Multiplicativity, from Lie groups to generalized geometry

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Aim of this talk

I shall survey the notion of "multiplicativity" from its inception in Drinfel'd's 1983 paper to recent developments in the theory of Lie groupoids and in that of generalized geometry.

• It all started with Drinfel'd's, "Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations", *Dokl. Akad. Nauk SSSR* 268 (1983), translated in *Soviet Math. Doklady* 27 (1983).

• At the time that Kirill Mackenzie was publishing his first book, "Lie groupoids and Lie algebroids in differential geometry" (1987), groupoids entered the picture in a very important way with Alan Weinstein's introduction of symplectic groupoids (1987) and Poisson groupoids (1988).

This talk

• The main part of this talk will deal with the theory of Poisson (and symplectic) groupoids, synthesized from the work of many people, in particular, Weinstein, Mackenzie, and Ping Xu.

• I shall introduce elements of the generalized geometry of manifolds, and of Lie groupoids and Lie algebroids.

• I shall show how properties of various structures (Poisson, presymplectic, holomorphic) on groupoids can be viewed as particular cases of the theorems on multiplicative generalized complex structures on Lie groupoids in the paper (to appear) of Madeleine Jotz, Mathieu Stiénon and Ping Xu.

References?

Many authors, including several in the audience, have made important contributions to the subject, and I shall not attempt to cite them all.

But I want to emphasize the role of our colleagues Janusz Grabowski, Paweł Urbański, Katarzyna Grabowska, and all their co-authors, in Poland and elsewhere, in the development of the theory of Lie algebroids and Lie groupoids and its applications to mechanics, and to recall the pioneering role of Włodzimierz Tulczyjew and that of our late colleague and friend Stanisław Zakrzewski.

This talk could be viewed as a partial introduction to the recent preprint, "Graded bundles in the category of Lie groupoids" by Grabowska and Grabowski with James Bruce, available on arXiv since February.

Why multiplicativity?

Sophus Lie (1842-1899)

On a Lie group, there is a multiplication.

It is natural to ask that any additional structure on the group be compatible with that multiplication.

Multiplicativity is what expresses this compatibility.

Later on, we shall extend the consideration of Lie groups to that of Lie groupoids.

Multiplicativity, a very simple concept

• Consider a Lie group G.
By definition, G is a smooth manifold with a group structure such that the multiplication, $m : G \times G \to G, \quad (g, h) \mapsto m(g, h) = gh$, is a smooth map, and also the inversion, $g \mapsto g^{-1}$, is smooth.

• Let X be a vector field on G.

Question. Can we compare the value of X at gh with the values of X at g and at h?

Answer

• We can left-translate the value of X at h by the tangent of the left translation by g.

• We can right-translate the value of X at g by the tangent of the right translation by h.

• Both resulting vectors are tangent to G at gh.

• So we can add these two vectors.

Therefore...

Multiplicativity of vector fields on Lie groups

Definition

A vector field X is multiplicative if

$$\boxed{X_{gh} = g \cdot X_h + X_g \cdot h}$$

We have abreviated $(T_h\lambda_g)(X_h)$ to $g \cdot X_h$, where λ_g is the left translation by $g \in G$.
Similarly, we have abreviated $(T_g\rho_h)(X_g)$ to $X_g \cdot h$, where ρ_h is the right translation by $h \in G$.

Existence of multiplicative vector fields?

We now consider the Lie algebra $\mathfrak{g} = Lie(G)$ of G.
For any element $x \in \mathfrak{g}$, let x^λ be the left-invariant vector field defined by x so that, for all $g \in G$, $(x^\lambda)_g = g \cdot x$,
and let x^ρ be the right-invariant vector field defined by x, so that for all $g \in G$, $(x^\rho)_g = x \cdot g$.

Proposition

For any $x \in \mathfrak{g}$, the vector field $x^\lambda - x^\rho$ is multiplicative.

Proof. Use the properties $\lambda_{gh} = \lambda_g \circ \lambda_h$ and $\rho_{gh} = \rho_h \circ \rho_g$, and the fact that left- and right- translations commute.

A multiplicative vector field that can be written as $X = x^\lambda - x^\rho$ for some $x \in \mathfrak{g}$ is called exact or coboundary.

What is the flow of a multiplicative vector field?

The following proposition justifies the term "multiplicativity".

Proposition

Let ϕ_t be the flow of a multiplicative vector field X on a Lie group G. Then, for $g, h \in G$,

$$\boxed{\phi_t(gh) = \phi_t(g)\phi_t(h)}$$

Proof. In fact, $X = \frac{d}{dt}\phi_t|_{t=0}$ satisfies
$X_{gh} = \frac{d}{dt}\phi_t(g)\phi_t(h)_{|t=0} = \frac{d}{dt}\phi_t(g)_{|t=0} \cdot h + g \cdot \frac{d}{dt}\phi_t(h)_{|t=0}$,
and conversely by integration.

In the case of an exact multiplicative vector field, $X = x^\lambda - x^\rho$, the flow of X satisfies $\phi_t(g) = exp(-tx)\, g\, exp(tx)$.

Multiplicative multivectors

• The definition of multiplicativity is generalized to contravariant tensor fields X of order $k > 1$ in the obvious way: replacing $T_h\lambda_g$ by the tensor product of order k, $\otimes^k(T_h\lambda_g)$, and replacing $T_g\rho_h$ by $\otimes^k(T_g\rho_h)$.

• In particular, there are multiplicative multivectors.
Any $q \in \wedge^k\mathfrak{g}$ defines an exact multiplicative k-vector $Q = q^\lambda - q^\rho$.
For example, an element $r \in \wedge^2\mathfrak{g}$ defines an exact multiplicative bivector $\pi = r^\lambda - r^\rho$.

The general notion of "multiplicativity" for bivectors and more generally for multivectors appeared about 1990, together with the more general notion of affine multivectors, in the papers of Weinstein (1990), Lu and Weinstein (1990), Pierre Dazord and Daniel Sondaz (1991), yks (1991).

Another characterization of multiplicativity

Theorem

Let π be a Poisson bivector on a Lie group G. Then π is multiplicative if and only if the group multiplication $m : (g, h) \mapsto gh$ is a Poisson map from $G \times G$ to G, where $G \times G$ is equipped with the product Poisson structure.

Proof. Let $\{\ ,\ \}$ be the associated Poisson bracket defined by $\pi(df_1, df_2) = \{f_1, f_2\}$. A map, $m : G \to G \times G$, is a Poisson morphism if and only if
$\{f_1, f_2\}(gh) = \{f_1(g, .), f_2(g, .)\}(h) + \{f_1(., h), f_2(., h)\}(g)$
which is equivalent to the multiplicativity of π.

From QISM to multiplicativity

The theory of Poisson groups originated in

• the theory of the quantum inverse scattering method (QISM) and dressing transformations, that was the work of the Saint-Petersburg (then Leningrad) school, Ludwig Faddeev, Evgeny Sklyanin, P. P. Kulish, Leon Takhtajan, Mikhael Semenov-Tian-Shansky, and Alexei Reiman,

• the work of I. M. Gelfand and Irene Ya. Dorfman in Moscow on the relationship between solutions of the classical Yang-Baxter equation and Hamiltonian (i.e., Poisson) structures (1980, 1982).

Drinfel'd 1983

In his 3-page article (Doklady, 1983), motivated by Semenov-Tian-Shansky's "What is a classical r-matrix?" (1983), Drinfel'd introduced Lie groups with a "grouped Hamiltonian structure" which he called "Hamilton-Lie groups".

V. G. Drinfel'd (b. 1954) Fields medal 1990

Hamilton-Lie groups are defined by the requirement that the group multiplication be a Poisson map from the product manifold $G \times G$ with the product Poisson structure to G .

A little advertizing

In a lecture in Oberwolfach in the summer of 1986, I called Drinfel'd's Hamilton-Lie groups "Poisson-Drinfel'd groups". The full text appeared in 1987 in the proceedings of the conference.

N. Yu. Reshetikhin wrote in his review of that paper in *Mathematical Reviews* [MathSciNet] that it was an "attempt to understand the subject originally exposed by Drinfel'd in a very condensed form".

Reshetikhin was right!

Drinfel'd 1986

In his talk at the ICM in Berkeley in 1986, Drinfel'd called "Poisson-Lie groups" those groups in which "the Poisson bracket [is] compatible with the group operation".

[You may have heard the true story that, because Vladimir Drinfel'd was not allowed to travel to Berkeley, his celebrated address at the Congress in Berkeley in early August, "Quantum groups", was actually delivered by Pierre Cartier, who was given a few hours to master Drinfel'd's 32-page text which he had never seen before.
Cartier gave me a photocopy of a photocopy of Drinfel'd's typewritten manuscript shortly after he returned to Paris from Berkeley.]

We now use the term, Poisson Lie groups, following Drinfel'd and following Jiang-Hua Lu and Weinstein's influential paper (1990), or simply Poisson groups.

Definition

A Poisson group is a Lie group equipped with a multiplicative Poisson bivector.

Siméon-Denis Poisson (1781-1840)

Portrait by E. Marcellot, 1802. Copyright Collections École polytechnique

Multiplicativity as a cocycle condition

For any field of multivectors Q, define the map
$\rho(Q) : G \to \wedge^k \mathfrak{g} \quad \text{by} \quad \rho(Q)(g) = Q_g \cdot g^{-1}$,
and the map $\lambda(Q) : G \to \wedge^k \mathfrak{g} \quad \text{by} \quad \lambda(Q)(g) = -g^{-1} \cdot Q_g$.

Theorem

The following properties are equivalent:
(i) Q is multiplicative,
(ii) $\rho(Q)$ is a group cocycle,
(iii) $\lambda(Q)$ is a group cocycle.

Proof. If Q is multiplicative, then
$\rho(Q)(gh) = Q_g \cdot h \cdot (gh)^{-1} + g \cdot Q_h \cdot (gh)^{-1}$. Therefore,

$$\rho(Q)(gh) = \rho(Q)(g) + g \cdot \rho(Q)(h) \cdot g^{-1};$$

Thus $\rho(Q)$ is a 1-cocycle of the group G with values in $\wedge^k \mathfrak{g}$,
where G acts by the adjoint action.
The converse follows from the same calculation.
The computation for $\lambda(Q)$ is similar.

The exact case

If a multivector Q is multiplicative and exact, then $\rho(Q) = \lambda(Q)$. In fact, if $Q = q^\lambda - q^\rho$, then $\rho(Q) = g \cdot q \cdot g^{-1} - q = \lambda(Q)$.

A k-vector is multiplicative and exact if and only if $\rho(Q)$ (or $\lambda(Q)$) is a 1-coboundary of G with values in $\wedge^k \mathfrak{g}$.

These facts justify the terms "exact" and "coboundary".

Coboundary Poisson groups

When the bivector π is equal to $r^\lambda - r^\rho$, where $r \in \wedge^2\mathfrak{g}$, it is multiplicative. It remains to express the fact that it is "Poisson".

Theorem In order for $\pi = r^\lambda - r^\rho$ to be a Poisson bivector there is a conditon on r, known as the generalized classical Yang-Baxter equation: the element $[r, r] \in \wedge^3\mathfrak{g}$ is Ad_G-invariant. A sufficient condition is $[r, r] = 0$, which is known as the classical Yang-Baxter equation (CYBE).

What is $[r, r]$? It is the "algebraic Schouten bracket" defined as the extension of the Lie bracket of $\mathfrak{g}$ as a biderivation of the exterior algebra $\wedge^\bullet\mathfrak{g}$.
A Poisson group, defined by $r \in \wedge^2\mathfrak{g}$ satisfying the classical Yang-Baxter equation, is said to be triangular.

Questions

Are there multiplicative differential forms on Lie groups?
multiplicative symplectic forms?

Remark. Any multiplicative multivector Q vanishes at the identity of the group, because setting $g = h = e$ in the defining equation yields $Q_e = Q_e + Q_e$.
Since a multiplicative Poisson bivector vanishes at the identity of the group, it cannot be everywhere non degenerate, which implies that it cannot correspond to a symplectic structure.

We need to introduce Lie groupoids, following Reinhold Baer, Ronald Brown, Charles Ehresmann, Jean Pradines, Mackenzie, Weinstein, Xu and many others.

The tangent group of a group

The multiplication in TG is the tangent of the multiplication in G. In other words the multiplication $\times_{TG} : TG \times TG \to TG$ is defined as follows:
if $X \in T_g G$ and $Y \in T_h G$, then $\boxed{X \times_{(TG)} Y = g \cdot Y + X \cdot h}$.

Proposition
A vector field on G, $X : G \to TG$, is multiplicative if and only if X is a morphism of groups.

Proof. Vector field X, seen as a map from G to TG, is a morphism of groups if and only if

$$X_{gh} = X_g \times_{(TG)} X_h.$$

By the definition of the multiplication in the group TG, this condition coincides with the multiplicativity property for X.

Groupoids generalize groups

In a Lie groupoid, Γ and M are smooth manifolds.

$$\begin{array}{ccc} & \Gamma & \\ \alpha & \downarrow\downarrow & \beta \\ & M & \end{array}$$

Source $\alpha : \Gamma \to M$ and target $\beta : \Gamma \to M$ are surjective submersions.
There is a partially defined associative multiplication:
if $g \in \Gamma$, $h \in \Gamma$, then gh is defined if and only if $\alpha(g) = \beta(h)$.
Let $\Gamma^{(2)}$ be the submanifold of composable pairs in $\Gamma \times \Gamma$.
The multiplication map $(g, h) \in \Gamma^{(2)} \mapsto gh \in \Gamma$ is smooth.
Each element has an inverse.
The space M, called the base of the groupoid, is identified with the units of the multiplication.

Examples

Example. The trivial groupoid, $M \times M$ on base M, with the projections, $\alpha = pr_2$, $\beta = pr_1$, and multiplication map

$$(x, y)(y, z) = (x, z).$$

The inversion is $(x, y) \mapsto (y, x)$.
M is embedded in $M \times M$ as $m \mapsto (m, m)$.

Example. Any Lie group can be considered to be a groupoid over a point.

Example. The gauge groupoid of a principal bundle $P \to M$ whose structure group is a Lie group G. Let Γ be the quotient of $P \times P$ by the diagonal action of G on the right: $(g, (u, v)) \mapsto (ug, vg)$. The product of two elements in the quotient Γ is defined by choosing representatives (u, v) and (v, w) and setting

$$cl(u, v) \cdot cl(v, w) = cl(u, w).$$

The cotangent groupoid of a group

• The tangent bundle of a Lie group is a Lie group. It can be considered as a Lie groupoid over a point.

• What about the cotangent bundle of a Lie group?

• For any Lie group G, the cotangent bundle T^*G has a canonical structure of Lie groupoid with base $\mathfrak{g}^*$.

$$\begin{array}{ccc} & T^*G & \\ \alpha_{T^*G} & \downarrow\downarrow & \beta_{T^*G} \\ & \mathfrak{g}^* & \end{array}$$

For $\xi \in (T_g G)^*$, let $\alpha_{T^*G}(\xi) = \xi \circ T\lambda_g$ and $\beta_{T^*G}(\xi) = \xi \circ T\rho_g$.

If $\xi \in (T_g G)^*$ and $\eta \in (T_h G)^*$, their product is defined if and only if

$$\xi \circ T\lambda_g = \eta \circ T\rho_h,$$

and it is then

$$\xi \times_{(T^*G)} \eta = \xi \circ T\rho_{h^{-1}} = \eta \circ T\lambda_{g^{-1}} \in (T_{gh}G)^*.$$

Multiplicative bivectors as morphisms of groupoids

Theorem A

A Poisson bivector on a Lie group G is multiplicative if and only if it is a morphism from the Lie groupoid T^*G over $\mathfrak{g}^*$ to the Lie group TG considered as a Lie groupoid over a point.

$$\begin{array}{ccc} T^*G & \rightarrow & TG \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathfrak{g}^* & \rightarrow & \{pt\} \end{array}$$

This theorem is due to Mackenzie (1992).

Mackenzie's formulation permitted vast generalizations of the concept of multiplicativity.

From groups to groupoids: the tangent groupoid of a Lie groupoid

The construction of the tangent group of a group can be extended to the construction of the tangent groupoidof a groupoid, Γ,

$$\begin{array}{c} T\Gamma \\ \downarrow\downarrow \\ TM \end{array}$$

The groupoid multiplication in tangent bundle $T\Gamma$ is the tangent of the multiplication in Γ, and the source and target are also obtained by applying the tangent functor to the source and target of Γ.

From groups to groupoids: the cotangent groupoid of a Lie groupoid

For any Lie groupoid Γ, the cotangent bundle $T^*\Gamma$ has a canonical structure of a Lie groupoid, with, as its base, the dual A^* of the Lie algebroid A of Γ.

$$\begin{array}{ccc} & T^*\Gamma & \\ \alpha_{T^*\Gamma} & \downarrow\downarrow & \beta_{T^*\Gamma} \\ & A^* & \end{array}$$

Let $g \in \Gamma$. The source of an element $\xi \in (T_g\Gamma)^*$ is the element of $A^*_{\alpha g}$ defined by

$$\alpha_{T^*\Gamma}\xi(X) = \xi(g \cdot (X - \rho X)), \text{ for } X \in A_{\alpha g},$$

and its target is the element of $A^*_{\beta g}$ defined by

$$\beta_{T^*\Gamma}\xi(X) = \xi(X \cdot g), \text{ for } X \in A_{\beta g},$$

where ρ is the anchor of A. (Recall that $g \cdot X$ is short for $(T\lambda_g)(X)$ and $X \cdot g$ is short for $(T\rho_g)(X)$.)

The multiplication in $T^*\Gamma$

Let $g, h \in \Gamma$. If $\xi \in (T_g\Gamma)^*$ and $\eta \in (T_h\Gamma)^*$, their product is defined if and only if $\alpha_{T^*\Gamma}\xi = \beta_{T^*\Gamma}\eta$, and then $\alpha g = \beta h$. Their product is the element $\xi \times_{(T^*\Gamma)} \eta \in (T_{gh}\Gamma)^*$ such that

$$\boxed{(\xi \times_{(T^*\Gamma)} \eta)(X \times_{(T\Gamma)} Y) = \xi(X) + \eta(Y)}$$

for all $X \in T_g\Gamma$ and $Y \in T_h\Gamma$.

Poisson groupoids

Question. How can we generalize the definition of Poisson groups to groupoids?

The most straightforward method is to consider a groupoid analogue of the characterization of multiplicativity given in Theorem A and set it as the definition of a "multiplicative" Poisson structure on a Lie groupoid.

Definition

A Poisson groupoid is a Lie groupoid equipped with a Poisson bivector that defines a map, $\pi^\sharp : T^*\Gamma \to T\Gamma$, over a map $A^* \to TM$, which is a morphism of Lie groupoids,

$$\begin{array}{ccc} T^*\Gamma & \to & T\Gamma \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^* & \to & TM \end{array}$$

Coisotropic and Lagrangian submanifolds

By definition, a submanifold of a Poisson manifold is coisotropic if the Poisson bracket of two functions that vanish on the submanifold itself vanishes on the submanifold.

A submanifold of a Poisson manifold is coisotropic if and only if, at each point in the submanifold, the image by the Poisson map of the orthogonal of the tangent space is contained in the tangent space.

By definition, a submanifold of a symplectic manifold is Lagrangian if and only if it is coisotropic and of minimal dimension (half the dimension of the ambient symplectic manifold).

A submanifold of a symplectic manifold is Lagrangian if the symplectic orthogonal of each tangent space is equal to that tangent space.

Original definition of Poisson groupoids

Poisson groupoids were introduced by Weinstein in "Coisotropic calculus and Poisson groupoids" (1988) as a generalization of both the Poisson groups and the symplectic groupoids.

They were defined as Lie groupoids with a Poisson structure, π, which is multiplicative, in the sense that the graph of the multiplication is a coisotropic submanifold of the Poisson manifold, $\Gamma_\pi \times \Gamma_\pi \times \Gamma_{-\pi}$.

It is clear that Poisson groupoids, defined in this way, generalize Poisson groups. In fact, recall that a bivector on a Lie group, G, is multiplicative if and only if the multiplication is a Poisson map from $G \times G$, with the product Poisson structure, to G and apply the following lemma.

Lemma.

A map from $G_\pi \times G_\pi$ to G_π is Poisson if and only if its graph is coisotropic in $G_\pi \times G_\pi \times G_{-\pi}$.

Equivalence of definitions

The definition of Poisson groupoids in terms of morphisms of groupoids is indeed equivalent to Weinstein's original definition, as was proved by Claude Albert and Dazord (1990) and by Mackenzie and Xu in 1994.

* * *

A bisection of a Lie groupoid $\Gamma \rightrightarrows M$ is a section, $M \to \Gamma$, of the source map such that its composition with the target map is a diffeomorphism of the base.
Bisections act on tensor fields and diffferential forms on the groupoid by the associated left and right translations.

The role of the translations on a Lie group defined by an element in the group is played by the translations defined by the bisections in a Lie groupoid.

Multiplicativity in groupoids

There is a relation that generalizes the relation satisfied by an affine bivector on a Lie group.

Theorem (Xu, 1995)
Let π be a bivector on a Lie groupoid. If π satisfies the morphism property, then

$$\boxed{\pi_{gh} = \hat{g} \cdot \pi_h + \pi_g \cdot \hat{h} - \hat{g} \cdot \pi_{\alpha g} \cdot \hat{h}}$$

for all $g, h \in \Gamma$ such that $\alpha g = \beta h$, where $\hat{g}$ is a bisection, $M \to \Gamma$, that takes the value g at αg, and $\hat{h}$ is a bisection, $M \to \Gamma$, that takes the value h at αh.

Symplectic groupoids as Poisson groupoids

Originally defined circa 1985, independently, by Mikhail Karasev and V. P. Maslov, by Weinstein, and by Zakrzewski, symplectic groupoids are Lie groupoids with a symplectic structure, ω, which is multiplicative, in the sense that the graph of the multiplication is a Lagrangian submanifold of the symplectic manifold $\mathit{\Gamma}_\omega \times \mathit{\Gamma}_\omega \times \mathit{\Gamma}_{-\omega}$.

It follows from the definitions that symplectic groupoids coincide with those Poisson groupoids whose Poisson structure is non-degenerate.

Although it is particularly important and has many features not present in the Poisson case in general, the case of symplectic groupoids can thus be treated as a particular case of the more general theory of Poisson groupoids.

Remark on "Symplectic groups"

Since any multiplicative Poisson bivector on a Lie group vanishes at the identity, there are no multiplicative symplectic forms on Lie groups.

On the other hand, in the 1980's, André Lichnerowicz with Alberto Medina studied the Lie groups equipped with a left-invariant symplectic form and called them "symplectic groups".

See Lichnerowicz's lecture at the Colloque Souriau of 1989 (Prog. Math. 99, 1991).

Despite the terminology, these "symplectic groups" are not a particular case of the symplectic groupoids defined above.

The infinitesimal of a Poisson groupoid

- The infinitesimal of a Poisson group is a Lie bialgebra.

- In a Poisson groupoid Γ, the base is a coisotropic submanifold.

The Poisson bivector on Γ induces a unique Poisson structure on the base such that the source map is a Poisson map and the target map is anti-Poisson.

The Lie algebroid A inherits a linear Poisson structure, as the normal bundle of a coisotropic submanifold of a Poisson manifold, and therefore the dual vector bundle A^* inherits a Lie algebroid structure.

From the multiplicativity of the Poisson structure of Γ, it follows that the pair (A, A^*) is a Lie bialgebroid (Mackenzie and Xu, 1994).

The infinitesimal of a symplectic groupoid

• In a symplectic groupoid, Γ, the base, M, is a Lagrangian submanifold of Γ. The Lie algebroid of Γ is isomorphic to the cotangent bundle of the base (equipped with the Lie algebroid structure of M with the induced Poisson structure)
and (A, A^*) is a Lie bialgebroid isomorphic to (T^*M, TM).

• Conversely, "integrating" a Poisson manifold into a symplectic groupoid was the subject of an important paper by Crainic and Rui Fernandes in 2004.

Multiplicative multivectors and forms on Lie groupoids

We have defined multiplicative multivectors on groups, multiplicative bivectors on groupoids, and multiplicative 2-forms on groupoids.
Are there, more generally, multiplicative k-vectors and k-forms on groupoids? and are there infinitesimal versions on Lie algebroids?

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For 0-forms and 1-forms, the following was already known in 1998.

A function, f, on a Lie groupoid, $\mathit{\Gamma}$, is called multiplicative if it defines a morphism of groupoids from $\mathit{\Gamma}$ to $M \times \mathbb{R}$, i.e.,
for g and h composable elements of $\mathit{\Gamma}$, $f(g \cdot h) = f(g) + f(h)$.

A 1-form, ω, on a Lie groupoid, $\mathit{\Gamma}$, is called multiplicative if it is a morphism of groupoids from $\mathit{\Gamma}$ to $T^*\mathit{\Gamma}$, i.e.,
for X and Y composable in $T\mathit{\Gamma}$, $\omega(X \times_{(TG)} Y) = \omega(X) + \omega(Y)$.

The contraction of a multiplicative 1-form with a multiplicative vector is a multiplicative function.

Multiplicative 2-forms

Multiplicative 2-forms on a Lie groupoid $\Gamma \rightrightarrows M$ appeared when Weinstein defined symplectic groupoids in 1987 (although he did not use the term "multiplicative" in this context). Recall that they were defined as 2-forms, ω, such that the graph of the partially defined groupoid multiplication, $m : \Gamma^{(2)} \subset \Gamma \times \Gamma \to \Gamma$, is Lagrangian in $\Gamma_\omega \times \Gamma_\omega \times \Gamma_{-\omega}$.

We now give equivalent characterizations of multiplicative 2-forms in terms of morphisms of Lie groupoids.

Multiplicative 2-forms as morphisms of groupoids

Theorem

Let ω be a 2-form on a Lie groupoid Γ over M, with multiplication m. The following properties are equivalent.

(i) graph(m) is Lagrangian in $\Gamma_\omega \times \Gamma_\omega \times \Gamma_{-\omega}$.

(ii) $m^*(\omega) = pr_1^*(\omega) + pr_2^*(\omega)$.

(iii) $\omega^\flat : T\Gamma \to T^*\Gamma$ is a morphism of Lie groupoids,

$$\begin{array}{ccc} T\Gamma & \stackrel{\omega^\flat}{\to} & T^*\Gamma \\ \downarrow\downarrow & & \downarrow\downarrow \\ TM & \to & A^* \end{array}$$

(iv) For all sections X_1, X_2 of $T\Gamma \to \Gamma$,

$$\omega^\flat(X_2 \times_{(T\Gamma)} X_1) = \omega^\flat(X_2) \times_{(T^*\Gamma)} \omega^\flat(X_1),$$

where the multiplication on the left-hand side (resp., right-hand side) is that of $T\Gamma$ (resp., $T^*\Gamma$).

(v) graph($\omega^\flat$) is a Lie subgroupoid of the direct sum Lie groupoid $T\Gamma \oplus T^*\Gamma \rightrightarrows TM \oplus A^*$ over some vector subbundle of $TM \oplus A^*$.

Multiplicative k-forms on Lie groupoids

It is straightforward to extend the definition of multiplicative forms to the case of k-forms, $k > 2$, and to formulate equivalent characterizations.

By definition, on a Lie groupoid Γ with multiplication m, a k-form, ω, is multiplicative if

$$m^*(\omega) = pr_1^*(\omega) + pr_2^*(\omega).$$

A k-form ω is multiplicative if and only if it defines a groupoid morphism from $T\Gamma \oplus T\Gamma \oplus \ldots \oplus T\Gamma$ (k terms) to $M \times \mathbb{R}$.

A k-form ω on Γ is multiplicative if and only if, for all pairs of composable elements of $T\Gamma$, (X_1, Y_1), (X_2, Y_2), ..., (X_k, Y_k),

$$\omega(X_1 \times_{(T\Gamma)} Y_1, X_2 \times_{(T\Gamma)} Y_2, ..., X_k \times_{(T\Gamma)} Y_k)$$

$$= \omega(X_1, X_2, ..., X_k) + \omega(Y_1, Y_2, ..., Y_k).$$

Multiplicative k-vectors on Lie groupoids

Similarly, by definition, a k-vector on Γ is multiplicative if it defines a groupoid morphism from $T^*\Gamma \oplus T^*\Gamma \oplus \ldots \oplus T^*\Gamma$ (k terms) to $M \times \mathbb{R}$.

Applying a multiplicative k-vector to a multiplicative k-form yields a multiplicative function.

Questions
Are the multiplicative multivectors closed under the Schouten bracket ?
Is the differential of a multiplicative form multiplicative ?

The group of bisections

The bisections of Γ form a group, denoted by $\mathcal{G}(\Gamma)$, with multiplication defined by:

$$(\Sigma_1 * \Sigma_2)(m) = \Sigma_1(\beta\Sigma_2(m)) \,.\, \Sigma_2(m),$$

for $\Sigma_1, \Sigma_2 \in \mathcal{G}(\Gamma)$, and $m \in M$. Here β is the target map of Γ and the dot is the groupoid multiplication.

By definition, each bisection $\Sigma \in \mathcal{G}(\Gamma)$ induces a diffeomorphism $\phi_\Sigma = \beta \circ \Sigma$ of M. Therefore a bisection Σ acts on $\Omega^k(M)$, the k-forms on M, by $(\Sigma, \lambda) \mapsto \Sigma.\lambda = \phi^*_\Sigma(\lambda)$, for $\lambda \in \Omega^k(M)$.
In other words, for vector fields, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$, tangent to M,

$$(\Sigma.\lambda)(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) = \lambda(T\phi_\Sigma(\varepsilon_1), T\phi_\Sigma(\varepsilon_2), ..., T\phi_\Sigma(\varepsilon_k)).$$

Since $\phi_{\Sigma_1 * \Sigma_2} = \phi_{\Sigma_1} \circ \phi_{\Sigma_2}$, the action thus defined is an action on $\Omega^k(M)$ of the opposite group $\mathcal{G}(\Gamma)^{opp}$ of $\mathcal{G}(\Gamma)$.

Multiplicative forms induce cocycles on the group of bisections

We denote by α the source map of Γ. For $\Sigma \in \mathcal{G}(\Gamma)$, $\alpha_{|\Sigma}$ is a diffeomorphism from Σ to M.
Let $\omega \in \Omega^k(\Gamma)$ be a k-form on Γ. We define a map

$$c_\omega : \Sigma \in \mathcal{G}(\Gamma)^{opp} \mapsto (\alpha_{|\Sigma}^{-1})^* \omega \in \Omega^k(M).$$

Theorem (Camille Laurent-Gengoux,unpublished)
If ω is a multiplicative k-form on Γ, then c_ω is a 1-cocycle on the group $\mathcal{G}(\Gamma)^{opp}$ with values in the $\mathcal{G}(\Gamma)^{opp}$-module $\Omega^k(M)$,

$$c_\omega(\Sigma_1 * \Sigma_2) = c_\omega(\Sigma_2) + \Sigma_2 . c_\omega(\Sigma_1).$$

Proof

Let ϵ be a vector field on M.
To evaluate $T(\alpha^{-1}_{|\Sigma_1 * \Sigma_2})\epsilon$, for ϵ tangent to M, we write ϵ as the tangent at $t = 0$ of a curve m_t on M.
Then the image of m_t under $\alpha^{-1}_{|\Sigma_1 * \Sigma_2}$ is $(\alpha^{-1}_{|\Sigma_1}(\beta\Sigma_2 m_t).(\alpha^{-1}_{|\Sigma_2} m_t)$.
Therefore the image of ϵ under $T(\alpha^{-1}_{|\Sigma_1 * \Sigma_2})$ is the tangent at $t = 0$ of this curve on Γ, defined by the product in $T\Gamma$,

$$T(\alpha^{-1}_{|\Sigma_1 * \Sigma_2})\epsilon = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Big((\alpha^{-1}_{|\Sigma_1}(\beta\Sigma_2 m_t) \times_{(T\Gamma)} (\alpha^{-1}_{|\Sigma_2} m_t \Big)$$

$$= T(\alpha^{-1}_{|\Sigma_1}) T\varphi_{\Sigma_2}\epsilon \times_{(T\Gamma)} T(\alpha^{-1}_{|\Sigma_2})\epsilon,$$

where $\varphi_{\Sigma_2} = \beta\Sigma_2$ is the diffeomorphism of M defined by Σ_2.

End of proof

Now, assume that ω is multiplicative.
Let $\epsilon_1, \epsilon_2, ..., \epsilon_k$ be vector fields on M. Apply the lemma, with $u_i = T(\alpha^{-1}_{|\Sigma_1}) T\varphi_{\Sigma_2}\epsilon_i$ and $v_i = T(\alpha^{-1}_{|\Sigma_2})\epsilon_i$, to obtain,

$$c_\omega(\Sigma_1 * \Sigma_2)(\epsilon_1, \epsilon_2, ..., \epsilon_k)$$

$$= \omega(T(\alpha^{-1}_{|\Sigma_1}) T\varphi_{\Sigma_2}\epsilon_1, T(\alpha^{-1}_{|\Sigma_1}) T\varphi_{\Sigma_2}\epsilon_2, ..., T(\alpha^{-1}_{|\Sigma_1}) T\varphi_{\Sigma_2}\epsilon_k)$$

$$+\omega(T(\alpha^{-1}_{|\Sigma_2})\epsilon_1, T(\alpha^{-1}_{|\Sigma_2})\epsilon_2, ..., T(\alpha^{-1}_{|\Sigma_2})\epsilon_k).$$

The first term of the right-hand side is $\Sigma_2.c_\omega(\Sigma_1)(\epsilon_1, \epsilon_2, ..., \epsilon_k)$, and the second term is $c_\omega(\Sigma_2)(\epsilon_1, \epsilon_2, ..., \epsilon_k)$. □

Unifying results in generalized geometry

We shall now report on recent work that shows that "generalized geometry" plays the role, among others, of unifying many scattered results.

We shall not deal with Courant algebroids in general, but only with the generalized tangent bundles of manifolds, in particular those of groupoids and of Lie algebroids.

Generalized tangent bundle

The generalized tangent bundle, $\tau M = TM \oplus T^*M$, of an arbitrary smooth manifold, M, is a framework that permits treating Poisson structures, presymplectic structures (defined by closed 2-forms) and complex structures as particular cases of generalized complex structures.

Consider the bracket on the sections of the vector bundle $\tau M = TM \oplus T^*M$ defined by

$$\boxed{[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi}$$

for all $X, Y \in C^\infty(TM), \xi, \eta \in C^\infty(T^*M)$.
In particular, $[X, \xi] = \mathcal{L}_X \xi$ and $[\xi, X] = -i_X d\xi$.

This non-skewsymmetric bracket satisfies the Jacobi identity in its Leibniz form,

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]],$$

for all sections u, v, w of the vector bundle $\tau M = TM \oplus T^*M$.
This bracket is called the Dorfman bracket.

The Courant bracket and Courant algebroids

• The generalized tangent bundle was the subject of Ted Courant's thesis (1990). He defined a bracket on $TM \oplus T^*M$ which is skewsymmetric but does not satisfy the Jacobi identity. This bracket is now called the Courant bracket.

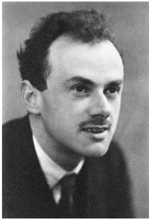
• The Courant bracket was generalized to the double of a Lie bialgebroid by Zhang-Ju Liu, Weinstein and Ping Xu, who then defined the general concept of a Courant algebroid (1997).

• The Dorfman bracket had appeared in Irene Dorfman's papers on integrable systems and in her book on Dirac structures (1993), but it was only introduced in the theory of Courant algebroids circa 1998, independently by Pavol Ševera, Ping Xu, and yks.

• The Courant bracket is the skew-symmetrization of the Dorfman bracket.

• The generalized tangent bundles are also called standard Courant algebroids, or Pontryagin bundles. They are the framework of Nigel Hitchin's generalized geometry (2003).

Dirac structures

 P.A.M. Dirac (1902-1984) Nobel prize in Physics 1933

A Dirac bundle (or Dirac structure) in $TM \oplus T^*M$ is a maximally isotropic vector subbundle (with respect to the canonical fibrewise bilinear symmetric form) whose space of sections is closed under the Dorfman bracket. A Dirac bundle in $TM \oplus T^*M$ is said to define a Dirac structure on M.

Examples

- The graph of a Poisson bivector on M is a Dirac structure.
- The graph of a closed 2-form (presymplectic structure) on M is a Dirac structure.
- Dirac pairs generalize bihamiltonian structures (yks 2012).

Multiplicative Dirac structures on Lie groupoids

Recall that, for any Lie group G, the generalized tangent bundle $TG \oplus T^*G$ is a Lie groupoid over $\mathfrak{g}^*$, the sum of the Lie group TG (considered as a groupoid) and the Lie groupoid $T^*G \rightrightarrows \mathfrak{g}^*$.
Recall also that in a Poisson group, G, the bivector π is such that
(i) graph(π) $\subset TG \oplus T^*G$ is a Dirac structure (since π is Poisson),
(ii) graph(π) is a subgroupoid of $TG \oplus T^*G \rightrightarrows \mathfrak{g}^*$
(since π is multiplicative, or, equivalently, $\pi^\sharp : T^*G \to TG$ is a morphism of groupoids).

Whence the following definition (Cristián Ortiz, 2009, 2013).

Definition

A Dirac structure on a Lie groupoid $\varGamma$ is multiplicative if it is defined by a vector subbundle $L \subset T\varGamma \oplus T^*\varGamma$ which is a subgroupoid of the Lie groupoid $T\varGamma \oplus T^*\varGamma \rightrightarrows TM \oplus A^*$ over a vector subbundle $L_0 \subset TM \oplus A^*$.
A Lie groupoid equipped with a multiplicative Dirac structure is called a Dirac groupoid.

Examples of Dirac groupoids

• Poisson groupoids. In this case, L is the graph of $\pi^\sharp$ and the vector subbundle $L_0 \subset TM \oplus A^*$ is the graph of the anchor $\mathcal{A}^* \to TM$ of the Lie algebroid A^*.

• Presymplectic groupoids. A presymplectic groupoid is a groupoid $\Gamma \rightrightarrows M$ with $\dim \Gamma = 2 \dim M$, equipped with a multiplicative, closed 2-form ω.
In some papers, the definition includes an additional condition on the 2-form ω, $\ker \omega_m \cap \ker T_m\alpha \cap \ker T_m\beta = \{0\}$.

In the case of a presymplectic groupoid, L is the graph of $\omega^\flat$ and the vector subbundle $L_0 \subset TM \oplus A^*$ is the graph of the map $\omega_0 : TM \to A^*$ defined by the restriction to $TM \subset T\Gamma$ of $\omega^\flat : T\Gamma \to T^*\Gamma$.
The map $\sigma : A \to T^*M$ dual to $-\omega_0$ is called infinitesimally multiplicative in Bursztyn-Crainic-Weinstein-Zhu, 2004, Bursztyn-Cabrera, 2009 and 2012, and elsewhere.

The structure of $TA \oplus T^*A$

Let A be a Lie algebroid with base M. Then

- TA is a Lie algebroid with base TM,
- T^*A is a Lie algebroid with base A^*.

In fact, since A is a Lie algebroid, A^* has a linear Poisson structure and therefore $T^*A^* \to A^*$ is a Lie algebroid.
Composing with the canonical map $T^*A \to T^*A^*$ yields the Lie algebroid structure of $T^*A \to A^*$.

Taking the direct sum $TA \oplus T^*A$ yields a Lie algebroid with base $TM \oplus A^*$.

Theorem

Let Γ be a Lie groupoid with Lie algebroid A. The Lie algebroid of the Lie groupoid $T\Gamma \oplus T^*\Gamma \rightrightarrows TM \oplus A^*$ is the Lie algebroid $TA \oplus T^*A \to TM \oplus A^*$.

Towards the study of the infinitesimal of a Dirac groupoid

On a vector bundle A, a linear bivector π defines a map, $\pi^\sharp : T^*A \to TA$, which is a morphism of double vector bundles

$$\text{from}\quad \begin{array}{ccc} T^*A & \to & A \\ \downarrow & & \downarrow \\ A^* & \to & M \end{array} \quad\text{to}\quad \begin{array}{ccc} TA & \to & A \\ \downarrow & & \downarrow \\ TM & \to & M \end{array}.$$

Therefore the graph of $\pi^\sharp$ is a double vector subbundle of

$$\begin{array}{ccc} TA \oplus T^*A & \to & A \\ \downarrow & & \downarrow \\ TM \oplus A^* & \to & M \end{array} \quad \text{over a vector subbundle of } TM \oplus A^*.$$

If the bivector π is Poisson, the graph of π is a Dirac structure on A, $\mathrm{graph}(\pi) \subset TA \oplus T^*A \to A$.

Almost Dirac and Dirac algebroids

With a view to applications to Lagrangian and Hamiltonian mechanics, Grabowska and Grabowski (2011) defined a Dirac algebroid (resp., Dirac-Lie algebroid) structure on a vector bundle E to be an almost Dirac (resp., Dirac) structure on E^* that satisfies the above double vector subbundle property.

The notion of Dirac-Lie algebroid clearly extends the notion of Lie algebroid since, whenever A is a Lie algebroid, the graph of the associated linear Poisson bivector on A^* defines a Dirac-Lie algebroid structure on A.

Morphic Dirac algebroids

With the different aim of characterizing the infinitesimal of a "Dirac groupoid", thus generalizing the Lie bialgebroids which are the infinitesimals of Lie groupoids, Ortiz defined a "Dirac algebroid" to be a Lie algebroid, A, with a Dirac structure, $\ell \subset TA \oplus T^*A$, satisfying the double vector subbundle property with the additional requirement that ℓ is a Lie subalgebroid of the Lie algebroid $TA \oplus T^*A \to TM \oplus A^*$ over a vector subbundle $\ell_0 \subset TM \oplus A^*$.

To avoid confusion, we shall call such Lie algebroids "morphic Dirac algebroids".

Lie bialgebroids as morphic Dirac algebroids

• Lie bialgebroids are examples of "morphic Dirac algebroids" since in a Lie bialgebroid (A, A^*), A is a Lie algebroid and the Lie algebroid structure of A^* defines a linear Poisson bivector $\pi_{(A^*)}$ on the vector bundle A (whence the double vector subbundle property) and the compatibility of the two Lie algebroid structures, on A and on its dual, is expressed by the fact that the Poisson bivector $\pi_{(A^*)}$ on A is a Lie algebroid morphism from T^*A to TA (whence its graph is a Lie subalgebroid of the Lie algebroid $TA \oplus T^*A$).

• The graph of a linear closed 2-form ω on a Lie algebroid A (a double vector bundle map form $TA \to TM$ to $T^*A \to A^*$) defines a morphic Dirac structure if and only the map $\omega^\flat : TA \to T^*A$ is a morphism of Lie algebroids.

Infinitesimal multiplicativity

Theorem (Ortiz)

There is a bijective correspondence between multiplicative Dirac structures on a source connected and source simply-connected Lie groupoid and morphic Dirac structures on its Lie algebroid.

Dirac groups as examples of Dirac groupoids

Dirac groups are, by definition, Lie groups G equipped with a multiplicative Dirac structure (Ortiz 2008, Madeleine Jotz 2011).

Dirac groups generalize Poisson groups.

The isotropy subgroups of a Dirac groupoid are Dirac groups.

There is a bijective correspondence between multiplicative Dirac structures on a connected and simply-connected Lie group G and ideals $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ and its dual form a Lie bialgebra.

To conclude, I shall now consider multiplicative complex structures on Lie groupoids.

Complex structures

Recall that the Nijenhuis torsion of a $(1,1)$-tensor on a manifold is a $(1,2)$-tensor defined by means of the Lie bracket of vector fields, and that complex structures on manifolds are endomorphisms of the tangent bundle whose square is $-\mathrm{Id}$ and whose Nijenhuis torsion vanishes.

The Nijenhuis torsion is named after Albert Nijenhuis.

Nijenhuis, born in 1926, died in Seattle on 13 February 2015.
He had been a student of Jan A. Schouten (1883 - 1971) in Amsterdam.

Generalized complex structures

• The generalized complex structures on a manifold M are defined just as the complex sructures, replacing the tangent bundle by the generalized tangent bundle and the Lie bracket of vector fields by the Dorfman bracket of sections of the generalized tangent bundle. They are skew-symmetric endomorphisms of $\tau M = TM \oplus T^*M$, whose square is $-\mathrm{Id}_{\tau M}$, and whose Nijenhuis torsion vanishes.

Remark. Indeed, the Nijenhuis torsion of a skew-symmetric endomorphism of τM of square $\pm\mathrm{Id}_{\tau M}$ is again a tensor! (cf. yks 2011)

• Poisson structures, presymplectic structures, and complex structures are particular cases of the generalized complex structures.

• A generalized complex structure is equivalent to a pair of complex conjugate Dirac structures in the complexified generalized tangent bundle.

Multiplicative generalized complex structures

Question. What are multiplicative generalized complex structures on Lie groupoids?

When Γ is a Lie groupoid over a manifold M, $T\Gamma \oplus T^*\Gamma$, the generalized tangent bundle of Γ, is a Lie groupoid over $TM \oplus A^*$.

Definition

A generalized complex structure on a Lie groupoid Γ is called multiplicative if it is a Lie groupoid automorphism of $T\Gamma \oplus T^*\Gamma$.

Madeleine Jotz, Mathieu Stiénon, and Ping Xu coined the name "Glanon groupoids" (see arXiv 2013).

The notion of a multiplicative generalized complex structure on a Lie groupoid incorporates many particular cases.

Symplectic groupoids

- Let $\omega^\flat : T\Gamma \to T^*\Gamma$ be skew-symmetric and non-degenerate. The skew-symmetric endomorphism of $T\Gamma \oplus T^*\Gamma$,

$$\mathcal{N} = \begin{pmatrix} 0 & (\omega^\flat)^{-1} \\ \omega^\flat & 0 \end{pmatrix},$$

is a multiplicative generalized complex structure if and only if (Γ, ω) is a symplectic groupoid.

- Multiplicative generalized complex structures of the form

$$\mathcal{N} = \begin{pmatrix} N & (\omega^\flat)^{-1} \\ \omega^\flat & -{}^tN \end{pmatrix},$$

where $N : T\Gamma \to T\Gamma$ and ω is a non-degenerate 2-form, correspond to symplectic-Nijenhuis structures.

- Multiplicative generalized complex structures of the form

$$\mathcal{N} = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -{}^tN \end{pmatrix},$$

where $N : T\Gamma \to T\Gamma$, ω is a 2-form, and π is a bivector, correspond to Poisson-quasi-Nijenhuis structures.

Holomorphic groupoids

• Let N be a vector bundle endomorphism of $T\Gamma \to \Gamma$.
The skew-symmetric endomorphisms of $T\Gamma \oplus T^*\Gamma$

$$\mathcal{N} = \begin{pmatrix} N & 0 \\ 0 & -{}^tN \end{pmatrix}$$

which are generalized multiplicative complex structures correspond to multiplicative holomorphic Lie groupoid structures, N, on Γ.

A holomorphic groupoid, (Γ, N), is a Lie groupoid, Γ, equipped with a vector bundle endomorphism, $N : T\Gamma \to T\Gamma$, that satisfies $N^2 = -\mathrm{Id}$, $\mathcal{T}N = 0$ and is multiplicative, i.e., is a Lie groupoid endomorphism over a map, $N_M : TM \to TM$ that satisfies $N_M^2 = -\mathrm{Id}$ and $\mathcal{T}N_M = 0$ (Laurent-Gengoux, Stiénon, Xu, 2009).

The infinitesimal of a multiplicative generalized complex structure

Theorem (Jotz-Stiénon-Xu)
Let A be the Lie algebroid of a Lie groupoid Γ.
A multiplicative generalized complex structure on Γ induces a multiplicative generalized complex structure on A, i.e., a skew-symmetric vector bundle endomorphism, $\nu : TA \oplus T^*A \to TA \oplus T^*A$, such that $\nu^2 = -\mathrm{Id}$, $\mathcal{T}\nu = 0$, and ν is a Lie algebroid endomorphism.

Integration theorem (Jotz-Stiénon-Xu)
If Γ is a source-connected and source simply-connected Lie groupoid integrating a Lie algebroid A with a multiplicative generalized complex structure, then there is a multiplicative generalized complex structure on Γ integrating that of A.

Induced Poisson groupoid structure (Jotz-Stiénon-Xu)

Proposition. A Lie groupoid equipped with a multiplicative generalized complex structure has an induced Poisson groupoid structure.

In fact, $\mathcal{N}$ is necessarily of the form $\mathcal{N} = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -{}^tN \end{pmatrix}$, with $\pi^\sharp : T^*\Gamma \to T\Gamma$ a morphism of Lie groupoids, i.e., the associated bivector, π, is multiplicative, and π is a Poisson bivector.

Proposition. If a Lie algebroid A is equipped with a multiplicative generalized complex structure, then there is an associated Lie algebroid structure on A^* such that (A, A^*) is a Lie bialgeboid.

Theorem Let Γ be a Lie groupoid equipped with a multiplicative generalized complex structure $\mathcal{N}$. Let A be the Lie algebroid of Γ. Then the Lie bialgebroid of Γ, viewed as the Poisson groupoid with the Poisson structure induced by $\mathcal{N}$, is the Lie bialgebroid (A, A^*) associated to the multiplicative generalized complex structure on A induced by $\mathcal{N}$.

And...

There is a host of developments related to the theme of this talk:

• groupoid cohomology and multiplicative forms as groupoid cocycles,

• Dirac Lie groups, multiplicative Manin pairs,

• quasi-Poisson groupoids,

• twisted multiplicative forms on groupoids,

• Poisson actions,

• infinite-dimensional case and integrable systems,

• multiplicative forms with non-trivial coefficients and Spencer operators,

• the general problem of the infinitesimal descripton of structures and of their integration,

• contact groupoids and Courant algebroids,

• applications to Lagrangian and Hamiltonian mechanics...

... as in the papers of, in particular, Janusz Grabowski, Paweł Urbański, Katarzyna Grabowska, ...

Bon anniversaire!

Best wishes to Prof. Janusz Grabowski!