EMBEDDING TOPOLOGICAL DYNAMICAL SYSTEMS WITH PERIODIC POINTS IN CUBICAL SHIFTS

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Abstract. According to a conjecture of Lindenstrauss and Tsukamoto, a topological dynamical system (X, T) is embeddable in the d-cubical shift $(([0,1]^d)^{\mathbb{Z}},\ shift)$ if both its *mean dimension* and *periodic dimension* are strictly bounded by $\frac{d}{2}$. We verify the conjecture for the class of systems admitting a finite dimensional non-wandering set and a closed set of periodic points. This class of systems is closely related to systems arising in physics. In particular we prove an embedding theorem for systems associated with the two dimensional Navier-Stokes equations of fluid mechanics. The main tool in the proof of the embedding result is the new concept of *local markers*. Continuing the investigation of (global) markers initiated in previous work it is shown that the *marker property* is equivalent to a topological version of the *Rokhlin Lemma*. Moreover new classes of systems are found to have the marker property, in particular, extensions of aperiodic systems with a countable number of minimal subsystems. Extending work of Lindenstrauss we show that for systems with the marker property, vanishing mean dimension is equivalent to the small boundary property.

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1. Introduction

The question under which conditions a topological dynamical system (X,T) is embeddable in the *d-cubical shift* $(([0,1]^d)^{\mathbb{Z}},\ shift)$ stems from Auslander's 1988 influential book. According to Jaworski's Theorem (1974), for X finite-dimensional and T aperiodic, embedding is possible with $d = 1$. Auslander posed the question if for $d = 1$, it is sufficient that X is minimal. The question was solved in the negative by Lindenstrauss and Weiss (2000), adroitly using the invariant of *mean dimension* introduced by Gromov (1999). Around the same time Lindenstruass (1999) showed that if X is an extension of an aperiodic minimal system and $mdim(X,T) < \frac{d}{36}$, then (X,T) is embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$. Recently Lindenstrauss and Tsukamoto (2012) have introduced a unifying conjecture and several cases of this conjecture have been verified. According to this conjecture the only obstructions for embeddability are given by the invariants of *mean dimension* and *periodic dimension,* the later quantifying the natural obstruction due to the set of periodic points. A precise statement of the conjecture is that $mdim(X,T) < \frac{d}{2}$ and $perdim(X,T) < \frac{d}{2}$ imply (X,T) is embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$. In Gutman and Tsukamoto (2012) the conjecture was verified for extensions of aperiodic subshifs and in Gutman (2012) the conjecture was verified for finite-dimensional systems. In the same article it was shown that for extensions of aperiodic finite-dimensional systems $mdim(X,T) < \frac{d}{16}$ implies (X,T) is embeddable in $(([0,1]^{d+1})^{\mathbb{Z}},\ shift)$.

A keen observer will notice that all embedding results mentioned above, involving infinite-dimensional systems, require the assumption of aperiodicity. This is due to a common device used in the proofs: existence of *markers* (of all orders). Markers can be thought of as a suitable generalization of the familiar markers of symbolic dynamics, introduced by Krieger (1982), to the setting of arbitrary dynamical systems. As a necessary condition for the existence of markers of all orders is the aperiodicity of the system, one is confined to the category of aperiodic systems.

In this article we resolve this difficulty by introducing the concept of *local markers.* It has the desired consequence of allowing us to treat some infinite-dimensional systems admitting periodic points. In particular we verify the Lindenstrauss-Tsukamoto Conjecture for the class of systems whose non-wondering set is finite dimensional and set of periodic points is closed and discuss some examples. The result is a consequence of a more general embedding theorem stating that systems with the local marker property verifying $mdim(X,T) < \frac{d}{36}$ and $perdim(X,T) < \frac{d}{2}$ are embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$.

The embedding results we obtain can be fitted into a framework not uncommon in physics: infinite-dimensional systems which exhibit finite-dimensional global attractors. We demonstrate this by proving the embeddability into cubical shift of a discrete version of a model of the two-dimensional Navier-Stokes equations from fluid mechanics.

Recognizing the importance of markers, both local and global, we continue the investigation of markers as carried out in Gutman (2012) which itself was a generalization of previous work by Bonatti and Crovisier (2004), and prove in particular that an aperiodic system with a countable number of minimal subsystems admits the marker property.

In Gutman (2011) the notion of the topological Rokhlin property was introduced. This is a dynamical topological analogue of the Rokhlin Lemma of measured dynamics. Here we show that the marker property is equivalent to a strong version of the topological Rokhlin property. Following closely Lindenstrauss (1999) this characterization results with several fruitful applications: Systems with the marker property admit a compatible metric with respect to which the *metric mean dimension* equals the (topological) mean dimension. Moreover for such systems, vanishing mean dimension is equivalent to the small boundary property and to being an inverse limit of finite entropy systems. Preliminary results for this work were reported in [Gut13].

2. Preliminaries

The following article is closely related to the article [Gut15] and we recommend the reader to familiarize herself or himself with the Introduction and Preliminaries sections of that article.

2.1. **Conventions.** Throughout the article with the exception of Section 8 and Appendix **??**, a **topological dynamical system** (t.d.s) (X,T), also denoted $(\mathbb{Z},X)$, consists of a *metric* compact space (X,d) and a *homeomorphism* $T:X\to X$. In Subsections 2.2, 2.5, 2.6, Section 8 and Appendix **??** (and only there) we relax this requirement and assume only $T:X\to X$ is continuous. These systems are denoted by $(\mathbb{N},X)$. If it is desired to emphasize T is a homemorphism we say (X,T) is an **invertible topological dynamical system**. $P=P(X,T)$ denotes the set of periodic points and P_n denotes the set of periodic points of period $\leq n$. In addition we use the notation $H_n=P_n\setminus P_{n-1}$. $\triangle=\{(x,x)|\,x\in X\}$ denotes the *diagonal* of $X\times X$. If $x\in X$ and $\epsilon>0$, let $B_\epsilon(x)=\{y\in X|\,d(y,x)<\epsilon\}$ denote the open ball around x. We denote $\overline{B}_\epsilon(x)=\overline{B_\epsilon(x)}$. Note $\overline{B}_\epsilon(x)\subseteq\{y\in X|\,d(y,x)\leq\epsilon\}$ but equality does not necessary hold. For A and B compact subsets of the same metric space we denote $d(A,B)=\min_{a\in A,b\in B}d(a,b)$. For $f,g\in(C(X,[0,1]^d)$, we define $||f-g||_\infty\triangleq sup_{x\in X}||f(x)-g(x)||_\infty$. In §8 $||\cdot||$ is used to denote the norm in a Hilbert space.

2.2. **The Non-Wandering Set.** Let (X,T) be a t.d.s where T is not necessarily invertible. The following definition is based on [Wal82, Theorem 5.7]. A point $x\in X$ is said to be **non-wandering** if for every open set

$x \in U$ and every $N \geq 1$ there is $k \geq N$ so that $U \cap T^{-k}U \neq \emptyset$. The **non-wandering set** $\Omega(X)$ is the collection of all non-wandering points. Note $\Omega(X)$ is a non-empty, closed and $T\Omega(X) \subset \Omega(X)$. If T is invertible then $T\Omega(X) = \Omega(X)$.

2.3. **Covers.** A collection τ of sets in X **covers** a set $A \subset X$ if $A \subset \bigcup_{S \in \tau} S$. The **restriction** of τ to A is defined by $\tau_{|A} = \{S \cap A\}_{S \in \tau}$. An **open** respectively **closed cover** is a finite collection of open respectively closed sets which covers X. Let α, β be collections of sets. One says that β is a **refinement** of α, denoted $\beta \succ \alpha$, if for every $V \in \beta$, there is $U \in \alpha$ so that $V \subset U$.

2.4. **Dimension.** Let $\mathcal{C}$ denote the collection of open covers of X. For $\alpha \in \mathcal{C}$ define its **order** by $ord(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$. Let $D(\alpha) = \min_{\beta \in \mathcal{C}: \beta \succ \alpha} ord(\beta)$ The **Lebesgue covering dimension** is defined by $dim(X) = \sup_{\alpha \in \mathcal{C}} D(\alpha)$.

2.5. **Periodic Dimension.** Let (X, T) be a t.d.s where T is not necessarily invertible. Let P_m denote the set of points of period $\leq m$. Introduce the infinite vector $\overrightarrow{perdim}(X, T) = (\frac{dim(P_m)}{m})_{m \in \mathbb{N}}$. This vector is clearly a topological dynamical invariant. Let $d > 0$. We write $perdim(X, T) < d$, if for every $m \in \mathbb{N}$, $\overrightarrow{perdim}(X, T)|_m < d$.

2.6. **Mean Dimension.** Let (X, T) be a t.d.s where T is not necessarily invertible. Define:

$$mdim(X, T) = \sup_{\alpha \in \mathcal{C}} \lim_{n \to \infty} \frac{D(\alpha^n)}{n}$$

where $\alpha^n = \bigvee_{i=0}^{n-1} T^{-i}\alpha$. Mean dimension was introduced by Gromov [Gro99] and systematically investigated by Lindenstrauss and Weiss in [LW00].

The following two subsections follow closely the corresponding subsections in [Gut11]:

2.7. **The Topological Rokhlin Property.** The classical Rokhlin lemma states that given an aperiodic invertible measure-preserving system (X, T, μ) and given $\epsilon > 0$ and $n \in \mathbb{N}$, one can find $A \subset X$ so that $A, TA, \ldots, T^{n-1}A$ are pairwise disjoint and $\mu\big(\bigcup_{k=0}^{n-1} T^k A\big) > 1 - \epsilon$. It easily follows that given an aperiodic invertible measure-preserving system (X, T, μ) and given $\epsilon > 0$, one can find a measurable function $f : X \to \{0, 1, \ldots, n-1\}$ so that if we define the *exceptional set* $E_f = \{x \in X \,|\, f(Tx) \neq f(x)+1\}$, then $\mu(E_f) < \epsilon$. The new formulation allows us to generalize to the topological category. Indeed following [SW91], given a t.d.s (X, T) and a set $E \subset X$, we define the orbit-capacity of a set E in the following manner (the limit exists):

$$ocap(E) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} \sum_{k=0}^{n-1} 1_E(T^k x)$$

(X, T) is said to have the **topological Rokhlin property (TRP)** if and only if for every $\epsilon > 0$ there exists a continuous function $f : X \to \mathbb{R}$ so that for the *exceptional set* $E_f = \{x \in X \,|\, f(Tx) \neq f(x) + 1\}$, one has $ocap(E_f) < \epsilon$.

2.8. **The Small Boundary Property.** Following [SW91] we call $E \subset X$ **small** if $ocap(E) = 0$. For closed sets this has a simple interpretation. Indeed a closed set $A \subset X$ is small if and only if for any T-invariant measure μ of X, one has $\mu(A) = 0$. When X has a basis of open sets with small boundaries, (X, T) is said to have the **small boundary property (SBP)**. In [LW00] it was shown that SBP implies mean dimension zero. In [Gut11] it was shown that if (X, T) is an extension of an aperiodic space with SBP then it has TRP.

2.9. **The Metric Mean Dimension.** A set $S \subset X$ is called (n, ϵ, d)-spanning if for every $x \in X$ there is a $y \in S$ so that for all $0 \leq k < n$, $d(T^k x, T^k y) < \epsilon$. Define $A(n, \epsilon, d)$ to be the cardinality of a minimal (n, ϵ, d)-spanning set. Define:

$$s(\epsilon, d) = \limsup_{n \to \infty} \frac{\log(A(n, \epsilon, d))}{n}$$

$$mdim_d(X, T) = \liminf_{\epsilon \to 0} \frac{s(\epsilon, d)}{|\log(\epsilon)|}$$

In [LW00] it was shown that $mdim_d(X, T) \leq mdim(X, T)$. By a classical theorem of Bowen and Dinaburg, the topological entropy is given by $h_{top}(X, T) = \lim_{\epsilon \to 0} s(\epsilon, d)$. Thus it was concluded in [LW00] that finite topological entropy implies mean dimension zero.

2.10. **The Baire Category Theorem Framework.** Let $G = \mathbb{Z}$ or $G = \mathbb{N}$. We are interested in the question under which conditions a topological dynamical system (G, X)[1] is embeddable in the $(G-)$ d-**cubical shift** $(([0,1]^d)^G, G - shift)$ for some $d \in \mathbb{N}$. Notice that a continuous function $f : X \to [0,1]^d$ induces a continuous G-equivariant mapping $I_f : (X, T) \to (([0,1]^d)^G, G - shift)$ given by $x \mapsto (f(T^k x))_{k \in G}$, also known as the **orbit-map**. Conversely, any G-equivariant continuous factor map $\pi : (X, T) \to (([0,1]^d)^G, G - shift)$ is induced in this way by $\pi_0 : X \to [0,1]^d$, the projection on the zeroth coordinate. We therefore study the space of continuous functions $C(X, [0,1]^d)$. Instead of explicitly constructing a $f \in C(X, [0,1]^d)$ so that $I_f : (X, T) \hookrightarrow (([0,1]^d)^G, G - shift)$ is an embedding, we show that the property of being an embedding $I_f : (X, T) \hookrightarrow (([0,1]^d)^G, shift)$ is *generic* in $C(X, [0,1]^d)$ (but without exhibiting an explicit embedding). To make this precise introduce the following definition:

[1] In most of this article $G = \mathbb{Z}$ and therefore we do not usually specify G. Note however that in Section 8, $G = \mathbb{N}$.

Definition 2.1. Let $G = \mathbb{Z}$ or $G = \mathbb{N}$. Suppose $K \subset (X \times X) \setminus \Delta$ is a compact set and $f \in C(X, [0,1]^d)$. We say that I_f is $(G-)K$**-compatible** if for every $(x, y) \in K$, $I_f(x) \neq I_f(y)$, or equivalently if for every $(x, y) \in K$, there exists $n \in G$ so that $f(T^n x) \neq f(T^n y)$. Define:

$$D_K = \{f \in C(X, [0,1]^d) | \, I_f \text{ is K} - \text{compatible}\}$$

By Lemma A.2 of [Gut15], D_K is open in $C(X, [0,1]^d)$. Suppose we have shown that there exists a a closed countable cover $\mathcal{K}$ of $(X \times X) \setminus \Delta$ so that D_K is dense for all $K \in \mathcal{K}$. By the Baire category theorem $(C(X, [0,1]^d), || \cdot ||_\infty)$, is a *Baire space*, i.e., a topological space where the intersection of countably many dense open sets is dense. This implies $\bigcap_{K \in \mathcal{K}} D_K$ is dense in $(C(X, [0,1]^d), || \cdot ||_\infty)$. Any $f \in \bigcap_{K \in \mathcal{K}} D_K$ is K-compatible for all $K \in \mathcal{K}$ simultaneously and therefore induces an embedding $I_f : (X, T) \hookrightarrow (([0,1]^d)^G, G-shift)$. A set in a topological space is said to be **comeagre** or **generic** if it is the complement of a countable union of nowhere dense sets. A set is said to be G_δ if it is the countable intersection of open sets. As a dense G_δ set is comeagre, the above argument shows that the set $\mathcal{A} \subset C(X, [0,1]^d)$ for which $I_f : (X, T) \hookrightarrow (([0,1]^d)^G, G - shift)$ is an embedding is comeagre, or equivalently, that the fact of I_f being an embedding is generic in $(C(X, [0,1]^d), || \cdot ||_\infty)$.

2.11. **Overview of the Article.** In Section 3 the definition of the marker property is recalled and new classes of system admitting the marker property are exhibited, in particular, extensions of aperiodic systems with a countable number of minimal subsystems. Additionally some simple examples are discussed. In Section 4 the local marker property is defined and verified for systems of finite dimensional systems with closed sets of periodic points. In Section 5 the local and global strong topological Rokhlin properties are introduced and investigated. In particular it is shown that the marker property is equivalent to the (global) strong topological Rokhlin property. In Section 6 the following embedding theorem is proven: If (X,T) has the local marker property, $mdim(X,T) < \frac{d}{36}$ and $perdim(X,T) < \frac{d}{2}$, then (X,T) is generically embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$. In Section 7 various applications of the embedding theorem are given, in particular, the verification of the Lindenstrauss-Tsukamoto Conjecture for the class of systems admitting a finite dimensional non-wandering set and a closed set of periodic points. In Section 8 the last result is applied to a family of dynamical systems associated with the Navier-Stokes equations for a two-dimensional incompressible viscous flow. In particular it is shown that after a finite and calculable time a discretization of the system modelling the flow can be embedded in a cubical shift. In Appendix A it is shown that systems with the marker property admit a compatible metric with respect to which the *metric mean dimension* equals the (topological) mean dimension. Moreover for systems with the marker property, vanishing mean dimension is equivalent to having the small boundary property and to being an inverse limit of finite entropy systems.

3. The Marker Property

Definition 3.1. A subset F of a t.d.s (X,T) is called an n**-marker** $(n \in \mathbb{N})$ if:

(1) $F \cap T^i(F) = \emptyset$ for $i = 1, 2, \ldots, n-1$.
(2) The sets $\{T^i(F)\}_{i=1}^m$ cover X for some $m \in \mathbb{N}$.

The system (X,T) is said to have the **marker property** if there exist *open* n-markers for all $n \in \mathbb{N}$.

Remark 3.2. Clearly the marker property is stable under extension, i.e. if (X,T) has the marker property and $(Y,S) \to (X,T)$ is an extension, then (Y,S) has the marker property.

Remark 3.3. By Lemma A.1 of [Gut15] (X,T) has a closed n-marker iff (X,T) has an open n-marker.

The marker property was first defined in [Dow06] (Definition 2), where one requires the n-markers to be clopen. In the same article it was proven that an extension of an aperiodic zero-dimensional (non necessarily invertible) t.d.s has the marker property. This was essentially based on the "Krieger Marker Lemma" (Lemma 2 of [Kri82]). In [Gut15] Theorem 6.1 it was proven that aperiodic finite dimensional t.d.s have the marker property. From [Lin99, Lemma 3.3] it follows that an extension of an aperiodic minimal system has the marker property. Given these results it is natural to ask the following question:

Problem 3.4. Does any aperiodic system have the marker property?

We do not know the answer of the previous problem. However we are able to prove two theorems establishing the existence of the marker property under natural assumptions. We also discuss examples.

Theorem 3.5. *(Downarowicz & Gutman) If (X,T) is an extension of an aperiodic t.d.s which has a countable number of minimal subsystems then it has the marker property.*

Proof. We may assume w.l.o.g that (X,T) is aperiodic and has a countable number of minimal subsystems. Let $n \in \mathbb{N}$. We will construct inductively an open set $U \subset X$ so that the sets $\{T^i(U)\}_{i=1}^n$ are pairwise disjoint and $\{T^i(U)\}_{i=1}^m$ cover X for some m. Let $M_1, M_2, \ldots$ be an enumeration of the minimal subsystems of X. Using the fact there is only a countable number of minimal subsystems find $m_1 \in M_1$, $r_1 > 0$ so that $\{T^i B_{r_1}(m_1)\}_{i=-n}^n$ are pairwise disjoint and for all $l \geq 2$, $M_l \not\subseteq \bigcup_{i=-n}^n T^i \partial B_{r_1}(m_1)$ (here we use that $\{\bigcup_{i=-n}^n T^i \partial B_r(m_1)\}_{r>0}$ is a uncountable collection of pairwise disjoint sets). Define $U_1 = B_{r_1}(m_1)$. Assume one has defined an open set $U_k \subset X$ so that:

(1) For any $i = 1, \dots, k$ there exists $j = j(i) \in \mathbb{Z}$ so that $U_k \cap T^j M_i \neq \emptyset$.
(2) For all $l \geq k+1$, $M_l \not\subseteq \bigcup_{i=-n}^n T^i \partial U_k$.
(3) $\{T^i(U_k)\}_{i=1}^n$ are pairwise disjoint

If $U_k \cap T^j M_{k+1} \neq \emptyset$ for some $j \in \mathbb{Z}$, define $U_{k+1} = U_k$. We now assume that $U_k \cap T^j M_{k+1} = \emptyset$ for all $j \in \mathbb{Z}$. By assumption $M_{k+1} \not\subseteq \bigcup_{i=-n}^n T^i \partial U_k$. Conclude $M_{k+1} \not\subseteq \bigcup_{i=-n}^n T^i \overline{U}_k$. Using the fact there is only a countable number of minimal subsystems, we can find $m_{k+1} \in M_{k+1}$ and $r_{k+1} > 0$ so that $\{T^i B_{r_{k+1}}(m_{k+1})\}_{i=-n}^n$ are pairwise disjoint and so that it holds:

$$B_{r_{k+1}}(m_{k+1}) \cap \bigcup_{i=-n}^n T^i U_k = \emptyset, \tag{3.1}$$

$$\forall l > k+1 \quad M_l \setminus \bigcup_{i=-n}^n T^i \partial U_k \not\subseteq \bigcup_{i=-n}^n T^i \partial B_{r_1}(m_1) \tag{3.2}$$

Define $U_{k+1} = U_k \cup B_{r_{k+1}}(m_{k+1})$. We now verify that the desired properties hold:

(1) For any $i = 1, \dots, k+1$ there exists $j = j(i) \in \mathbb{Z}$ so that $U_{k+1} \cap T^j M_i \neq \emptyset$. Indeed if $i \leq k$, this follows from property (1) above. For $i = k+1$ it is trivial.
(2) For all $l \geq k+2$, $M_l \not\subseteq \bigcup_{i=-n}^n T^i \partial U_{k+1}$. Indeed if follows from $\partial U_{k+1} \subset \partial U_k \cup \partial B_{r_{k+1}}(m_{k+1})$ and (3.2).

(3) $\{T^i(U_{k+1})\}_{i=1}^n$ are pairwise disjoint. Indeed it is enough to show $T^{i_1}U_k \cap T^{i_2}B_{r_{k+1}}(m_{k+1}) = \emptyset$ for all $1 \leq i_1, i_2 \leq n$. This follows from (3.1).

Finally we define $U = \bigcup_{k=1}^{\infty} U_k$. As $U_1 \subset U_2 \subset \cdots$, it holds that $\{T^i(U)\}_{i=1}^n$ are pairwise disjoint. Clearly for any $i \in \mathbb{N}$ there exists $j = j(i) \in \mathbb{Z}$ so that $U \cap T^j M_i \neq \emptyset$. As U is open and M_i is compact this implies there exists $m(i) \in \mathbb{N}$ so that $M_i \subset \bigcup_{l=0}^{m(i)} T^l U$. Fix $x \in X$. There exists $i \in \mathbb{N}$ so that $\overline{orb(x)} \cap M_i \neq \emptyset$. Conclude there exists $k \in \mathbb{Z}$, so that $T^k x \in \bigcup_{l=0}^{m(i)} T^l U$. By a simple compactness argument we deduce the existence of $m \in \mathbb{N}$ so that$\{T^i(U)\}_{i=1}^m$ cover X. $\square$

Example 3.6. Clearly the previous theorem applies to every t.d.s which consists of a finite union of minimal systems. We now present a simple example of an aperiodic t.d.s with an infinite countable number of minimal systems. Let $C_r = \{(x,y) |\, x^2 + y^2 = r\} \subset \mathbb{R}^2$, be a circle of radius r around the origin. Select a strictly decreasing sequence of positive numbers $r_1 > r_2 > \cdots$ with $r_i \to r_0 > 0$. Let $X = \bigcup_{i=0}^{\infty} C_{r_i} \subset \mathbb{R}^2$ and define $T : X \to X$ by rotating by α on each circle where α is some irrational number.

Example 3.7. Not every aperiodic system has a countable number of minimal subsystems. Indeed consider $X = \mathbb{T}^2$, the two-dimensional torus equipped with $T(x,y) = (x + \alpha, y)$ for some α irrational number.

Definition 3.8. Let (X,T) be a t.d.s and denote by $\mathcal{M}$ the collection of all minimal subspaces of (X,T). (X,T) has a **compact minimal subsystems selector** if there exists a compact L so that for every $M \in \mathcal{M}$, $|L \cap M| = 1$ and $L \subset \bigcup \mathcal{M}$.

Theorem 3.9. *(Downarowicz) If (X,T) is an extension of an aperiodic t.d.s with a compact minimal subsystems selector than it has the marker property.*

Proof. We may assume w.l.o.g that (X,T) has a compact minimal subsystems selector L. Denote by $\mathcal{M}$ the collection of all minimal subspaces of (X,T). If $x,y \in L$, $x \neq y$, then there exists distinct $M_x, M_y \in \mathcal{M}$ so that $x \in M_x$ and $y \in M_y$. This implies $orb(x) \cap orb(y) = \emptyset$. As (X,T) is aperiodic conclude the closed sets $\{T^i L\}_{i=-\infty}^{\infty}$ are pairwise disjoint. Let $n \in \mathbb{N}$. There exists $\epsilon > 0$ so that $T^i B_\epsilon(L)$ $(1 \leq i \leq n)$ are pairwise disjoint. For every $M \in \mathcal{M}$ there exists by minimality $m = m(M)$ so that:

$$M \subset \bigcup_{i=0}^{m} T^i B_\epsilon(L)$$

For every $z \in X$, there exists $M_z \in \mathcal{M}$ so that $\overline{orb(z)} \cap M_z \neq \emptyset$. We conclude by a compactness argument that $B_\epsilon(L)$, $T^1 B_\epsilon(L)$, $T^2 B_\epsilon(L), \ldots$, eventually cover X. $\square$

Example 3.10. A simple example of an aperiodic system with a compact minimal subsystems selector is given by Example 3.7. A selector is given by $\{0\} \times \mathbb{T}$. An aperiodic system with a compact minimal subsystems selector without a non trivial minimal factor is given by taking a disjoint union of the previous example X with a circle, $Y = X \mathring{\cup} \mathbb{T}$ where the circle is equipped with a rotation by β, such that α and β are incommensurable.

Example 3.11. Not all aperiodic t.d.s have a compact minimal subsystems selector. Indeed let $X = \mathbb{T}^2$ be the two-dimensional torus and $T : X \to X$ be given by $T(x,y) = (x + \frac{1}{2}, y + \alpha)$ for some α irrational. Let $\mathcal{M}$ the collection of all minimal subspaces of (X,T). Note $\mathcal{M} = \{\{t, t + \frac{1}{2}\} \times \mathbb{T} | \, t \in [0, \frac{1}{2})\}$. Assume for a contradiction (X,T) has a compact minimal selector L. Let L_2 be the projection of L on the first coordinate. L_2 is a closed set so that $\mathbb{T} = L_2 \mathring{\cup} (L_2 + \frac{1}{2})$. Contradiction.

4. The Local Marker Property

Definition 4.1. Let Z, W be closed sets with $Z \times W \subset (X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$. A subset F of a t.d.s (X, T) is called a **local n-marker** ($n \in \mathbb{N}$) for $Z \times W$ if:

(1) $F \cap T^i(F) = \emptyset$ for $i = 1, 2, \ldots, n-1$.
(2) The sets $\{T^i(F)\}_{i=1}^{m}$, $i = 0, 1, \ldots, m-1$, cover $Z \cup W$ for some m.

We say $Z \times W$ has **local markers** if it has *open* n-markers for all $n \in \mathbb{N}$. We say that a cover of $Y \subset (X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets $\{Z_i \times W_i\}_{i=1}^{\infty}$ has the **local marker property relatively to** Y if for every i, $Z_i \times W_i$ has local markers. We say (X, T) has the **local marker property** if there is a cover with the local marker property relatively to $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$.

Remark 4.2. If (X, T) has the marker property than it has the local marker property.

Remark 4.3. Just as in the case of the marker property, $Z \times W$ has local markers iff it has closed n-markers for all $n \in \mathbb{N}$.

Theorem 4.4. *If* $\dim(X) < \infty$ *and P is closed, then* (X, T) *has the local marker property.*

Proof. The proof follows closely the proof of Theorem 6.1 of [Gut15] where it is shown that an aperiodic finite dimensional t.d.s has the marker property. Theorem 6.1 of [Gut15] is based on a certain generalization of Lemma 3.7 of [BC04] which is one of the building blocks in the proof of the Bonatti-Crovisier Tower Theorem for C^1-diffeomorphisms on manifolds [BC04, Theorem 3.1]. Let $\{Z_i \times V_i\}_{i=1}^{\infty}$ be an arbitrary countable cover of $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets so that for every i, $Z_i \cap P = \emptyset$ and $V_i \cap P = \emptyset$ (here we use that P

is closed) and $Z_i \cap V_i = \emptyset$. Fix $n, k \in \mathbb{N}$. For every $x \in Z_k \cup V_k$ choose an open set U_x so that $x \in U_x$, $\overline{U}_x \subset X \setminus P$ and $\overline{U}_x \cap T^i \overline{U}_x = \emptyset$ for $i = 1, 2, \ldots, m = (2dim(X) + 2)n - 1$. Let $U_{x_1}, U_{x_2}, \ldots, U_{x_s}$ be a finite cover of $Z_k \cup V_k$. We now continue exactly as in the proof of Theorem 6.1 of [Gut15], to find a W, so that $\overline{W} \cap T^i \overline{W} = \emptyset$, $i = 1, 2, \ldots, n-1$ and $Z_k \cup V_k \subset \bigcup_{i=1}^{s} \overline{U}_{x_i} \subset \bigcup_{i=0}^{m} T^i(\overline{W})$ (we use the fact that P is closed in order to invoke Lemma 6.2 of [Gut15]). The existence of an open n-marker for $Z_k \cup V_k$ follows easily. $\square$

Proposition 4.5. *If $(\Omega(X), T)$ has the local marker property then (X, T) has the local marker property.*

Proof. We will show there is a closed countable cover which has the local marker property relatively to $S = (X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by defining $S_1, S_2, S_3 \subset X \times X$ so that $S = S_1 \cup S_2 \cup S_2^* \cup S_3$ where $S_2^* = \{(y, x) | \, (x, y) \in S_2\}$, and exhibiting 3 closed countable covers which have the local marker property relatively to S_1, S_2 and S_3 respectively (the case of S_2^* will follow from the case of S_2). As $(\Omega(X), T)$ has the local marker property, there exists a a countable cover $\{Z_i \times W_i\}_{i=1}^{\infty}$ which has the local marker property relatively to $(\Omega(X) \times \Omega(X)) \setminus (\triangle \cup (\Omega(X) \times P) \cup (P \times \Omega(X)))$, however it is important to note this is w.r.t the topology induced by $\Omega(X)$. Also note that for all i, $Z_i \cup W_i \subset \Omega(X)$. We now define S_1, S_2, S_3:

(1) $S_1 = (\Omega(X) \times \Omega(X)) \cap S = (\Omega(X) \times \Omega(X)) \setminus (\triangle \cup (\Omega(X) \times P) \cup (P \times \Omega(X)))$. We claim $\{Z_i \times W_i\}_{i=1}^{\infty}$ has the local marker property relatively to S_1 (w.r.t the topology induced by X). Indeed fix k. Let F be a closed (in $\Omega(X)$ and therefore in X) n-marker for $Z_k \times W_k$ in $\Omega(X)$. Clearly one can find an $\epsilon > 0$ so that $B_\epsilon(F) \subset X$ is an open n-marker for $Z_k \times W_k$ in X.

(2) $S_2 = ((X \setminus \Omega(X)) \times \Omega(X)) \cap S$. As $X \times X$ is second-countable, every subspace is Lindelöf, i.e. every open cover has a countable subcover. Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $X \setminus \Omega(X)$ such that for each i there is an $\epsilon_i > 0$ so that $\{T^k B_{\epsilon_i}(U_i)\}_{k\in\mathbb{Z}}$ are pairwise disjoint. We claim the countable closed cover $\{\overline{U}_i \times W_k\}_{i,k=1}^{\infty}$ has the local marker property relatively to S_2. First observe that as $B_{\epsilon_i}(U_i) \subset X \setminus \Omega(X)$, $\overline{U}_i \cap W_k = \emptyset$. Now fix i, k, n. Let $F \subset \Omega(X)$ be a closed n-marker for $Z_k \times W_k$. Let $0 < \delta < d(\bigcup_{j=-(n-1)}^{n-1} T^j B_{\epsilon_i/2}(U_i), \Omega(X))$ so that $B_\delta(F)$ is still an open n-marker for $Z_k \times W_k$. We claim $B_\delta(F) \cup B_{\epsilon_i/2}(U_i)$ is an open n-marker for $\overline{U}_i \times W_k$. Indeed as $F \subset \Omega(X)$, the choice of δ guarantees $T^{j_1} B_\delta(F) \cap T^{j_2} B_{\epsilon_{i/2}}(U_i) = \emptyset$ for all $0 \le j_1, j_2 \le n-1$.

(3) $S_3 = (X \setminus \Omega(X))^2 \cap S$. Let $\{U_i\}_{i=1}^{\infty}$ be the open cover of the previous case. We can assume w.l.o.g that for each $x, y \in (X \setminus \Omega(X))^2$ with $x \neq y$ there are $i \neq k$ so that $x \in U_i$ and $y \in U_k$ and $\overline{U_i} \cap \overline{U_k} = \emptyset$. Note $\mathcal{C} = \{\overline{U}_i \times \overline{U}_k | i, k \in \mathbb{Z}, \overline{U_i} \cap \overline{U_k} = \emptyset\}$ is a closed cover of $(X \setminus \Omega(X))^2 \cap S$. A countable closed cover which has the local marker property relatively to S_3, will be achieved by splitting each member of the cover $\mathcal{C}$ to an union of at most a countable number of closed products. Fix $i \neq k$ so that $\overline{U}_i \times \overline{U}_k \in \mathcal{C}$. If for all $j \in \mathbb{Z}$, $T^j B_{\epsilon_i}(U_i) \cap B_{\epsilon_k}(U_k) = \emptyset$, then $B_{\epsilon_i}(U_i) \cup B_{\epsilon_k}(U_k)$ is an open n-marker of $\overline{U}_i \times \overline{U}_k$ for all n. Assume this is not the case. Note that $\{\overline{U}_i \times (\overline{U}_k \cap T^j \overline{B}_{\epsilon_i/2}(U_i))\}_{j\in\mathbb{Z}} \cup \{\overline{U}_i \times (\overline{U}_k \setminus \bigcup_{j\in\mathbb{Z}} T^j B_{\epsilon_i/2}(U_i))\}$ is a countable closed cover of $\overline{U}_i \times \overline{U}_k$. Let $j \in \mathbb{Z}$. Note that $B_{\epsilon_i}(U_i)$ is an open n-marker of $\overline{U}_i \times (\overline{U}_k \cap T^j \overline{B}_{\epsilon_i/2}(U_i))$ for all n. Fix n. As $d(\bigcup_{j=-(n-1)}^{n-1} \bigcup T^k(\overline{U}_i), \overline{U}_k \setminus \bigcup_{j\in\mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)) > 0$, there is $\delta > 0$ so that $T^{j_1} B_\delta(\overline{U}_i) \cap T^{j_2} B_\delta(\overline{U}_k \setminus \bigcup_{j\in\mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)) = \emptyset$ for all $0 \le j_1, j_2 \le n-1$. Taking $\delta < min\{\epsilon_i, \epsilon_k\}$, guarantees

that $B_\delta(\overline{U}_i) \cup B_\delta(\overline{U}_k \setminus \bigcup_{j\in\mathbb{Z}} T^j B_{\epsilon_i/2}(U_i))$ is an open n-marker for $\overline{U}_i \times \overline{U}_k \setminus \bigcup_{j\in\mathbb{Z}} T^j B_{\epsilon_i/2}(U_i)$.

□

5. The Strong Topological Rokhlin Property

In [Gut11, Subsection 1.9] the topological Rokhlin property was introduced (see also Subsection 2.7). Here is a stronger variant, originating in [Lin99]:

Definition 5.1. We say that (X,T) has the **(global) strong topological Rokhlin property** if for every $n \in \mathbb{N}$ there exists a continuous function $f : X \to \mathbb{R}$ so that if we define the *exceptional set* $E_f = \{x \in X \mid f(Tx) \neq f(x) + 1\}$, then $T^{-i}(E_f)$, $i = 0, 1, \ldots, n-1$, are pairwise disjoint.

Remark 5.2. Assume $b - a \leq n - 1$. Under the above conditions consider $x \in X$. Then there exists at most one index $a \leq l \leq b$ so that $f(T^{l+1}x) \neq f(T^l x)+1$. Indeed if $T^l x, T^{l'} x \in E_f$ for $a \leq l < l' \leq b$, then $E_f \cap T^{l-l'} E_f \neq \emptyset$ contradicting the definition.

Definition 5.3. We say that (X,T) has the **local strong topological Rokhlin property** if one can cover $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets $\{Z_i \times W_i\}_{k=1}^{\infty}$, where for every k, $Z_k \cap W_k = \emptyset$ and for every $m \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ with $a < b$ there exists a continuous function $f : \bigcup_{i=a}^{b} T^i(Z_k \cup W_k) \to \mathbb{R}$ so that if we define the *exceptional set* $E_f = \{x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k) \mid f(Tx) \neq f(x) + 1\}$, then $T^{-i}(E_f)$, $i = 0, \ldots, m-1$, are pairwise disjoint (note $x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k)$ implies that both $f(x)$ and $f(Tx)$ are defined). In this context $\{Z_i \times W_i\}_{k=1}^{\infty}$ is also said to have the **local strong topological Rokhlin property**.

Remark 5.4. Assume $m \geq b - a - 2$. Under the above conditions consider $x \in (Z_k \cup W_k)$. Then there exists at most one index $a \leq l \leq b - 1$ so that

$f(T^{l+1}x) \neq f(T^l x) + 1$. Indeed $T^l x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k)$ and $f(T^{l+1}x)) \neq f(T^l x) + 1$ imply $T^l x \in E_f$. If $T^l x, T^{l'} x \in E_f$ for $a \leq l < l' \leq b-1$, then $E_f \cap T^{l-l'} E_f \neq \emptyset$ contradicting the definition.

The following proposition is based on Lemma 3.3 of [Lin99], which is the statement that a system with an aperiodic minimal factor has the strong topological Rokhlin property:

Proposition 5.5. *If (X,T) has the local marker property then (X,T) has the local strong topological Rokhlin property.*

Proof. By assumption one can cover $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ by a countable collection of products of closed sets $\{Z_i \times W_i\}_{i=1}^{\infty}$, with the local marker property. We will now show that $\{Z_i \times W_i\}_{i=1}^{\infty}$ has the local strong topological Rokhlin property. Fix $m, k \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ with $a < b$. Let F be an open m marker for $Z_k \cup W_k$. It will be convenient to write $Z_k \cup W_k \subset \bigcup_{i=-a}^{q-b} T^i(F)$ for some $q > b - a$. Choose a closed $R \subset F$ so that $Z_k \cup W_k \subset \bigcup_{i=-a}^{q-b} T^i R$. Conclude:

$$\bigcup_{i=a}^{b} T^i(Z_k \cup W_k) \subset \bigcup_{i=0}^{q} T^i R \tag{5.1}$$

Let $\omega : X \to [0,1]$ be a continuous function so that $\omega_{|R} \equiv 1$ and $\omega_{|F^c} \equiv 0$. We define a random walk for $z \in X$. At any point p we arrive during the random walk, the walk terminates with probability $\omega(p)$ and moves to $T^{-1}p$ with probability $1 - \omega(p)$. Notice that for every point $z \in \bigcup_{i=a}^{b} T^i(Z_k \cup W_k)$ the random walk will terminate after a finite number of steps. Indeed by (5.1) there is an $i \in \{0, \ldots, q\}$ so that $z \in T^i R$, which implies the walk terminates in at most q steps (when the point hits R). Conclude there is a finite number of possible walks starting at z and we denote by $f(z)$ the expected length of the walk starting at z. As there is a uniform bound on the length of walks, $f : \bigcup_{i=a}^{b} T^i(Z_k \cup W_k) \to \mathbb{R}_+$ is continuous. Note that if $y \notin F$ and $y \in \bigcup_{i=a+1}^{b} T^i(Z_k \cup W_k)$, then $f(T^{-1}y) = f(y) - 1$. Notice that $x \in \bigcup_{i=a}^{b-1} T^i(Z_k \cup W_k)$ and $f(Tx) \neq f(x) + 1$ implies $y \triangleq Tx \in \bigcup_{i=a+1}^{b} T^i(Z_k \cup W_k)$ and $f(T^{-1}y) = f(TT^{-1}x) = f(x) \neq f(Tx) - 1 = f(y) - 1$. Therefore in such a case one must have $Tx = y \in F$. Conclude $E_f = \{x \in \bigcup_{i=a}^{b-1} T^i(\overline{U} \cup \overline{V}) \,|\, f(Tx) \neq f(x) + 1\} \subset T^{-1}F$. We therefore have $T^{-i}(E_f) \cap E_f = \emptyset$ $i = 1, \ldots, m-1$.

□

The following question is interesting:

Problem 5.6. Does the the local strong topological Rokhlin property imply the local marker property?

The question can be answered assuming the global strong topological Rokhlin property:

Theorem 5.7. *(X, T) has the strong topological Rokhlin property iff (X, T) has the marker property.*

Proof. The fact that the marker property implies the strong topological Rokhlin property follows by a similar argument to the proof of Proposition 5.5. To prove the other direction assume (X, T) has the strong topological Rokhlin property. Fix $n \in \mathbb{N}$ and let $f : X \to \mathbb{R}$ be a continuous function such that for the open set $E_f = \{x \in X \,|\, f(Tx) \neq f(x) + 1\}$, $T^{-i}(E_f)$, $i = 0, 1, \ldots, n-1$, are pairwise disjoint. We claim that the iterates E_f,$T^{-1}E_f, \ldots$ eventually cover X. Indeed as f is bounded from above for any $x \in X$, the series $f(T^i x)$, $i = 1, 2, \ldots$ cannot increase indefinitely.

□

6. An Embedding Theorem for systems with the local marker property

6.1. **The Embedding Theorem.** We now state our main embedding theorem. The proof has two parts. The first part (Proposition 6.3) deals with the non-periodic points whereas the second part deals with the periodic points (Proposition 6.4). Both parts are based on a highly technical extension of the proof of Theorem 5.1 of [Lin99].

Theorem 6.1. *Assume (X,T) has the local marker property. Let $d \in \mathbb{N}$ be such that $mdim(X,T) < \frac{d}{36}$ and $perdim(X,T) < \frac{d}{2}$, then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$ is an embedding is comeagre in $C(X,[0,1]^n)$.*

Proof. As explained in Subsection 2.10 we need to exhibit a closed countable cover $\mathcal{C}$ of $(X \times X) \setminus \Delta$ so that D_C is dense for all $C \in \mathcal{C}$. By Proposition 5.5 (X,T) has the local strong topological Rokhlin property. By Proposition 6.3 below one can cover $X \times X \setminus \big(\triangle \cup (X \times P) \cup (P \times X)\big)$ by a closed countable cover $\mathcal{W}$ so that for all $W \in \mathcal{W}$, D_W is dense in $C(X,[0,1]^d)$. Let P_n denote the set of points of period $\leq n$ and define $H_n = P_n \setminus P_{n-1}$. By Proposition 6.4 below for every $n \in \mathbb{N}$ there is a countable closed cover $\mathcal{K}_n$ of $\big((X \setminus P) \times H_n\big) \cup \big(H_n \times (X \setminus P)\big)$ so that for all $K \in \mathcal{K}_n$, D_K is dense in $C(X,[0,1]^d)$. By the proof of Theorem 4.1 of [Gut15] there is closed countable cover $\mathcal{P}$ of $(P \times P) \setminus \Delta$ so that D_K is dense for all $K \in \mathcal{K}$. Let $\mathcal{C} = \mathcal{W} \cup \mathcal{P} \cup \bigcup_n \mathcal{K}_n$. As $(X \times X) \setminus \Delta$ is the union of $(X \times X) \setminus \big(\triangle \cup (X \times P) \cup (P \times X)\big)$, $\bigcup_n \big((X \setminus P) \times H_n\big) \cup \big(H_n \times (X \setminus P)\big) = \big((X \setminus P) \times P\big) \cup \big(P \times (X \setminus P)\big)$ and $(P \times P) \setminus \Delta$, clearly $\mathcal{C}$ has the desired properties. We now proceed to prove Proposition 6.3 and Proposition 6.4. Throughout, it turns out to be convenient to define:

$$m_{dim} = \begin{cases} \frac{1}{72} & mdim(X,T) = 0 \\ mdim(X,T) & \text{otherwise} \end{cases}$$

Notice it holds $36m_{dim} < d$. □

We start with an auxiliary lemma:

Lemma 6.2. *Let $A \subset X$, $M \in \mathbb{N}$ be an even integer and $n : \bigcup_{i=-4M}^{\frac{M}{2}-1} T^i A \to \mathbb{R}$ a function. Assume there are $x_1, x_2 \in A$ so that for each x_i, $i = 1,2$ there is at most one index (depending on x_i) $-4M \leq j_i \leq \frac{M}{2} - 2$ for which $n(T^{j_i+1}x_i) \neq n(T^{j_i}x_i) + 1$. Then one can find an index $-4M \leq r \leq 0$ so that $\lfloor n(T^r x_i) \rfloor \mod M \leq \frac{M}{2}$ and for $r \leq s \leq r + \frac{M}{2} - 2$, $i = 1,2$:*

$$(\lceil n(T^s x_i) \rceil \mod M) = (\lceil n(T^r x_j) \rceil \mod M) + s - r \tag{6.1}$$

$$(\lfloor n(T^s x_i) \rfloor \mod M) = (\lfloor n(T^r x_j) \rfloor \mod M) + s - r \tag{6.2}$$

Proof. By the proof of Lemma 5.7 of [Lin99] one can find an index $-4M \leq r \leq 0$ so that for $r \leq s \leq r + \frac{M}{2} - 1$ one has for $i = 1,2$:

$$(n(T^s x_i) \mod M) = (n(T^r x_j) \mod M) + s - r$$

In particular for $r \leq s \leq r + \frac{M}{2} - 2$, $(n(T^s x_i) \mod M) \in [0, \frac{M}{2} + s - r + 1) \subset [0, M-1)$ and $(\lfloor n(T^r x_i) \rfloor \mod M) \leq \frac{M}{2}$. We therefore conclude (6.1) and (6.2) hold for this range of indices. □

Proposition 6.3. *Let $\mathcal{K}$ be a countable closed cover of $X \times X \setminus \big(\triangle \cup (X \times P) \cup (P \times X)\big)$ which has the local marker property, then for all $K \in \mathcal{K}$, D_K is dense in $C(X, [0,1]^d)$.*

Proof. This proof is heavily influenced by the proof of Theorem 5.1 of [Lin99]. Fix $K \in \mathcal{K}$ with $K = Z \times W$ with Z, W closed and $Z \cap W = \emptyset$. Fix $\epsilon > 0$. Let $\tilde{f} : X \to [0,1]^d$ be a continuous function. We will show that there exists a continuous function $f : X \to [0,1]^d$ so that $\|f - \tilde{f}\|_\infty < \epsilon$ and I_f is K-compatible. We start by a general construction and then relate it to $\tilde{f}$. Let $\delta = dist(Z, W) > 0$. Let α be a cover of X with $\max_{U\in\alpha} diam(\tilde{f}(U)) < \frac{\epsilon}{2}$ and $\max_{U\in\alpha} diam(U) < \delta$. Let $\epsilon' > 0$ be such that $36m_{dim}(1+2\epsilon') < d$. Let $N \in \mathbb{N}$ be such that it holds $\frac{1}{N}D(\alpha^{N+1}) < (1+\epsilon')m_{dim}$ (here we use $m_{dim} > 0$ and Remark 2.5.1 of [Gut15]), $36(1+2\epsilon') < \epsilon' N$ and N is divisible by 36. Let $\gamma \succ \alpha^{N+1}$ be an open cover so that $D(\alpha^{N+1}) = ord(\gamma)$. We have thus $ord(\gamma) < N(1+\epsilon')m_{dim}$. Let $M = \frac{2}{9}N$ and $\Delta = \frac{M}{8} - 1$. Notice $M, \Delta \in \mathbb{N}$. Notice $\Delta d > (\frac{N}{36} - 1)36m_{dim}(1+2\epsilon') = Nm_{dim}(1+\epsilon') + m_{dim}(N\epsilon' - 36(1+2\epsilon')) > Nm_{dim}(1+\epsilon')$. Conclude:

$$ord(\gamma) < \Delta d$$

For each $U \in \gamma$ choose $q_U \in U$ so that $\{q_U\}_{U\in\gamma}$ is a collection of distinct points in X, and define $\tilde{v}_U = (\tilde{f}(T^i q_U))_{i=0}^{N-1}$. According to Lemma 5.6 of [Lin99], one can find a continuous function $F : X \to ([0,1]^d)^N$, with the following properties:

1. $\forall U \in \gamma$, $||F(q_U) - \tilde{v}_U||_\infty < \frac{\epsilon}{2}$,
2. $\forall x \in X,\ F(x) \in co\{F(q_U) |\, x \in U \in \gamma\}$,
3. If for some $0 \le l, j < N - 4\Delta$ and $\lambda, \lambda' \in (0,1]$ and $x, y, x', y' \in X$ so that:

$$\lambda F(x)|_l^{l+4\Delta-1} + (1-\lambda)F(y)|_{l+1}^{l+4\Delta} = \lambda' F(x')|_j^{j+4\Delta-1} + (1-\lambda')F(y')|_{j+1}^{j+4\Delta}$$

then there exist $U \in \gamma$ so that $x, x' \in U$ and $l = j$ (note the statement $l = j$ is missing from Lemma 5.6 of [Lin99] but follows from the proof).

By Proposition 5.5, (X, T) has the local strong topological Rokhlin property. By Definition 5.3 one can find a continuous function $n : \bigcup_{-4M}^{\frac{M}{2}-1} T^i(Z \cup$

$W) \to \mathbb{R}$ so that for $E_n = \{x \in \bigcup_{-4M}^{\frac{M}{2}-2} T^i(Z \cup W) \,|\, n(Tx) \neq n(x)+1\}$ one has $E_n \cap T^i(E_n) = \emptyset$ for $1 \leq i \leq \frac{9}{2}M - 1$. Let $\underline{n}(x) = \lfloor n(x) \rfloor \mod M$, $\overline{n}(x) = \lceil n(x) \rceil \mod M$, $n'(x) = \{n(x)\}$. Let $A = \bigcup_{-4M}^{\frac{M}{2}-1} T^i(Z \cup W)$. Define:

$$f'(x) = (1 - n'(x) F(T^{-\underline{n}(x)}x)|_{\underline{n}(x)} + n'(x) F(T^{-\overline{n}(x)}x)|_{\overline{n}(x)} \quad x \in A \tag{6.3}$$

f' is continuous by the argument appearing on p. 241 of [Lin99]. By the argument of Claim 1 on p. 241 of [Lin99], as $\max_{U\in\alpha} diam(\tilde{f}(U)) < \frac{\epsilon}{2}$ and $\max_{U\in\gamma} ||F(q_U) - v_U||_\infty < \frac{\epsilon}{2}$ we have $||\tilde{f}_{|A} - f'||_{A,\infty} < \epsilon$. By Lemma A.5 of [Gut15] there is $f : X \to [0,1]^d$ so that $f_{|A} = f'_{|A}$ and $||f - \tilde{f}||_\infty < \epsilon$. We now show that $f \in D_K$. Fix $x' \in Z$ and $y' \in W$. Assume for a contradiction $f(T^a x') = f(T^a y')$ for all $a \in \mathbb{Z}$. Notice that by Remark 5.4 for both x', y' there is at most one index $-4M \leq j_{x'}, j_{y'} \leq \frac{M}{2} - 2$ for which $n(T^{j_{x'}+1}x') \neq n(T^{j_{x'}}x') + 1$, $n(T^{j_{y'}+1}y') \neq n(T^{j_{y'}}y') + 1$ respectively. By Lemma 6.2 one can find an index $-4M \leq r \leq 0$ so that for $r \leq s \leq r + \frac{M}{2} - 2$, for $z' = x', y'$, $\underline{n}(T^s z') = \underline{n}(T^r z') + s - r$, $\overline{n}(T^s z') = \overline{n}(T^r z') + s - r$ and $\underline{n}(T^r z') \leq \frac{M}{2}$. Denote $\lambda = n'(T^r x')$, $\lambda' = n'(T^r y')$, $a = \underline{n}(T^r x') \leq \frac{M}{2}$ and $a' = \underline{n}(T^r y')$. Substituting $T^s x', T^s y'$ for $r \leq s \leq r + 4\Delta - 1 = r + \frac{M}{2} - 5$ in equation (6.3) (note $T^s x', T^s y' \in A$), we conclude from the equality $I_f(x')|_r^{r+\frac{M}{2}-5} = I_f(y')|_r^{r+\frac{M}{2}-5}$:

$$(1-\lambda)F(T^{r-a}x')|_a^{a+4\Delta-1} + \lambda F(T^{r-a-1}x')|_{a+1}^{a+4\Delta} = (1-\lambda')F(T^{r-a'}y')|_{a'}^{a'+4\Delta-1} + \lambda' F(T^{r-a'-1}y')|_{a'+1}^{a'+4\Delta}$$

E.g. notice that for $0 \leq i \leq \frac{M}{2} - 5$ it holds that $T^{-\underline{n}(T^{r+i}x')}T^{r+i}x' = T^{-(a+i)+r+i}x' = T^{r-a}x'$. As the conditions of Lemma 5.6 of [Lin99] are fulfilled then by condition (3), one has that $a = a'$ and that there exist $U \in \gamma \succ \alpha^{N+1}$ so that $T^{r-a}x', T^{r-a}y' \in U$. As $N = -4M - \frac{M}{2} \leq r -$

$a \leq 0$ we can find $V \in \alpha$, so that $x', y' \in V$. This is a contradiction to $\max_{U\in\alpha} diam(U) < dist(Z, W) = \delta$.

□

Proposition 6.4. *Assume* (X, T) *has the local strong topological Rokhlin property and let* $n \in \mathbb{N}$, *then there is a countable closed cover* $\mathcal{K}$ *of* $(X \setminus P) \times H_n$ *so that for* $K \in \mathcal{K}$, D_K *is dense in* $C(X, [0,1]^d)$.

Proof. Let $\mathcal{C}$ be a cover of $(X \times X) \setminus (\triangle \cup (X \times P) \cup (P \times X))$ with the local strong topological Rokhlin property. Cover H_n by a countable collection. Let W be an open set in H_n (not necessarily open in X) with $y \in W \subset \overline{W} \subset H_n$. Let $Z \times R \in \mathcal{C}$. Fix $\epsilon > 0$. Let $\tilde{f} : X \to [0,1]^d$ be a continuous function. We will show that there exists a continuous function $f : X \to [0,1]^d$ so that $\|f - \tilde{f}\|_\infty < \epsilon$ and I_f is K-compatible for $K = Z \times \overline{W}$. Let α be a cover of X with $\max_{U\in\alpha} diam(\tilde{f}(U)) < \frac{\epsilon}{2}$. Let $\epsilon' > 0$ be such that $36 m_{dim}(1+2\epsilon') < d$. We will see it is enough to assume $18 m_{dim}(1+2\epsilon') < d$ (actually it is enough to assume $8 m_{dim}(1+2\epsilon') < d$ but we will not use this fact). Let $N \in \mathbb{N}$, divisible by 18, be such that it holds $\frac{1}{N} D(\alpha^N) < (1+\epsilon') m_{dim}$ and $N\epsilon' - 9n(1+2\epsilon') > \frac{1}{m_{dim}}$. Let $\gamma \succ \alpha^N$ be an open cover so that $D(\alpha^N) = ord(\gamma)$. Let $M = \frac{2}{9}N$ and $S = \frac{M}{4}$. Notice $(S - \frac{n}{2})d > (\frac{N}{18} - \frac{n}{2}) 18 m_{dim}(1 + 2\epsilon') = N m_{dim}(1+\epsilon') + m_{dim}(N\epsilon' - 9n(1+2\epsilon')) > 1 + N m_{dim}(1+\epsilon')$. As $ord(\gamma) < N(1+\epsilon') m_{dim}$, conclude:

$$ord(\gamma) + 1 < (S - \frac{n}{2})d$$

For each $U \in \gamma$ choose $q_U \in U$ so that $\{q_U\}_{U\in\gamma}$ is a collection of distinct points in X, and define $\tilde{v}_U = (\tilde{f}(T^i q_U))_{i=0}^{N-1}$. According to Lemma 6.5 one can find a continuous function $F : X \to ([0,1]^d)^N$, with the following properties:

(1) $\forall U \in \gamma$, $||F(q_U) - \tilde{v}_U||_\infty < \frac{\epsilon}{2}$,

(2) $\forall x \in X,\ F(x) \in co\{F(q_U) | \, x \in U \in \gamma\}$,

(3) For any $0 \le l, j < N - 2S$, and $\lambda \in (0,1]$ and $x, y \in X$ it holds:

$$(1-\lambda)F(x)|_l^{l+2S-1} + \lambda F(y)|_{l+1}^{l+2S} \notin V_{2S}^n$$

where,

$$V_{2S}^n \triangleq \{w = (w_0, \ldots, w_{2\Delta-1}) \in ([0,1]^d)^{2S} | \forall 0 \le a, b \le 2S-1, a = b \mod n \to w_a = w_b\}.$$

By Definition 5.3 one can find a continuous function $n : \bigcup_{-4M}^{\frac{M}{2}-1} T^i Z \to \mathbb{R}$ so that for $E_n = \{x \in \bigcup_{-4M}^{\frac{M}{2}-2} T^i Z \,|\, n(Tx) \neq n(x)+1\}$ one has $E_n \cap T^i(E_n) = \emptyset$ for $1 \le i \le \frac{9}{2}M - 1$. Let $\underline{n}(x) = \lfloor n(x) \rfloor \mod M$, $\overline{n}(x) = \lceil n(x) \rceil \mod M$, $n'(x) = \{n(x)\}$. Let $A = \bigcup_{-4M}^{\frac{M}{2}-1} T^i Z$. Define:

$$f'(x) = (1 - n'(x)F(T^{-\underline{n}(x)}x)|_{\underline{n}(x)} + n'(x)F(T^{-\overline{n}(x)}x)|_{\overline{n}(x)} \quad x \in A \tag{6.4}$$

f' is continuous by the argument appearing on p. 241 of [Lin99]. By the argument of Claim 1 on p. 241 of [Lin99], as $\max_{U\in\alpha} diam(\tilde{f}(U)) < \frac{\epsilon}{2}$ and $\max_{U\in\gamma} ||F(q_U) - \tilde{v}_U||_\infty < \frac{\epsilon}{2}$ we have $||f' - \tilde{f}_{|A}||_\infty < \epsilon$. By Lemma A.5 of [Gut15] there is $f : X \to [0,1]^d$ so that $f_{|A} = f'_{|A}$ and $||f - \tilde{f}||_\infty < \epsilon$. We now show that $f \in D_K$. Fix $x' \in Z$ and $y' \in \overline{W}$. Assume for a contradiction $f(T^a x') = f(T^a y')$ for all $a \in \mathbb{Z}$. Notice that by Remark 5.4 there is at most one index $-4M \le j_{x'} \le \frac{M}{2} - 2$ for which $n(T^{j_{x'}+1}x') \neq n(T^{j_{x'}}x') + 1$. By Lemma 6.2 one can find an index $-4M \le r \le 0$ so that for $r \le s \le r + \frac{M}{2} - 1$, $\underline{n}(T^s x') = \underline{n}(T^r x') + s - r$ and $\overline{n}(T^s x') = \overline{n}(T^r x') + s - r$. Denote $\lambda = n'(T^r x')$ and $a = \underline{n}(T^r x')$. Substituting $T^s x'$ for $r \le s \le r + 2S - 1 = r + \frac{M}{2} - 1$ in equation (6.4) (note $T^s x' \in A$), we conclude from the equality $I_f(x')|_r^{r+\frac{M}{2}-1} = I_f(y')|_r^{r+\frac{M}{2}-1}$ (compare with the analogue part in the proof of Proposition 6.3):

$$(1-\lambda)F(T^{r-a}x')|_a^{a+2S-1} + \lambda F(T^{r-a-1}x')|_{a+1}^{a+2S} = I_f(y')|_r^{r+2S-1}$$

As $y' \in \overline{W} \subset H_n$, one clearly has $I_f(y')|_r^{r+2S-1} \in V_{2S}^n$. This is a contradiction to property (3).

□

Lemma 6.5. *Let $\epsilon > 0$. Let $N, n, d, S \in \mathbb{N}$ with $N > 2S$. Let γ be an open cover of R with $ord(\gamma) + 1 \leq (S - \frac{n}{2})d$. Assume $\{q_U\}_{U\in\gamma}$ is a collection of distinct points in R and $\tilde{v}_U \in ([0,1]^d)^N$ for every $U \in \gamma$, then there exists a continuous function $F : R \to ([0,1]^d)^N$, with the following properties:*

1. *$\forall U \in \gamma$, $||F(q_U) - \tilde{v}_U||_\infty < \frac{\epsilon}{2}$,*
2. *$\forall x \in R$, $F(x) \in co\{F(q_U)|\, x \in U \in \gamma\}$,*
3. *For any $0 \leq l < N - 2S$, and $\lambda \in [0,1]$ and $x_0, x_1 \in R$ it holds:*

$$(1-\lambda)F(x_0)|_l^{l+2S-1} + \lambda F(x_1)|_{l+1}^{l+2S} \notin V_{2S}^n \tag{6.5}$$

where $V_{2S}^n \triangleq \{y = (y_0, \ldots, y_{2S-1}) \in ([0,1]^d)^{2S}| \quad \forall 0 \leq a, b \leq 2S-1,\ (a = b \mod n) \to y_a = y_b\}$.

Proof. Let $\{\psi_U\}_{U\in\gamma}$ be a partition of unity subordinate to γ so that $\psi_U(q_U) = 1$. Let $\vec{v}_U \in ([0,1]^d)^N$, $U \in \gamma$ be vectors that will be specified later. Define:

$$F(x) = \sum_{U\in\gamma} \psi_U(x)\vec{v}_U$$

For $x \in R$ define $\gamma_x = \{U \in \gamma|\, \psi_U(x) > 0\}$. Let $\lambda_0 = 1 - \lambda$, $\lambda_1 = \lambda$. Write (6.5) explicitly as:

$$\sum_{j=0}^{1} \sum_{U\in\gamma_{x_j}} \lambda_j \psi_U(x_j)\vec{v}_U|_{l+j}^{l+2S-1+j} \notin V_{2S}^n \tag{6.6}$$

Note that $dim(V^n_{2S}) = nd$ and $nd+|\gamma_{x_1}|+|\gamma_{x_2}| \leq nd+2(ord(\gamma)+1) \leq 2Sd$ which will be used repeatedly. Denote $W = V^n_{2S}+span\big(\{\vec{v}_U|_l^{l+2S-1}\}_{U\in\gamma_{x_0}\setminus\gamma_{x_1}}\cup \{\vec{v}_U|_{l+1}^{l+2S}\}_{U\in\gamma_{x_1}\setminus\gamma_{x_0}}\big)$. By Lemma A.6 of [Gut15], almost surely in $\big([0,1]^d)^{2S}\big)^{|\gamma_{x_1}\triangle\gamma_{x_2}|}$, $dim(W) = nd + |\gamma_{x_1}\triangle\gamma_{x_2}|$. Let $U \in \gamma_{x_0}\cap\gamma_{x_1}$. In equation (6.6) we find an expression of the form $\lambda_0\psi_U(x_0)\vec{v}_U|_l^{l+2S-1}+\lambda_1\psi_U(x_1)\vec{v}_U|_{l+1}^{l+2S}$. Using Lemma 6.6 repeatedly (exactly $|\gamma_{x_0}\cap\gamma_{x_1}|$ times) we conclude that for any fixed W (i.e where we have chosen the parameters defining W), almost surely in $\big([0,1]^d)^{2S}\big)^{|\gamma_{x_0}\cap\gamma_{x_1}|}$, $dim(W + span\big(\{\vec{v}_U|_l^{l+2S-1}, \vec{v}_U|_{l+1}^{l+2S}\}_{U\in\gamma_{x_0}\cap\gamma_{x_1}}\big)) = nd + |\gamma_{x_1}\triangle\gamma_{x_2}| + 2|\gamma_{x_1}\cap\gamma_{x_2}| = nd + |\gamma_{x_1}| + |\gamma_{x_2}|$. As the left-hand side of (6.6) is a convex combination of almost surely independent vectors belonging to a linear subspace which intersects V^n_{2S} trivially , we conclude Equation (6.6) holds almost surely in the parameters involved. As there is a finite number of constraints of the form (6.6), we can therefore choose $\vec{v}_U \in ([0,1]^d)^N$, $U \in \gamma$ so that properties (1) and (3) hold. Finally property (2) holds trivially as $F(q_U) = \vec{v}_U$. □

Lemma 6.6. *Let $m, r \in \mathbb{N}$ with $r < m$. Let $V \subset \mathbb{R}^m$ be a linear subspace with $dim(V) \leq m-2$. Then almost surely w.r.t Lebesgue measure for $(x_1, x_2, \ldots x_{r+m}) \in [0,1]^{r+m}$,*

$$W(x_1, x_2, \ldots x_{r+m}) = span\{(x_1, \ldots x_m), (x_{r+1}, x_{r+2}, \ldots x_{r+m})\} \subset \mathbb{R}^m$$

is a linear subspace such that $dim(V + W) = dim(V) + 2$.

Proof. Clearly one can assume w.l.o.g $dim(V) = m-2$. Choose a basis for V, $v_1, \ldots v_{m-2}$ and consider the square $m \times m$ matrix

$$\mathcal{M} = [(x_1, \ldots x_m), (x_{r+1}, x_{r+2}, \ldots x_{r+m}), v_1, \ldots v_{m-2}]$$

where the vectors should be understood as the columns of M. Let c_{tl}, $t < l$ denote the determinant of the submatrix corresponding to erasing the first two columns and rows t and l. Note it is not possible that all $c_{tl} = 0$. Indeed as $v_1, \dots v_{m-2}$ are linearly independent we can add vectors v_{m-1}, v_m so that $v_1, \dots v_m$ spans $\mathbb{R}^m$. Using Leibniz formula for the determinant of $[v_{m-1}, v_m, v_1, \dots v_{m-2}]$ we see that not all $c_{tl} = 0$. Order $\{c_{tl}\}_{tl}$ lexigoraphically and let c_{ij} be the minimal non-zero element. Notice $\det(M) = \pm x_i(x_{j+r}c_{ij} + f) + g$ where f does not depend on x_i, x_{j+r} and g does not depend on x_i. The crucial fact that f does not depend on x_i, x_{j+r} follows from the minimality of c_{ij}. We now derive conditions which guarantee $\det(\mathcal{M}) \neq 0$. We impose no conditions on $\{x_1, x_2, \dots x_{r+m}\} \setminus \{x_i, x_{j+r}\}$. For a fixed choice of values for $\{x_1, x_2, \dots x_{r+m}\} \setminus \{x_i, x_{j+r}\}$ which determines f we require $x_{j+r} \neq \frac{f}{c_{ij}}$. Fixing additionally x_{j+r} according to this condition g is determined and we require $x_i \neq \frac{g}{x_{j+r}c_{ij}+f}$. By Fubini's Theorem we conclude $\det(\mathcal{M}) \neq 0$ almost surely w.r.t Lebesgue measure for $(x_1, x_2, \dots x_{r+m}) \in [0,1]^{r+m}$.

$\square$

7. Applications of the embedding theorem

In this section we present two general applications of Theorem 6.1. Recall that Lindenstrauss proved in [Lin99] that an extension of an aperiodic minimal system with $mdim(X,T) < \frac{d}{36}$ is embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$. In Theorem 7.1 we replace the condition of being an extension of an aperiodic minimal system by being an extension of an aperiodic t.d.s which either is finite-dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector. These classes correspond exactly to the classes where we have proved the marker property to hold. Although establishing the marker property in these cases is far from trivial, we believe this theorem is an indication of the usefulness of the marker property point of view.

The second application of Theorem 6.1, Theorem 7.3, differs qualitatively from 7.1. According to this theorem a t.d.s with a finite dimensional non-wondering set and a closed set of periodic points such that $perdim(X,T) < \frac{d}{2}$ is embeddable in $(([0,1]^d)^{\mathbb{Z}},\ shift)$. The theorem has several striking qualities. Firstly it is a true example of embedding in the presence of periodic points, which is the the title of this work. Secondly it does not involve the notion of mean dimension due to an application of a highly non-trivial "mean-dimension addition formula" due to Tsukamoto. Finally, as will be discussed in the next section, it can be applied in the realm of fluid mechanics.

Theorem 7.1. *Assume (X,T) is an extension of an aperiodic t.d.s which either is finite-dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector. Then (X,T) has the strong Rokhlin property. If in addition $d \in \mathbb{N}$ is such that $mdim(X,T) < \frac{d}{36}$, then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$ is an embedding is comeagre in $C(X,[0,1]^d)$.*

Proof. In those cases, by Theorems 3.5, 3.9 as well as Theorem 6.1 of [Gut15], (X,T) has the marker property. We can therefore conclude by Theorem 5.7 that (X,T) has the strong Rokhlin property. By Theorem 6.1 , as (X,T) is aperiodic the second part of the theorem holds. □

Lemma 7.2. $mdim(X,T) = mdim(\Omega(X),T)$

Proof. Clearly $mdim(X,T) \geq mdim(\Omega(X),T)$. To see the reversed inequality let $Y = X/\Omega(X)$ (i.e. the quotient space where the closed and T-invariant subspace $\Omega(X)$ is identified with a point) and let $\pi : (X,T) \to (Y,T')$ be the quotient map, where T' is the induced transformation. Note that $\pi_{X\setminus\Omega(X)}$ is injective. We can therefore use the remarkable "mean-dimension addition formula" [Tsu08, Theorem 4.6] in order to conclude $mdim(X,T) \leq$

$mdim(Y,T') + mdim(\Omega(X),T)$. As $\pi(\Omega(X)) \simeq \{\bullet\}$ is the only closed invariant subsystem of Y, it holds $h_{top}(Y,T') = 0$ which implies $mdim(Y,T') = 0$ (see Subsection 2.9). We therefore conclude $mdim(X,T) \leq mdim(\Omega(X),T)$ as desired.

□

Theorem 7.3. *Let (X,T) be a t.d.s so that $\Omega(X)$ is finite dimensional, the set of periodic points $P(X,T)$ is closed and $perdim(X,T) < \frac{d}{2}$. Then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$ is an embedding is comeagre in $C(X,[0,1]^d)$.*

Proof. By Lemma 7.2 as $\Omega(X)$ is finite dimensional, $mdim(X,T) = mdim(\Omega(X),T) = 0$. By Theorem 4.4 $(\Omega(X),T)$ has the local marker property. By Proposition 4.5 (X,T) has the local marker property. Combining all of these facts, we conclude by Theorem 6.1 that the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X,T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$ is an embedding is comeagre in $C(X,[0,1]^d)$. □

Remark 7.4. Note that the statement of Theorem 7.3 does not involve mean-dimension but its proof does. It would be interesting to find a direct proof.

Recall the Lindenstrauss-Tsukamoto Conjecture from the Introduction.

Corollary 7.5. *Let (X,T) be a t.d.s so that $\Omega(X)$ is finite dimensional and the set of periodic points $P(X,T)$ is closed, then the Lindenstrauss-Tsukamoto Conjecture holds for (X,T).*

Proof. It is sufficient to notice that $mdim(X,T) = 0$ (as pointed out in the proof of Theorem 7.3) and apply Theorem 7.3.

□

Example 7.6. We now construct a family of examples for which the previous theorem is applicable. Let $R : [0,1] \to [0,1]$ be a continuous invertible map such that $R(0) = 0$, $R(1) = 1$ and such there are no other fixed points. It easily follows that for all $0 < x < 1$, $\lim_{n\to\infty} R^n(x) = 1$, $\lim_{n\to-\infty} R^n(x) = 0$ or $\lim_{n\to\infty} R^n(x) = 0$, $\lim_{n\to-\infty} R^n(x) = 1$, e.g. $R(x) = \sqrt{x}, x^2$. Let $Q = [0,1]^{\mathbb{N}}$, be the Hilbert cube, equipped with the product topology. Define $\mathbf{R} : Q \to Q$, by $\mathbf{R}((x_i)_{i=1}^{\infty}) = (R(x_i))_{i=1}^{\infty}$. It is easy to see $\Omega(Q, \mathbf{R}) = \{0,1\}^{\mathbb{N}}$. Let (Y, S) be a finite dimensional t.d.s with a closed set of periodic points. It follows easily that $\Omega(Y \times Q, S \times \mathbf{R}) = \Omega(Y, S) \times \Omega(Q, \mathbf{R}) = \Omega(Y, S) \times \{0,1\}^{\mathbb{N}}$. As $\{0,1\}^{\mathbb{N}}$ is zero-dimensional and $\Omega(Y, S) \subset Y$, we conclude $\Omega(Y \times Q, S \times \mathbf{R})$ is finite-dimensional. Moreover $P(Y \times Q, S \times \mathbf{R}) = P(Y, S) \times \{0,1\}^{\mathbb{N}}$, which is closed. We have thus verified all prerequisites that enable us to apply the previous theorem for the infinite-dimensional system $(Y \times Q, S \times \mathbf{R})$. Additionally notice that as $\{0,1\}^{\mathbb{N}}$ consists of fixed points of $(Q, \mathbf{R})$ and is zero-dimensional, $\overrightarrow{perdim}(Y \times Q, S \times \mathbf{R}) = \overrightarrow{perdim}(Y, S)$.

8. The two-dimensional Navier-Stokes Equations

8.1. **Overview of the Section.** Theorem 7.3 is closely related to a situation not uncommon in dynamical systems arising in physics - the existence of a finite dimensional global attractor (see definition below). Good references to this and related subjects are [Hal88, Lad91, Tem97]. A case in point are the Navier-Stokes equations which describe the motion of fluid. We will concentrate on the case where the flow is confined to a two-dimensional domain as it is much better understood than the general three-dimensional case. Two dimensional models for flows may sound unrealistic but actually bear some importance both as an approximation to certain real life phenomena and as a gateway to the three dimensional case (see p. 13 of Chapter I of [FMRT01]). Our goal is to embed a discrete model of the Navier-Stokes equations into a cubical shift. In Subsection 8.2 we introduce the Navier-Stokes equations and an associated (Hilbert) space of solutions. In Subsection 8.3 we introduce a discrete model on a compact set which is absorbing. This means that every initial state will end up in this set after finite time. The discrete model which we obtain is not invertible. In Subsection 8.4 we embed the discrete model inside an invertible t.d.s. In Subsection 8.5 we verify that that the non-wandering set of this invertible t.d.s is finite dimensional. In Subsection 8.6 we apply Theorem 7.3 to the invertible system and obtain as a consequence sufficient conditions for embedding of the discrete model in a cubical $\mathbb{Z}$-shift. These conditions include the requirement that the set of periodic points is closed. In Subsection 8.7 we improve the embedding theorem by removing this condition.

8.2. **Background.** Following [Rob11, Rob13], consider the Navier-Stokes equations for a two-dimensional incompressible viscous flow:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x) \\ \nabla u = 0 \\ \int_\Omega u\,dx = 0\ , \qquad \int_\Omega f(x)\,dx = 0 \\ u(x,0) = u_0(x), \end{cases}$$

subject to periodic boundary conditions with basic domain $\Omega = [0,L]^2$, $L > 0$. The velocity field, $u = (u', u'')$, and the pressure, p, are the unknown functions, while $f(x)$ is a given forcing term and $\nu > 0$ is a given constant viscosity.

Let us denote by

$$\mathcal{V} = \{u \in [C^\infty_{per}(\Omega)]^2 |\ \nabla\cdot u = 0 \text{ and } \int_\Omega u\,dx = 0\},$$

Here:

$$C^{\infty}_{per}(\Omega) = \{u \in C^{\infty}(\mathbb{R}^2) |\ \forall x \in \mathbb{R}^2\ u(x + Le_i) = u(x) \quad i = 1, 2\}$$

where $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$.

Denote by H the Hilbert space which is the closure of $\mathcal{V}$ in $[L^2_{per}(\Omega)]^2$ and denote the induced norm by $||\cdot||$. The so called *functional equation for the Navier-Stokes equations* can be written as an evolution equation in the Hilbert space H as:

$$\begin{cases} \frac{du}{dt} + \nu Au(t) + B(u(t), u(t)) = f, \text{ for } t > 0, \\ u(0) = u_0 \end{cases} \tag{8.1}$$

where A is a certain linear operator and B is a certain bi-linear form and $f \in H$. It can be shown that given $u_0 \in H$, there is a unique solution $u = u_{u_0}(x, t) \in C^0([0, \infty), H)$. We define a **semigroup of solution operators** (also known as a *transformation monoid*), $\mathbb{S} = \{S(t)\}_{t \geq 0}$ by $S(t) : H \to H$ for $t \geq 0$ by:

$$S(t)u_0 = u_{u_0}(\cdot, t)$$

Fixing $T = S(t)$ for some $t > 0$, we get a continuous map $\mathbb{N} \times H \to H$ given by $(n, u_0) \mapsto T^n u_0$. One of the remarkable properties of $\mathbb{S}$ is the existence of a finite-dimensional global attractor as defined next:

Definition 8.1. $A \subset H$ is a called a **global attractor** for $S(t_0)$ for some $t_0 > 0$ if

(1) A is compact,
(2) $S(t_0)(A) = A$; and

(3) A attracts bounded sets, i.e. for every bounded subset B of H,

$$\lim_{n\to\infty} \operatorname{dist}(S(nt_0)(B), A) = 0, \tag{8.2}$$

where $\operatorname{dist}(C, D) = \sup_{c\in C} \inf_{d\in D} \|c - d\|$ is the Hausdorff semidistance.

$A \subset H$ is a called a **global attractor** for $\mathbb{S}$ if A is a global attractor for $S(t_0)$ for all $t_0 > 0$[2]. Notice that if a global attractor exists then it is unique.

In [Rob11, Subsection 11.4] the existence of the global attractor is proven by showing the existence of a *compact* absorbing set for $\mathbb{S}$ as defined next:

Definition 8.2. X is called **absorbing** for $\mathbb{S}$ if for every bounded subset B of E there exists $t_B \geq 0$ such that,

$$\forall t \geq t_B \; S(t)B \subset X \tag{8.3}$$

Moreover it is shown that if $B = \overline{B}_M(0)$ (the closed ball of radius M around the origin) then it is sufficient to take $t_B = \max\{0, -\log \frac{||f||^2}{M^2}\} + 1$ in order to guarantee 8.3. This means that in practice (i.e. in a real experiment) we may guarantee that after a certain calculable time the system is in an absorbing compact set. Whereas the system tends to the global attractor as time tends to *infinity*, it is guaranteed to belong to an absorbing compact set after a *finite* time. Thus understanding absorbing compact sets is interesting.

8.3. **A discrete model.** In order to align ourselves with the material developed in previous chapters we proceed to discretize the action specified by $\mathbb{S}$. Let X be a compact absorbing set. As X is bounded, by Equation (8.3) there exists $t_X \geq 0$ such that:

$$\forall t \geq t_X \; S(t)X \subset X \tag{8.4}$$

[2][Rob11, Rob13] use an a priori stronger definition but for our purposes the given definition is sufficient.

Thus if we choose any $t \geq t_X$ and define $T = S(t)$ then $T : X \to X$ defines a t.d.s[3]. However as will be seen in Remark 8.13, it sometime desirable to be able to choose $t > 0$ as small as we wish. The next lemma show how to modify X so that this is possible.

Lemma 8.3. *Suppose X is a compact absorbing set and $t_X \geq 0$ such that for all $t \geq t_X$, $S(t)X \subset X$. Let $t > 0$ and denote $T = S(t)$ and $n = \lceil \frac{t_X}{t} \rceil$. Define $X' = \bigcup_{k=0}^{n} T^k X$, then X' is a compact absorbing set such that $TX' \subset X'$.*

Proof. As X' is a finite union of continuous images of X, X' is compact. As X' contains X, it is absorbing. Finally $TX' = \bigcup_{k=0}^{n} T^{k+1}X \subset X' \cup T^{n+1}X$. However by Equation (8.4), $T^{n+1}X \subset X \subset X'$. □

From now on we assume $t_X = 0$. We choose some $t > 0$ that will be specified later and consider the t.d.s (X, T) where $T = S(t)$.

8.4. **An invertible model.** As pointed out before the continuous map $T : X \to X$ is not necessarily invertible. However by subsection 2.5 of [Rob13], T is injective. In order to align ourselves with the material developed in previous chapters we proceed to equivariantly embed (X, T) inside an invertible t.d.s with almost unchanged non-wandering and periodic points sets. In Subsection 8.7, we exhibit a different approach which yields a stronger result. As a consequence we leave some of the topological details of the argument in this section to the interested reader.

Lemma 8.4. *Let (X, T) be a t.d.s consisting of a metric compact space X and a continuous injective map $T : X \to X$. Then there exists an invertible t.d.s (X_∞, T_∞) consisting of a metric compact space X_∞ and a continuous invertible map $T_\infty : X_\infty \to X_\infty$, as well as an equivariant embedding ϕ :*

[3] In this section a topological dynamical system (t.d.s) (X, T) consists of a compact metric space X and a continuous (not necessarily invertible) transformation $T : X \to X$.

$(X,T) \hookrightarrow (X_\infty, T_\infty)$, i.e. an embedding $\phi : X \hookrightarrow X_\infty$ such that $T_\infty \phi = \phi T$, with the following properties:

(1) There exists an element $\infty \in X_\infty \setminus X$ such that $T_\infty \infty = \infty$ (∞ is a fixed point).
(2) $P(X_\infty, T_\infty) = \phi(P(X,T)) \cup \{\infty\}$.
(3) $\Omega(X_\infty, T_\infty) = \phi(\Omega(X,T)) \cup \{\infty\}$.

Proof. The proof of the lemma is straightforward. Inductively one adds to X preimages of order $1, 2, \ldots$ for those points of X which do not have a preimage to start with. Finally one takes the one-point compactification of the resulting locally compact space. We assume w.l.o.g that $X \setminus TX \neq \emptyset$ as otherwise the lemma is trivial. Let us now describe the proof in several steps:

Constructing an increasing chain of embeddings: In this step we construct injective but not invertible t.d.s $(X,T) = (X_1, T_1), (X_2, T_2), \ldots$ and equivariant embeddings $\phi_i : (X_i, T_i) \hookrightarrow (X_{i+1}, T_{i+1})$ $(T_{i+1}\phi_i = \phi_i T_i)$ for $i = 1, 2, \ldots$. Assume we have constructed (X_n, T_n). Let us construct (X_{n+1}, T_{n+1}). Define $L_n^1 = \overline{X_n \setminus T_n X_n}$. This is the closure of the set of points which do not have a preimage under T_n. Define $D_n^1 = L_n^1 \cap T_n X_n$ and $C_n^1 = T^{-1} D_n^1$. Trivially $D_n^1 \subset L_n^1$. Let L_n^2 be a (homeomorphic and disjoint) copy of L_n^1 with D_n^2 a copy of D_n^1. Denote the natural isomorphism $i_n : L_n^2 \leftrightarrow L_n^1$. Notice $T_{n|C_n^1} : C_n^1 \to D_n^1$ induces a homeomorphism $T_n' : C_n^1 \to D_n^2$ by $T_n' = i_n \circ T_{n|C_n^1}$. Define X_{n+1} as the adjunction space $\big(X_n \mathring{\cup} L_n^2\big)/\sim$ where $\sim$ is the closed equivalence relation induced by the identification map T_n'. One now checks X_{n+1} is compact and metric and a natural embedding $\phi_n : X_n \hookrightarrow X_{n+1}$ is induced. Abusing notation we write $X_n \subset X_{n+1}$.

We now define the continuous map $T_{n+1} : X_{n+1} \to X_{n+1}$ such that $T_{n+1|X_n} = T_n$ by the the following:

$$T_{n+1}x = \begin{cases} T_n x & x \in X_n \\ \\ i_n(x) & x \notin X_n \end{cases}$$

Note that T_{n+1} is well defined as if $[x]_\sim \in X_n \cap L_n^2$ with $[x] = \{x_1, x_2\}$ with $x_1 \in C_n^1$ and $x_2 \in D_n^2$, then by definition of the adjunction space $i_n(x_2) = T_n x_1$.

Embedding the original system inside a locally compact space equipped with a homeomorphism: Define Y as the direct limit of the X_i, $Y = \varinjlim X_i$. One checks that Y is locally compact and natural embeddings $j_n : X_n \hookrightarrow Y$ are induced. Abusing notation we write $Y = \bigcup X_i$. We now define the map $S : Y \to Y$ by $Sy = T_n x$ if $y \in X_n$. One checks this is well defined and that the resulting map S is a homeomorphism. Moreover $j_1 : (X, T) \hookrightarrow (Y, S)$ is an equivariant embedding, i.e. $S \circ j_1 = j_1 \circ T$.

One-point compactification of the locally compact space: Define $X_\infty = Y \cup \{\infty\}$ to be the one-point compactification of Y. Define the homeomorphism $T_\infty : X_\infty \to X_\infty$ by $T_\infty x = Sx$ if $x \neq \infty$ and $T_\infty \infty = \infty$. (X_∞, T_∞) is the invertible t.d.s described in the statement of the lemma. As Y embeds in X_∞, we have an equivariant embedding $\phi : (X, T) \hookrightarrow (X_\infty, T_\infty)$. Abusing notation we will write $X = X_1 \subset X_2 \subset \cdots X_\infty$ and identify T and T_∞ on X. It is easy to see $P(X_\infty, T_\infty) = P(X, T) \cup \{\infty\}$. Notice that for all $n \in \mathbb{N}$, $T_\infty^{n-1} X_n \subset X$. Let $x \in X_\infty \setminus (X \cup \{\infty\})$. As X_∞ is regular and $X \cup \{\infty\}$ is closed we may find an open set U such that $x \in U \subset \overline{U}$ and $\overline{U} \cap (X \cup \{\infty\}) = \emptyset$. As $\overline{U}$ is compact we conclude there exists $N \in \mathbb{N}$ so that $\overline{U} \subset X_N$. This implies that for all $k \geq N - 1$, $T^k U \cap U = \emptyset$ which implies in turn $x \notin \Omega(X_\infty, T_\infty)$ (recall the definition of the non-wandering set in Subsection 2.2). We thus conclude $\Omega(X_\infty, T_\infty) = \phi(\Omega(X, T)) \cup \{\infty\}$ as desired. $\square$

8.5. **Verification that the non-wandering set is finite-dimensional.** A remarkable property of $\mathbb{S}$ is the fact that the global attractor has finite upper box-counting dimension (see [Rob11, Subsection 12.4]). This certainly implies that the global attractor has finite Lebesgue covering dimension[4]. Let us denote the global attractor by A. Clearly $A \subset X$. Our next goal is to show $\Omega(X,T) \subset A$ which will imply $\Omega(X,T)$ is finite dimensional. To do so we introduce a variant of the classical notion of an omega-limit of a point:

Definition 8.5. Let (X,T) be a t.d.s and let $C \subset X$. The **omega-limit of** C is defined by:

$$\omega(C) \triangleq \bigcap_{k\geq 0} \overline{\bigcup_{n\geq k} T^n(C)} = \{x \in X|\, x = \lim_{j\to\infty} T^{n_j}(b_j),\ n_j \to \infty,\ b_j \in C\}$$

Lemma 8.6. *$\Omega(X,T) \subset A$.*

Proof. By Theorem 11.3 of [Rob11] slightly adapted to the case of $T = S(t)$ we have $A = \omega(X)$. Assume $x \in \Omega(X,T)$. For any open set U such that $x \in U$ we may find n as a large as we wish so that $T^{-n}U \cap U \neq \emptyset$. This implies one may find a sequence $k_n \to \infty$ and $b_n \in X$, such that $T^n b_n \to x$. Conclude that $x \in \omega(X) = A$. $\square$

8.6. **A conditional embedding theorem.** As a result of the previous subsections and Theorem 7.3 we have the following theorem:

Theorem 8.7. *Let $\mathbb{S}$ be a semigroup of solutions operators associated with Equation (8.1). Let X be a compact absorbing set for $\mathbb{S}$. Let $T = S(t)$ such that (X,T) is a t.d.s. Let (X_∞, T_∞) be the invertible model constructed in Subsection 8.4 in which (X,T) equivariantly embeds. If the set of periodic points $P(X,T)$ is closed and perdim$(A,T) < \frac{d}{2}$ for some $d \in \mathbb{N}$, then the*

[4]The Lebesgue covering dimension of a compact metric space is less or equal its upper box-counting dimension ([Rob11, p. 85]).

collection of continuous functions $f : X_\infty \to [0,1]^d$ *so that* $I_f : (X_\infty, T_\infty) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$ *is an embedding is comeagre in* $C(X, [0,1]^d)$*.*

Proof. According to Lemma 8.6, $\Omega(X,T)$ is finite dimensional. According to Lemma 8.4 we may embed (X,T) inside an invertible t.d.s (X_∞, T_∞) such that $P(X_\infty, T_\infty)$ is closed, $\Omega(X_\infty, T_\infty)$ is finite dimensional and $perdim(X_\infty, T_\infty) = perdim(X,T) = perdim(A,T)$. The last equality holds as $P(X,T) \subset \Omega(X,T) \subset A$ by Lemma 8.6. We now apply Theorem 7.3. □

The previous theorem is conditional in the sense that for given $f \in H$ and $\nu > 0$, one has to verify that the set of periodic points $P(X,T)$ is closed. As far as the author is aware of this has not been verified or invalidated in the literature on the Navier-Stokes equations. It is therefore desirable to prove a similar theorem which does not require this condition. This is the subject of the next subsection.

8.7. **An unconditional embedding theorem.** Recall that in [Gut15] it was proven that if (X,T) is a finite dimensional t.d.s with $perdim(X,T) < \frac{d}{2}$, then (X,T) is embeddable in $(([0,1]^d)^{\mathbb{Z}}, shift)$ comeagrely. Moreover as X is finite dimensional, the inequality $perdim(X,T) < \frac{d}{2}$ can be verified even if one can only control a finite number of quantities of the form $dim(P_k(X,T))$ and in particular if $d \geq 2\dim(X) + 1$. In this subsection we show that the same is true for (X,T), where $T = S(t)$, due to existence of a finite dimensional global attractor. We start with a simple lemma:

Lemma 8.8. *Let* $\mathbb{S}$ *be a semigroup of solutions operators associated with Equation (8.1). Let* $t > 0$ *and suppose* X *is a compact absorbing set for* $T = S(t)$*, and* A *is a finite dimensional global attractor for* T*. Let* U *be an open set such that* $A \subset U$ *then there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$*,* $T^n(X) \subset U$*.*

Proof. One can assume w.l.o.g $U \neq X$. As X is compact it is bounded in H. Therefore by Definition 8.1 $\lim_{n\to\infty} \operatorname{dist}(T^n X, A) = 0$, where $\operatorname{dist}(T^n X, A) = \sup_{c\in T^n X} \inf_{a\in A} \|c - a\|$. It is therefore enough to show that there exists $\epsilon > 0$ such that for any $c \in U^c$, $\inf_{a\in A} \|c - a\| > \epsilon$. Assume this is not true. Choose a sequence $c_n \in U^c$ with $\operatorname{dist}(c_n, A) \leq \frac{1}{n}$. Assume w.l.o.g $c_\epsilon \to c$. Then $c \in U^c \cap A$ which is a contradition. □

Lemma 8.9. *Let $F \subset X$ be a closed set and $n \in \mathbb{N}$. Let $\alpha' = \{A'_j\}_{j\in J}$ be a finite open cover of X. Let $\beta = \{B_i\}_{i\in I}$ be a finite refinement of $\alpha'_{|F} = \{A'_j \cap F\}_{j\in J}$ such that $ord(\beta) \leq n$. Then there exists a finite collection τ, of open sets in X, such that τ refines α', covers F and $ord(\tau) \leq n$.*

Proof. As β is a refinement of α', we may find a mapping $i \mapsto j_i$ such that $B_i \subset A'_{j_i}$ for all $i \in I$. By [Dug66, Theorem 6.1 of Chapter VII] the normality of X implies we may find a closed cover of F, $\gamma = \{C_i\}_{i\in I}$ (for all i, $C_i \subset F$), such that $C_i \subset B_i \subset A'_{j_i|F}$. Let $0 < \epsilon = \min_{\{i|\, A'_{j_i} \neq X\}} d(C_i, A'^c_{j_i})$ (this distance is measured in X). As $ord(\beta) \leq n$, for any distinct $i_1, i_2, \ldots i_{n+2}$, one has $\bigcap_{k=1}^{n+2} C_{i_k} = \emptyset$. We claim there exists $0 < \delta < \epsilon$ such that for any distinct $i_1, i_2, \ldots i_{n+2}$, $\bigcap_{k=1}^{n+2} B_\delta(C_{i_k}) = \emptyset$. Indeed if such a $\delta > 0$ does not exist one can find specific $i_1, i_2, \ldots i_{n+2}$ and $x_m \in B_{\frac{1}{m}}(C_{i_k})$ for $k = 1, \ldots n+2$ and $m \in \mathbb{N}$. Assume w.l.o.g $x_m \to x$ to conclude $x \in \bigcap_{k=1}^{n+2} C_{i_k}$ which is a contradiction. We finally define a cover of F, $\tau = \{B_\delta(C_i)\}_{i\in I}$. Clearly τ refines β and therefore refines α' and has $ord(\tau) \leq n$. □

Theorem 8.10. *Let $\mathbb{S}$ be a semigroup of solutions operators associated with Equation (8.1). Let $t > 0$ and suppose X is a compact absorbing set for $T = S(t)$, and A is a finite dimensional global attractor for T. If $perdim(A, T) < \frac{d}{2}$ for some $d \in \mathbb{N}$, then the collection of continuous functions $f : X \to [0,1]^d$ so that $I_f : (X, T) \hookrightarrow (([0,1]^d)^{\mathbb{N}}, \mathbb{N} - shift)$ is an embedding is comeagre in $C(X, [0,1]^d)$.*

Proof. The proof is achieved by adapting the proof of Theorem 8.1 in [Gut15] which uses the Baire category theorem framework (see Subsection 2.10). Just as in [Gut15, Theorem 8.1], we will use the following representation $(X \times X)\setminus(\triangle\cup(P\times P)) = D_1\cup D_2$ where $D_1 = (X\times X)\setminus(\triangle\cup(P_{6n}\times X)\cup(X\times P_{6n}))$ and $D_2 = (P_{6n}\times(X\setminus P))\cup((X\setminus P)\times P_{6n})$. The case of D_2 is similar to the case of D_1 so we give the details only for this case. This amounts to adopting Lemma 8.2 of [Gut15]. Let $(x, y) \in D_1$. We may find open sets $U_x, U_y \subset X$ so that $x \in U_x$, $y \in U_y$ and so that $\{T^{i_k}\overline{U}_x, T^{i_k}\overline{V}_y\}_{k=0}^{2n}$ are pairwise disjoint for $0 = i_0 < i < \cdots < i_{2n}$. Define $K_{(x,y)} = \overline{U}_x \times \overline{V}_y$. Let $\epsilon > 0$. Let $\tilde{f} : X \to [0, 1]^d$ be a continuous function. We will show that there exists a continuous function $f : X \to [0, 1]^d$ so that $\|f-\tilde{f}\|_\infty < \epsilon$ and I_f is $\mathbb{N}-K_{(x,y)}$-compatible. Let α be an open cover of X with $\max_{W\in\alpha,k\in\{0,1,\ldots,2n\}} diam(\tilde{f}(T^{i_k}W)) < \frac{\epsilon}{2}$, $\max_{W\in\alpha} diam(W) < \epsilon$. Denote $dim(A) = n$. By Lemma 8.9, one may find a a finite collection τ, of open sets in X, [Gut15] such that τ refines α, covers A and $ord(\tau) \leq n$. Denote $A \subset \bigcup\tau \triangleq U$. By Lemma 8.8, there is an $N \in \mathbb{N}$, such that $T^N K_{(x,y)} \subset U \times U$. We now continue by applying the proof of Proposition 8.2 of [Gut15] verbatim to $T^N K_{(x,y)}$ with one caveat. Indeed observe that although Proposition 8.2 of [Gut15] is stated for invertible $T : X \to X$, the proof only uses the fact T is injective. □

Remark 8.11. Note that the previous theorem used essentially the fact that a global finite dimensional attractor exists. Trying to remove the condition of a closed set of periodic points from the statement of Theorem 7.3 seems harder.

The next two remarks shows how the condition $perdim(A, T) < \frac{d}{2}$ can be dealt with.

Remark 8.12. Let $dim(A) = n$. As $\frac{n}{\lfloor\frac{2n}{d}\rfloor+1} < \frac{d}{2}$, in order to verify $perdim(A, T) < \frac{d}{2}$, it is enough to verify for all $m \leq m_0 \triangleq \lfloor\frac{2n}{d}\rfloor + 1$:

$$dim(P_m(A,T) < \frac{dm}{2}$$

Note this constitutes a finite number of inequalities. Moreover if $d > 2n$ the condition $perdim(A,T) < \frac{d}{2}$ is void.

Remark 8.13. According to [Kuk94] under certain conditions on the parameters, if $t > 0$ is small enough, it is possible to show that $P_1(X, S(t)) = P_2(X, S(t)) = \cdots = P_{m_0}(X, S(t))$. In a similar (but not identical) set-up to the one used here, [Tem80] shows that generically $P_1(X, S(t))$ is finite, in particular zero-dimensional. One wonders if the same is true here.

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Appendix A. The Equivalence of SBP and Vanishing Mean Dimension Under the Marker Property

Recall the definition of $mdim_d(X,T)$ in Subsection 2.9.

Theorem A.1. *If (X,T) has the marker property then there is a compatible metric d' such that $mdim(X,T) = mdim_{d'}(X,T)$.*

Proof. This is a straightforward generalization of Theorem 4.3 of [Lin99], which is the statement that the conclusion of the theorem holds if the system has an aperiodic minimal factor. ☐

As a corollary of the previous theorem we have the following theorem:

Theorem A.2. *Assume (X,T) is an extension of an aperiodic t.d.s which either is finite dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector then there is a compatible metric d' such that $mdim(X,T) = mdim_{d'}(X,T)$.*

Theorem A.3. *If (X,T) has the marker property then the following conditions are equivalent:*

(a) $mdim(X,T) = 0$

(b) (X,T) has the small boundary property (SBP)

(c)$(X,T) = \lim_{\overleftarrow{i \in \mathbb{N}}}(X_i,T_i)$ where $h_{top}(X_i,T_i) < \infty$ for $i \in \mathbb{N}$.

Proof. $(a) \Rightarrow (b)$ is straightforward generalization of Theorem 6.2 of [Lin99], which is the statement that $(a) \Rightarrow (b)$ holds if the system has an aperiodic minimal factor. $(c) \Rightarrow (a)$ follows from Proposition 2.8 of [LW00] (this implication is true for any system). $(b) \Rightarrow (a)$ is Theorem 5.4 of [LW00] (this implication is true for any system). $(a) \Rightarrow (c)$ is straightforward generalization of Proposition 6.14 of [Lin99], which is the statement that $(a) \Leftrightarrow (c)$ holds if the system has an aperiodic minimal factor. $\square$

As a corollary of the previous theorem we have the following theorem:

Theorem A.4. *Assume (X,T) is an extension of an aperiodic t.d.s which either is finite dimensional or has a countable number of minimal subsystems or has a compact minimal subsystems selector then then $mdim(X,T) = 0$ iff (X,T) has the small boundary property iff $(X,T) = \lim_{\overleftarrow{i \in \mathbb{N}}}(X_i,T_i)$ where $h_{top}(X_i,T_i) < \infty$ for $i \in \mathbb{N}$.*

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