X}(fe) = X(f)e + (-)^{|f|} f\mathbb{X}e, \quad \text{for some graded vector field } X.$$

Remark

$\omega \in \Omega(\mathcal{M}, \mathcal{E})$ can be *contracted with* and *Lie differentiated along* $\mathbb{X}$. Interior products and Lie derivatives satisfy usual Cartan identities:

$$[i_{\mathbb{X}}, i_{\mathbb{Y}}] = 0, \quad [L_{\mathbb{X}}, i_{\mathbb{Y}}] = i_{[\mathbb{X},\mathbb{Y}]}, \quad [L_{\mathbb{X}}, L_{\mathbb{Y}}] = L_{[\mathbb{X},\mathbb{Y}]}.$$

Definition: *vector $\mathbb{N}Q$-bundle*

A VB $\mathcal{E} \to \mathcal{M}$ + an homological derivation $\mathbb{Q}$, i.e.

$$\deg \mathbb{Q} = 1, \quad \text{and} \quad [\mathbb{Q}, \mathbb{Q}] = 0.$$

Spencer Data

Simplifying Assumption: $\Gamma(\mathcal{E})$ is generated in deg 0

I.e. $\mathcal{E} = \mathcal{M} \times_M E$ for some non-graded VB $E \to M$. Then *a negatively graded derivation $\mathbb{X}$ of $\mathcal{E}$ is determined by its symbol $X \in \mathfrak{X}_-(\mathcal{M})$*.

Key Remark

A degree $n > 0$ form $\omega \in \Omega^k(\mathcal{M}, \mathcal{E})$ is completely determined by interior products with and Lie derivatives along negatively graded derivations:

$$n\omega = L_\Delta \omega = |\alpha| \left(z^\alpha L_{\partial/\partial z^\alpha} \omega + dz^\alpha \wedge i_{\partial/\partial z^\alpha} \omega \right).$$

Definition: *Spencer data of a deg $n > 0$ form $\omega \in \Omega^k(\mathcal{M}, \mathcal{E})$*

$$D : \mathfrak{X}_-(\mathcal{M}) \longrightarrow \Omega^k(\mathcal{M}, \mathcal{E}), \quad X \longmapsto D(X) := L_X \omega,$$

and

$$\ell : \mathfrak{X}_-(\mathcal{M}) \longrightarrow \Omega^{k-1}(\mathcal{M}, \mathcal{E}), \quad X \longmapsto \ell(X) := i_X \omega.$$

Spencer Data

Theorem

Spencer data establish a one-to-one correspondence between degree $n > 0$ forms $\omega \in \Omega^k(\mathcal{M}, \mathcal{E})$ and pairs (D, ℓ), with

- $D : \mathfrak{X}_-(\mathcal{M}) \to \Omega^k(\mathcal{M}, \mathcal{E})$ *a degree n first order DO, and*
- $\ell : \mathfrak{X}_-(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M}, \mathcal{E})$ *a degree n $C^\infty(M)$-linear map,*

such that

$$D(fX) = fD(X) + (-)^X df \wedge \ell(X),$$

and, moreover,

$$L_X D(Y) - (-)^{XY} L_Y D(X) = D([X, Y]),$$
$$L_X \ell(Y) - (-)^{X(Y-1)} i_Y D(X) = \ell([X, Y]),$$
$$i_X \ell(Y) - (-)^{(X-1)(Y-1)} i_Y \ell(X) = 0.$$

One can describe (inductively on n) ω in terms of non-graded data!

A First Example: Degree 1 Symplectic $\mathbb{N}Q$-manifolds

Definition: *deg n symplectic $\mathbb{N}$-manifold*

A deg n $\mathbb{N}$-manifold $\mathcal{M}$ + a deg n symplectic form ω.

Example: *the shifted cotangent bundle $T^*[n]M$ of a deg 0 manifold M*

Notice that

$$\mathfrak{X}_-(T^*[n]M) = \Omega^1(M)[n].$$

$T^*[n]M$ is equipped with a deg n symplectic form ω determined by

$$L_{(df)^\vee}\omega = 0, \quad \text{and} \quad i_{(df)^\vee}\omega = df, \quad f \in C^\infty(M).$$

Hence $D = (-)^n d : \Omega^1(M) \to \Omega^2(M)$ and $\ell = \mathrm{id} : \Omega^1(M) \to \Omega^1(M)$.

Definition: *deg n symplectic $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a deg n symplectic form ω such that $L_Q\omega = 0$.

A First Example: Degree 1 Symplectic $\mathbb{N}Q$-manifolds

Theorem [Roytenberg 2002]

There is a "one-to-one" correspondence between deg 1 symplectic $\mathbb{N}Q$-manifolds $(\mathcal{M}, Q)$ and Poisson manifolds.

An alternative proof via Spencer data

Let $\mathcal{M} = A[1] \to M$. Then $\mathfrak{X}_-(\mathcal{M}) = \Gamma(A)[1]$.

$$\left.\begin{array}{rcc} \text{non-degeneracy} & \Rightarrow & \ell : \Gamma(A) \simeq \Omega^1(M) \\ \text{closedness} & \Rightarrow & D = -d \circ \ell \end{array}\right\} \Rightarrow (\mathcal{M}, \omega) \simeq (T^*[1]M, \omega),$$

Hence, $L_Q\omega = 0 \Leftrightarrow i_{(df)^\vee} i_{(dg)^\vee} L_Q \omega = L_{(df)^\vee} i_{(dg)^\vee} L_Q \omega = 0.$

From $Q = d_{T^*M}$ for a Lie algebroid $(T^*M, \rho_{T^*M}, [-,-]_{T^*M})$, follows

$$i_{(df)^\vee} i_{(dg)^\vee} L_Q \omega = -\rho_{T^*M}(df)(g) - \rho_{T^*M}(dg)(f)$$

and

$$L_{(df)^\vee} i_{(dg)^\vee} L_Q \omega = d\rho_{T^*M}(df)(g) - [df, dg]_{T^*M}.$$

Outline

Distributions on $\mathbb{N}Q$-Manifolds

Let $(\mathcal{M}, Q)$ be an $\mathbb{N}Q$-manifold, and $\mathcal{D} \subset T\mathcal{M}$ a distribution.

Simplifying Assumption: $T\mathcal{M}/\mathcal{D}$ is generated in deg $-k$

Projection

$$\theta_{\mathcal{D}} : T\mathcal{M} \to T\mathcal{M}/\mathcal{D} \to (T\mathcal{M}/\mathcal{D})[-k] =: \mathcal{E}_{\mathcal{D}}$$

can be seen as a deg k (surjective) $\mathcal{E}_{\mathcal{D}}$-valued 1-form.

Definition: *$\mathcal{D}$ compatible with Q*

If $[Q, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.

Proposition

$\mathcal{D}$ is compatible with Q iff there is an homological derivation $\mathbb{Q}$ of $\mathcal{E}_{\mathcal{D}}$ with symbol Q such that $\mathbb{L}_{\mathbb{Q}}\theta_{\mathcal{D}} = 0$.

Degree 1 Foliated $\mathbb{N}Q$-Manifolds

Definition: *deg n foliated $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + an involutive distribution $\mathcal{D}$ such that $T\mathcal{M}/\mathcal{D}$ is generated in deg $-n$ and $[Q, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.

Proposition [Zambon & Zhu 2012]

There is a "one-to-one" correspondence between deg 1 foliated $\mathbb{N}Q$-manifolds and Lie algebroids $A \to M$ + an infinitesimal ideal system covering TM.

Reminder: *infinitesimal ideal system covering TM*

- A Lie subalgebroid $B \subset A$ over M,
- a flat connection in A/B,

+ a certain compatibility condition.

There is a simple, alternative proof via Spencer data of $\theta_{\mathcal{D}}$.

Degree 1 Contact $\mathbb{N}Q$-Manifolds

A *contact structure on* $\mathcal{M}$ is an hyperplane distribution $\mathcal{C}$ with non-degenerate *curvature*: $\omega_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \longrightarrow T\mathcal{M}/\mathcal{C}$, $(X, Y) \longmapsto \theta_{\mathcal{C}}([X, Y])$.

Definition: *deg n contact $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a contact structure $\mathcal{C}$ such that $T\mathcal{M}/\mathcal{C}$ is generated in deg $-n$, and $[Q, \Gamma(\mathcal{C})] \subset \Gamma(\mathcal{C})$.

Proposition [Grabowski 2013], [Mehta 2013]

There is a "one-to-one" correspondence between deg 1 contact $\mathbb{N}Q$-manifolds and Jacobi bundles.

Reminder: *Jacobi bundle*

A line bundle $L \to M$ + a Lie bracket $\{-, -\}$ on $\Gamma(L)$ which is a first order DO in each entry.

There is a simple, alternative proof via Spencer data of $\theta_{\mathcal{C}}$.

Outline

Degree 1 LCS $\mathbb{N}Q$-Manifolds

A *lcs structure* on $\mathcal{M}$ is a pair consisting of a flat line bundle $(\mathcal{L}, \nabla)$ over $\mathcal{M}$ and a d_∇ -closed, non-degenerate, $\mathcal{L}$-valued, 2-form ω.

Definition: *deg n lcs $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}$-manifold $\mathcal{M}$ + a line $\mathbb{N}Q$-bundle $(\mathcal{L}, \mathbb{Q})$ generated in deg 0, and a lcs structure $((\mathcal{L}, \nabla), \omega)$ such that $\deg \omega = n$, and $L_{\mathbb{Q}}\omega = 0$.

Proposition

There is a "one-to-one" correspondence between deg 1 lcs $\mathbb{N}Q$-manifolds and lc Poisson manifolds.

Reminder: *lc Poisson manifold*

A manifold M with a flat line bundle (L, ∇) and a morphism $P : \wedge^2(T^*M \otimes L) \to L$ inducing a Lie bracket $\{-,-\}_P$ on $\Gamma(L)$:

$$\{\lambda, \mu\}_P = P(d_\nabla \lambda, d_\nabla \mu), \quad \lambda, \mu \in \Gamma(L).$$

Degree 1 Presymplectic $\mathbb{N}Q$-Manifolds

Definition: *deg n presymplectic $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a deg n presymplectic form ω such that $L_Q\omega = 0$.

Proposition

There is a "one-to-one" correspondence between deg 1, dim (m, m) *presymplectic $\mathbb{N}Q$-manifolds* [+ clean intersection] *and Dirac m-folds*.

Reminder: *Dirac manifold*

A manifold M + a maximal isotropic subbundle $D \subset TM \oplus T^*M$ whose sections are preserved by *Dorfman brackets*.

Dirac structures can be alternatively described within graded geometry as Lagrangian submanifolds in deg 2 symplectic $\mathbb{N}Q$-manifolds!

Degree 1 Poly-Symplectic $\mathbb{N}Q$-Manifolds

A *k-poly-symplectic structure* on $\mathcal{M}$ is a closed, $\mathbb{R}^k$-valued 2-form ω such that $\omega^\flat : X \mapsto i_X\omega$ is a VB embedding.

Definition: *deg n k-poly-symplectic* $\mathbb{N}Q$*-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a deg n k-poly-symplectic structure ω such that $L_Q\omega = 0$.

Proposition

There is a "one-to-one" correspondence between deg 1 k-poly-symplectic $\mathbb{N}Q$-manifolds and k-poly-Poisson manifolds.

Reminder: *k-poly-Poisson manifold* (in the sense of [Martinez 2015])

A manifold M + a subbundle $S \subset T^*M \otimes \mathbb{R}^k$ + a VB morphism $P : S \to TM$:

1. $i_{P(\alpha)}\beta + i_{P(\beta)}\alpha = 0$, with $\alpha, \beta \in \Gamma(S)$,
2. $\Gamma(S)$ is preserved by bracket $[\alpha, \beta]_S := L_{P(\alpha)}\beta - L_{P(\beta)}\alpha + di_{P(\beta)}\alpha$,
3. non degeneracy.

Half Step Behind: Degree 1 Cosymplectic $\mathbb{N}Q$-Manifolds

A *cosymplectic structure* on $\mathcal{M}$ is a pair (η, ω):

1. $\eta \in \Omega^1(\mathcal{M})$ and $\omega \in \Omega^2(\mathcal{M})$,
2. $\eta \neq 0$ and ω is non-degenerate on $\ker \eta$,
3. $d\eta = d\omega = 0$.

The "hard" part is to guess the definition of cosymplectic $\mathbb{N}Q$*-manifold!*

Definition: *deg n cosymplectic* $\mathbb{N}Q$*-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a deg n cosymplectic structure (η, ω) such that $L_Q \eta|_{\ker \eta} = L_Q \omega|_{\ker \eta} = 0$.

Proposition

There is a "one-to-one" correspondence between deg 1 *cosymplectic* $\mathbb{N}Q$*-manifolds and Poisson manifolds with a Poisson vector field.*

One Step Behind: Degree 1 Precontact $\mathbb{N}Q$-Manifolds

A *precontact structure* is an hyperplane distribution.

Definition: *deg n precontact $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a precontact structure $\mathcal{C}$ such that $T\mathcal{M}/\mathcal{C}$ is generated in deg $-n$, and $[Q, \Gamma(\mathcal{C})] \subset \Gamma(\mathcal{C})$.

Proposition

There is a "one-to-one" correspondence between deg 1, dim $(m, m+1)$ *precontact* $\mathbb{N}Q$*-manifolds* [+ clean intersection] *and Dirac-Jacobi bundles over a dim* m *manifold.*

Reminder: *Dirac-Jacobi bundle*

A line bundle $L \to M$ + a maximal isotropic subbundle $D \subset \operatorname{der} L \oplus J^1 L$ whose sections are preserved by Dorfman-Jacobi brackets.

Outline

Spencer Operators on Lie Algebroids

Spencer operators are infinitesimal counterparts of *multiplicative* VB valued differential forms on Lie groupoids.

Let $A \to M$ be a Lie algebroid and (E, ∇) a representation of A.

Definition: *E-valued k-Spencer operator on A*

A pair (D, ℓ) with

- $D : \Gamma(A) \to \Omega^k(M, E)$ a first order DO, and
- $\ell : \Gamma(A) \to \Omega^{k-1}(M, E)$ a $C^\infty(M)$-linear map,

such that

$$D(f\alpha) = fD(\alpha) - df \wedge \ell(\alpha), \quad \text{and, moreover}$$

$$L_{\nabla_\alpha} D(\beta) - L_{\nabla_\beta} D(\alpha) = D([\alpha, \beta]_A),$$
$$L_{\nabla_\alpha} \ell(\beta) + i_{\rho_A(\beta)} D(\alpha) = \ell([\alpha, \beta]_A),$$
$$i_{\rho_A(\alpha)} \ell(\beta) + i_{\rho_A(\beta)} \ell(\alpha) = 0.$$

deg 1 Higher Forms on $\mathbb{N}Q$-Manifolds

Theorem

There is a one-to-one correspondence between

1. - *deg* 1 $\mathbb{N}$*-manifolds* $\mathcal{M}$,
   - *an* $\mathbb{N}Q$*-vector bundle* $(\mathcal{E} \to \mathcal{M}, \mathbb{Q})$, *with* $\Gamma(\mathcal{E})$ *generated in deg* 0,
   - *a deg* 1 *form* $\omega \in \Omega^k(\mathcal{M}, \mathcal{E})$ *such that* $L_{\mathbb{Q}}\omega = 0$,
2. - *Lie algebroids* $A \to M$,
   - *a representation* (E, ∇) *of* A,
   - *an* E*-valued* k*-Spencer operator on* A.

Proof

$\mathcal{M} = A[1]$ and $\mathcal{E} = \mathcal{M} \times_M E$.

$\mathbb{Q} \Longleftrightarrow$ a Lie algebroid structure on A + a representation (E, ∇).

Spencer data of ω define an E-valued k-Spencer operator on A.

deg 1 Multisymplectic $\mathbb{N}Q$-manifolds

A *k-plectic structure* on $\mathcal{M}$ is $\omega \in \Omega^{k+1}(\mathcal{M})$ such that $\omega^\flat : X \mapsto i_X\omega$ is a VB embedding.

Definition: *deg n k-plectic $\mathbb{N}Q$-manifold*

A deg n $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ + a deg n k-plectic structure ω such that $L_Q\omega = 0$.

Corollary

There is a "one-to-one" correspondence between deg 1 k-plectic $\mathbb{N}Q$-manifolds and Lie algebroids + an IM k-plectic structure.

Reminder: *IM k-plectic structure on a Lie algebroid A*

A $C^\infty(M)$-linear map $\ell : \Gamma(A) \to \Omega^k(M)$ such that

$i_{\rho_A(\alpha)}\ell(\beta) + i_{\rho_A(\beta)}\ell(\alpha) = 0$, and $L_{\rho_A(\alpha)}\ell(\beta) - i_{\rho_A(\beta)}d\ell(\alpha) = \ell([X,Y]_A)$,

+ non-degeneracy conditions.

Conclusions

I presented a unified formalism for deg 1, contact, symplectic, and foliated ℕQ-manifolds.

New Examples in Degree 1

deg 1 ℕQ-manifold	standard geometry	(first) considered in
precontact	Dirac-Jacobi	[Wade 2000]
lcs	lc Poisson	[Vaisman 2007]
poly-symplectic	poly-Poisson	[Iglesias et al. 2013] [Martinez 2015]
presymplectic	Dirac	[Courant 1990]
cosymplectic	coPoisson	[Janyška & Modugno 2009]
higher form	Spencer operator	[Crainic et al. 2013]
multisymplectic	IM multisymplectic	[Bursztyn et al. 2013]

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Best Wishes, Janusz!