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Embedding $\mathbb{Z}^k$-actions in cubical shifts and $\mathbb{Z}^k$-symbolic extensions

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Abstract. *Mean dimension* is an invariant which makes it possible to distinguish between topological dynamical systems with infinite entropy. Extending in part the work of Lindenstrauss we show that if $(X, \mathbb{Z}^k)$ has a free zero-dimensional factor then it can be embedded in the $\mathbb{Z}^k$-shift on $([0, 1]^d)^{\mathbb{Z}^k}$, where $d = [C(k)\ \mathrm{mdim}(X, \mathbb{Z}^k)] + 1$ for some universal constant $C(k)$, and a topological version of the *Rokhlin lemma* holds. Furthermore, under the same assumptions, if $\mathrm{mdim}(X, \mathbb{Z}^k) = 0$, then $(X, \mathbb{Z}^k)$ has the *small boundary property*. One of the applications of this theory is related to Downarowicz's *entropy structure*, a master invariant for entropy theory, which captures the emergence of entropy on different scales. Indeed, we generalize this invariant and prove the Boyle–Downarowicz *symbolic extension entropy theorem* in the setting of $\mathbb{Z}^k$-actions. This theorem describes what entropies are achievable in symbolic extensions.

1. *Introduction*

1.1. *$\mathbb{Z}^k$-Actions.* Throughout this paper we denote a topological dynamical system by $(X, \mathbb{Z}^k)$. The action is denoted by $T^g x$ for $g \in \mathbb{Z}^k$. We call a $\mathbb{Z}^k$-dynamical system *free* if, for every $x \in X$ and every $\vec{0} \neq g \in \mathbb{Z}^k$, it holds that $T^g x \neq x$.

1.2. *Dimension.* Let X be a compact metric space. The Lebesgue covering dimension (or simply topological dimension) is an important invariant of topological spaces. Given a cover α of a space X consisting of open sets $U_1, U_2 \ldots, U_n$, define its *order* by $\mathrm{ord}(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$. Next let $D(\alpha)$ stand for the minimum order with respect to all covers β refining α (denoted by $\beta \succ \alpha$), i.e. $D(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta)$. This enables us to present the definition of topological dimension:

$$\dim(X) = \sup_{\alpha} D(\alpha).$$

1.3. *Mean dimension.* Let α, β be open covers of X. The *join* of α and β is the open cover by all sets of the form $A \cap B$ where $A \in \alpha$, $B \in \beta$. Similarly we can define the join $\bigvee_{i=1}^{n} \alpha_i$ of any finite collection of open covers of X. Write $F_n = \{-n, \ldots, n\}^k$ and define [**Gro99**]

$$\mathrm{mdim}(X, \mathbb{Z}^k) = \sup_{\alpha} \lim_{n\to\infty} \frac{D(\bigvee_{g\in F_n} T^g \alpha)}{|F_n|}.$$

One can show that the limit exists (see [**LW00**, Theorem 6.1]). In [**LW00**] Lindenstrauss and Weiss showed that $\mathrm{mdim}(([0, 1]^d)^{\mathbb{Z}}, \text{shift}) = d$. In addition they constructed minimal systems of arbitrary large mean dimension. For some time it was believed that any $\mathbb{Z}$-minimal system is embeddable in $([0, 1]^{\mathbb{Z}}, \text{shift})$. However their work showed this to be false. As embeddings do not raise mean dimension, it is clear that a dynamical system of mean dimension greater than one cannot be embedded in $([0, 1]^{\mathbb{Z}}, \text{shift})$.

1.4. *Embedding-dimension.* Define

$$\mathrm{edim}(X, \mathbb{Z}^k) = \min\{n \in \mathbb{N} \cup \{\infty\} \mid \exists \theta : (X, \mathbb{Z}^k) \hookrightarrow (([0, 1]^n)^{\mathbb{Z}^k}, \text{shift})\}.$$

This is the minimal n such that there is a continuous equivariant embedding of $(X, \mathbb{Z}^k)$ into the shift on the n-cube. Notice that any compact metric space can be embedded in $[0, 1]^{\mathbb{N}}$ which implies that $(X, \mathbb{Z}^k)$ is naturally embedded in $(([0, 1]^{\mathbb{N}})^{\mathbb{Z}^k}, \text{shift})$, hence $\mathrm{edim}(X, \mathbb{Z}^k)$ is well defined. In an ingenious article [**Lin99**] Lindenstrauss proved that $\mathrm{edim}(X, \mathbb{Z}) \leq 36\, \mathrm{mdim}(X, \mathbb{Z}) + 1$ for extensions of non-trivial minimal $\mathbb{Z}$-systems.

1.5. *Dense embedding.* As Baire category theorem plays a crucial role in the various proofs of this work, it is convenient to introduce the following definitions. To any mapping $f \in C(X, [0, 1]^d)$ we associate I_f. This is the continuous $\mathbb{Z}^k$-equivariant mapping $I_f : (X, \mathbb{Z}^k) \to (([0, 1]^d)^{\mathbb{Z}^k}, \text{shift})$ given by $I_f(x) = (f(T^g x))_{g\in\mathbb{Z}^k}$. Write

$$E^r = \{f \in C(X, [0, 1]^r) \mid I_f : (X, \mathbb{Z}^k) \hookrightarrow (([, 1]^r)^{\mathbb{Z}^k}, \text{shift) is an embedding}\}.$$

One says $\mathrm{edim}(X, \mathbb{Z}^k) \leq d$ *densely* if E^r is dense in $C(X, [0, 1]^r)$, for some $r \leq d$ (in $\|\cdot\|_\infty$ topology).

1.6. *Locally finite tilings.* Let $\mathcal{F}$ be the collection consisting of all compact convex polytopes of $\mathbb{R}^k$ with non-empty interiors. Denote by Φ the space of tilings by polytopes in $\mathcal{F}$, i.e. $\nu \in \Phi$ if and only if $\nu = \{V_n\}_{n\in\mathbb{N}}$ where the $V_n \in \mathcal{F}$ have pairwise disjoint interiors and their union equals $\mathbb{R}^k$. We also require the tilings $\nu \in \Phi$ to be *locally finite* in the sense that any compact region of $\mathbb{R}^k$ intersects only a finite number of members of ν. We introduce a topology for Φ. First notice that one can consider Φ to be a subset of $2^{\mathcal{F}}$, where $2^{\bullet}$ denotes power set. Given $A \subset \mathbb{R}^k$, define the restriction operator $|_A : \Phi \to 2^{\mathcal{F}}$ by $\nu|_A = \{V \in \nu \mid V \subset A\}$. Equip Φ with the topology generated by the basis consisting of all

$$\mathcal{U}(\nu, \epsilon) = \{\tau \in \Phi \mid \forall V \in \nu|_{B_{1/\epsilon}(\vec{0})}\ \exists V' \in \tau \text{ s.t. } d_H(V, V') < \epsilon\},$$

where $B_{1/\epsilon}(\vec{0})$ denotes the open ball of radius $1/\epsilon$ around the origin of $\mathbb{R}^k$ and

$$d_H(V, V') = \inf\{\epsilon > 0 \mid V \subset B_\epsilon(V') \wedge V' \subset B_\epsilon(V)\}$$

is the Hausdorff distance defined for pairs of closed sets (see the definition of $B_\epsilon(V)$ in §1.13).

There is a natural continuous action of $\mathbb{R}^k$ on Φ by translations. Indeed if $\nu = \{V_i\}_{i\in\mathbb{N}}$ then $T^g\nu = \{g + V_i\}_{i\in\mathbb{N}}$.

1.7. *Intersection property.* In order to tackle the $\mathbb{Z}^k$-case we introduce a new definition. Let $\theta_k = \sqrt{k}/2$ and for $A \subset \mathbb{R}^k$ define $A_{-r} = \{x \in \mathbb{R}^k \mid d(x, A^c) \geq r\}$.

Definition 1.7.1. A continuous $\mathbb{Z}^k$-equivariant mapping $\phi : (X, \mathbb{Z}^k) \to (\Phi, \mathbb{Z}^k)$ is called an (r, μ)-*intersection* function if for all $x, y \in X$ there exist $V^x \in \phi(x)$ and $V^y \in \phi(y)$ with $V^x, V^y \subset B_{r/2}(\vec{0})$ such that

$$|V^x_{-\theta_k} \cap V^y_{-\theta_k} \cap \mathbb{Z}^k| \geq \mu(2r + 1)^k.$$

$(X, \mathbb{Z}^k)$ has the *intersection property (IP)* if there exists a constant $0 < \mu < 1$, so that for any $N \in \mathbb{N}$ there exists $n \geq N$ and a continuous $\mathbb{Z}^k$-equivariant mapping $\phi : (X, \mathbb{Z}^k) \to (\Phi, \mathbb{Z}^k)$ which is a (n, μ)-intersection function. The constant μ is referred to as the *intersection ratio*.

In this work we extend the techniques of Lindenstrauss in order to prove the following theorem.

THEOREM 1.7.2. *If* $(X, \mathbb{Z}^k)$ *has the intersection property with intersection ratio* $0 < \mu < 1$ *then there exists* $C = C(k, \mu) > 0$ *such that* $\mathrm{edim}(X, \mathbb{Z}^k) \leq [C\,\mathrm{mdim}(X, \mathbb{Z}^k)] + 1$ *densely.*

Examples of spaces with the intersection property are provided by the following theorem.

THEOREM 1.7.3. *There exists* $0 < \mu = \mu(k) < 1$ *such that if* $(X, \mathbb{Z}^k)$ *is an extension of a free zero-dimensional system then it has the intersection property with intersection ratio* μ.

1.8. *A lower bound for embedding-dimension.* Theorem 1.7.2 gives an upper bound for edim in certain cases. With respect to the question of finding a lower bound for edim, one can show that there exist dynamical systems $(X, \mathbb{Z}^k)$ such that $\mathrm{edim}(X, \mathbb{Z}^k) \geq 2\,\mathrm{mdim}(X, \mathbb{Z}^k) + 1$. Here is an instructive example for $k = 1$. Let $X = K_5$ be the complete graph on five points, formed by taking five points in $\mathbb{R}^3$ in general position and joining all points by line segments. Notice $\dim(K) = 1$ and therefore by [**LW00**, Proposition 3.1] $\mathrm{mdim}(K^{\mathbb{Z}}, \mathrm{shift}) \leq 1$. As $[0, 1] \hookrightarrow K$ we have $\mathrm{mdim}(K^{\mathbb{Z}}, \mathrm{shift}) \geq 1$ and thus $\mathrm{mdim}(K^{\mathbb{Z}}, \mathrm{shift}) = 1$. We claim $\mathrm{edim}(K^{\mathbb{Z}}, \mathrm{shift}) = 3$. Assume for a contradiction that $\mathrm{edim}(K^{\mathbb{Z}}, \mathrm{shift}) = 2$, i.e. there exists $f : (K^{\mathbb{Z}}, \mathrm{shift}) \to (([0, 1]^2)^{\mathbb{Z}}, \mathrm{shift})$. Let $\pi : ([0, 1]^2)^{\mathbb{Z}} \to [0, 1]^2$ be the projection on the zeroth coordinate and $i : K \to K^{\mathbb{Z}}$ be the natural embedding given by $i(x) = (\ldots, x, x, x, \ldots)$. Conclude that the composition $\pi \circ f \circ i : K \to [0, 1]^2$ is one-to-one. This contradicts the known fact that K_5 is not a planar graph.

The last example can be generalized to $\mathbb{Z}^k$-systems with arbitrary large mean dimension. Flores (see [**Flo35**], a more accessible source is [**Eng78**, §1.11F]) proved that C_n, the union of all faces of dimension less than or equal to n of the simplex generated by $2n+3$ points in general position in $\mathbb{R}^{2n+2}$, cannot be embedded in $\mathbb{R}^{2n}$. As $\dim(C_n)=n$ and $[0,1]^n \hookrightarrow C_n$ we conclude similarly to the case $k=1$ that $\mathrm{edim}(C_n^{\mathbb{Z}^k}, \mathrm{shift}) = 2n+1$ whereas $\mathrm{mdim}(C_n^{\mathbb{Z}^k}, \mathrm{shift}) = n$.

1.9. *Topological Rokhlin property.* The classical Rokhlin lemma states that, given a free invertible measure-preserving system (X, T, μ) and given $\epsilon > 0$ and $n \in \mathbb{N}$, one can find $A \subset X$ such that $A, TA, \ldots, T^{n-1}A$ are disjoint and

$$\mu\bigg(\bigcup_{k=0}^{n-1} T^k A\bigg) > 1-\epsilon.$$

It easily follows that given a free invertible measure-preserving system (X, T, μ) and given $\epsilon > 0$, one can find a measurable mapping $f : X \to \{0, 1, \ldots, n-1\}$ so that if we define the *exceptional set* $E_f = \{x \in X \mid f(Tx) \neq f(x)+1\}$, then $\mu(E) < \epsilon$. The new formulation allows us to generalize to the topological category. Indeed following [**SW91**], given a dynamical system $(X, \mathbb{Z}^k)$ and a set $E \subset X$, we define the orbit-capacity of a set E in the following manner:

$$\mathrm{ocap}(E) = \lim_{n\to\infty} \sup_{x\in X} \frac{1}{|F_n|} \sum_{f\in F_n} 1_E(T^f x).$$

One can show that the limit exists. Denote by $\{e_j\}_{j=1}^k$ the standard unit vectors of $\mathbb{Z}^k$. We say that a free dynamical system $(X, \mathbb{Z}^k)$ has the *topological Rokhlin property (TRP)* if and only if for every $\epsilon > 0$ there exists a continuous function $f : X \to \mathbb{R}^k$ so that if we define the *exceptional set*

$$E_f = \{x \in X \mid \exists 1 \le j \le k \ f(T^{e_j}x) \neq f(x) + e_j\},$$

then $\mathrm{ocap}(E_f) < \epsilon$.

1.10. *Small boundary property.* Following [**SW91**] we call $E \subset X$ *small* if $\mathrm{ocap}(E) = 0$. For closed sets this has a simple interpretation. Indeed a closed set $A \subset X$ is small if and only if for any $\mathbb{Z}^k$-invariant measure of X, μ, one has $\mu(A) = 0$. When X has a basis of open sets with small boundaries, $(X, \mathbb{Z}^k)$ is said to have the *small boundary property (SBP)*. The topological Rokhlin property and the small boundary property are related by the following theorem.

THEOREM 1.10.1. *If $(X, \mathbb{Z}^k)$ is an extension of a free dynamical system with the small boundary property, then it has the topological Rokhlin property.*

As a zero-dimensional space has a basis consisting of open sets with empty boundaries, it has SBP and we conclude the following theorem.

THEOREM 1.10.2. *If $(X, \mathbb{Z}^k)$ is an extension of a free zero-dimensional system, then it has the topological Rokhlin property.*

In [**Lin99**] Lindenstrauss proved that a $\mathbb{Z}$-system with mean dimension zero admitting a non-trivial minimal factor has the small boundary property. Partially extending this result we show the following.

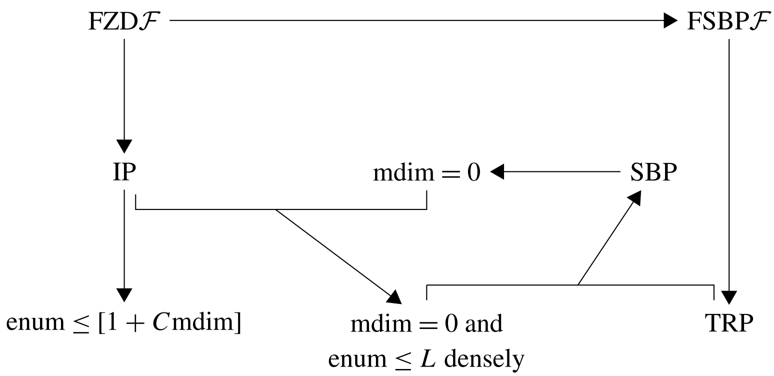
THEOREM 1.10.3. *If* $\mathrm{mdim}(X, \mathbb{Z}^k) = 0$, $\mathrm{edim}(X, \mathbb{Z}^k) \leq l$ *densely for some* $l \in \mathbb{N}$ *and* $(X, \mathbb{Z}^k)$ *has the topological Rokhlin property then* $(X, \mathbb{Z}^k)$ *has the small boundary property.*

Notice that a partial converse stating that if $(X, \mathbb{Z}^k)$ has the small boundary property, then it has mean dimension zero, follows from an easy generalization of the case $k = 1$ stated in [**LW00**].

1.11. *Summary of results.* We state a theorem that combines theorems that were mentioned in previous subsections.

THEOREM 1.11.1. *If* $(X, \mathbb{Z}^k)$ *is an extension of a free zero-dimensional system, then it has the topological Rokhlin property. Moreover there exists* $C = C(k) > 0$ *such that* $\mathrm{edim}(X, \mathbb{Z}^k) \leq [C\,\mathrm{mdim}(X, \mathbb{Z}^k)] + 1$. *Under the same assumptions, when* $\mathrm{mdim}(X, \mathbb{Z}^k) = 0$, $(X, \mathbb{Z}^k)$ *has the small boundary property.*

For the convenience of the reader we include a diagram summarizing the results appearing in §§1.7–1.10:



where FZD$\mathcal{F}$ = free zero-dimensional factor, TRP = topological Rokhlin property, SBP = small boundary property, and FSBP$\mathcal{F}$ = free factor with SBP.

1.12. *The $\mathbb{Z}^k$-symbolic extension entropy theorem.* In [**BD04**] Boyle and Downarowicz introduced *entropy structure* and prove the *symbolic extension entropy theorem* for $\mathbb{Z}$-actions. In [**Dow05**] Downarowicz replaced the definition of entropy structure by a new one, making it a topological invariant. In §5 we generalize the latter notion of entropy structure and prove the symbolic extension theorem for $\mathbb{Z}^k$-actions. In order to state the theorem we introduce the following definitions.

Definition 1.12.1. $(Y, \mathbb{Z}^k) \to (X, \mathbb{Z}^k)$ is called a *symbolic extension* of $(X, \mathbb{Z}^k)$ if $(Y, \mathbb{Z}^k)$ is a subshift over some *finite* alphabet $\mathcal{A}$.

Finite entropy is a necessary condition for the existence of a symbolic extension. However it is not a sufficient condition (see [**BFF02**, §3] for an example where $k = 1$). We introduce some definitions while using the convention that functions (e.g. the entropy function $h = h(\mu),\ \mu \in \mathcal{P}_{\mathbb{Z}^k}(X)$, where $\mathcal{P}_{\mathbb{Z}^k}(X)$ denotes the set of $\mathbb{Z}^k$-invariant probability measures of $(X, \mathbb{Z}^k)$) appear in regular font, whereas constants (e.g. $\mathbf{h}_{\mathrm{top}}$) appear in boldface.

For a given symbolic extension $\pi : (Y, \mathbb{Z}^k) \to (X, \mathbb{Z}^k)$ define

$$h^{\pi}_{\mathrm{ext}}(\mu) = \sup_{\nu \in \pi^{-1}(\mu)} h(\nu).$$

The *symbolic extension entropy* is given by (*symb. ext.* is an abbreviation of symbolic extension)

$$\mathbf{h}_{\mathrm{sex}} = \mathbf{h}_{\mathrm{sex}}(X, \mathbb{Z}^k) = \inf_{\substack{\pi : Y \to X \\ \text{symb. ext.}}} \mathbf{h}_{\mathrm{top}}(Y, \mathbb{Z}^k).$$

The *residual entropy* is given by

$$\mathbf{h}_{\mathrm{res}} = \mathbf{h}_{\mathrm{res}}(X, \mathbb{Z}^k) = \mathbf{h}_{\mathrm{sex}}(X, \mathbb{Z}^k) - \mathbf{h}_{\mathrm{top}}(X, \mathbb{Z}^k).$$

The importance of residual entropy is discussed in the introduction of [**Dow01**]. Following [**Dow05**, §2.1] we give the following definition.

Definition 1.12.2. Let f be a bounded function. Define the *upper-semi-continuous envelope of* f:

$$\tilde{f}(x) = \max\Big\{ f(x), \limsup_{x' \to x} f(x') \Big\}.$$

Define the *defect of upper semicontinuity*:

$$\dddot{f} = \tilde{f} - f.$$

Definition 1.12.3. Let $\mathcal{H} = (h_k)$ be an entropy structure for $(X, \mathbb{Z}^k)$ (see §5.2). A function $E : \mathcal{P}_{\mathbb{Z}^k}(X) \to \mathbb{R}$ is a *superenvelope* if and only if $E \geq h$ (h is the entropy function) and for every $\mu \in \mathcal{P}(X)_{\mathbb{Z}^k}$ one has

$$\lim_{k \to \infty} \overset{\cdots\cdots\cdots}{E - h_k} = 0.$$

One also allows the constant ∞ function as a superenvelope of $\mathcal{H}$. Denote by $E\mathcal{H}$ the infimum of all superenvelopes of $\mathcal{H}$. It is easy to see that $E\mathcal{H}$ is also a superenvelope of $\mathcal{H}$ [**Dow05**, Lemma 2.1.5, p. 63].

In the celebrated symbolic extension entropy theorem [**Dow05**, Theorem 5.1.1, p. 76] Downarowicz characterized the existence of symbolic extensions in terms of the entropy structure. We state the generalization to the $\mathbb{Z}^k$ setting.

THEOREM 1.12.4. (The $\mathbb{Z}^k$-symbolic extension entropy theorem) *Let $(X, \mathbb{Z}^k)$ be a dynamical system with finite topological entropy. A function $E : \mathcal{P}_{\mathbb{Z}^k}(X) \to \mathbb{R}$ equals h^{π}_{ext} for some symbolic extension π of $(X, \mathbb{Z}^k)$ if and only if E is a bounded, affine superenvelope of the entropy structure $\mathcal{H}$ of $(X, \mathbb{Z}^k)$. In particular* $\mathbf{h}_{\mathrm{sex}}(X, \mathbb{Z}^k) = \sup E\mathcal{H}$.

1.13. *Conventions.* We use $d(\cdot,\cdot)$ to denote distance in various spaces, e.g. distance in the underlying space X or in Euclidean space $\mathbb{R}^k$. Given $A \subset X$, $x \in X$ we define $d(x, V) = \inf_{v\in V} d(x, v)$ and the open and close δ-fattening around V respectively, $B_\delta(V) = \{x \in X \mid d(x, V) < \delta\}$, $\overline{B}_\delta(V) = \overline{B_\delta(V)}$. $B_\delta(x)$, $\overline{B}_\delta(x)$ denote the open and closed ball of radius δ around $x \in X$ respectively. $B_\delta(\vec{0})$ is the open ball of radius δ around the origin in Euclidean space. $v(\cdot)$ denotes volume in Euclidean space and dv is the standard volume element. $\|\cdot\|_\infty$ denotes the L_∞ norm both in $C(X, \mathbb{R})$ and in $\mathbb{R}^n$, i.e. for $f \in C(X, \mathbb{R})$, $\|f\|_\infty = \sup_{x\in X} |f(x)|$ and for $v \in \mathbb{R}^n$ $\|\vec{v}\| = \max_{i=1,\ldots,n} |v_i|$. For $r \in \mathbb{R}$, $[r] = \lfloor r\rfloor = \max\{n \in \mathbb{Z} \mid n \le r\}$ denotes the floor function. $\lceil r\rceil = \min\{n \in \mathbb{Z} \mid n \ge r\}$ denotes the ceiling function. $\pi_q : ([0, 1]^d)^{F_n} \to [0, 1]^d$ for $q \in F_n$ denotes the natural projection $(x_g)_{g\in F_n} \mapsto x_q$. Given $H : X \to ([0, 1]^d)^{F_n}$ and $T \subset F_n$ we use the notation $H(x)|_T = (\pi_q \circ H(x))_{q\in T}$. $\mathcal{P}(X)$ denotes the set of probability measures of X whereas $\mathcal{P}_{\mathbb{Z}^k}(X)$ denotes the set of $\mathbb{Z}^k$-invariant probability measures of $(X, \mathbb{Z}^k)$.

2. *Embedding in cubical shifts*

2.1. *Regular Voronoi tilings.* The following definition is based on [**Lig03**, Definition 3.1].

Definition 2.1.1. Let $A \subset \mathbb{Z}^k$, seen as a discrete subset of $\mathbb{R}^k$. For each $a \in A$ define $V(a) = \{x \in \mathbb{R}^k \mid d(x, a) \le d(x, A)\}$. One can easily show that $V(a)$ is a convex polytope. $\mathcal{V} = \mathcal{V}(A) = \{V(a)\}_{a\in A}$ is called a *Voronoi tiling* of $\mathbb{R}^k$. The point a is called the *Voronoi center* of $V(a)$. If for all $z \in \mathbb{Z}^k$ there is $a \in A$ so that $d(a, z) \le m$ and for all $a, b \in A$, $a \ne b$, $d(a, b) > m$ then A is called an *m-regular subset* of $\mathbb{R}^k$. A Voronoi tiling induced by an m-regular subset is called an *m-regular Voronoi tiling*. The collection of m-regular Voronoi tilings is denoted by $\mathcal{M}_m$. Its topology is induced by the inclusion $\mathcal{M}_m \subset \Phi$. The natural action of $\mathbb{Z}^k$ on Φ induces an action of $\mathbb{Z}^k$ on $\mathcal{M}_m$. One can easily prove from the above definition that for each $a \in A$ ($\theta_k = \sqrt{k}/2$)

$$\overline{B}_{m/2}(a) \subset V(a) \subset \overline{B}_{m+\theta_k}(a). \tag{2.1}$$

2.2. *The standard tiling.* The Voronoi tiling induced by $\mathbb{Z}^k$ is called the standard tiling of $\mathbb{R}^k$ and is denoted by $\mathcal{Z}^k$. The tiles of $\mathcal{Z}^k$ consist of

$$V_{(a_1,\ldots,a_k)} = [a_1 - \tfrac{1}{2}, a_1 + \tfrac{1}{2}] \times \cdots [a_k - \tfrac{1}{2}, a_k + \tfrac{1}{2}] \quad \text{where } (a_1, \ldots, a_k) \in \mathbb{Z}^k.$$

Given a compact set $A \subset \mathbb{R}^k$ we can cover it by a finite number of tiles from $\mathcal{Z}^k$. Using such covers one can easily show that $v(A) \le |A_{\theta_k} \cap \mathbb{Z}^k|$ where $v(\cdot)$ denotes the (k-dimensional) volume in $\mathbb{R}^k$ and $A_{\theta_k} = \{x \in \mathbb{R}^k \mid d(x, A) \le \theta_k\}$.

2.3. *Intersection property.* We prove Theorem 1.7.3.

Proof. From the definition of the intersection property it is clear that if $(X, \mathbb{Z}^k)$ is an extension of a system having the intersection property then $(X, \mathbb{Z}^k)$ has the intersection property too. In order to prove the theorem it is therefore enough to prove that any free zero-dimensional system has the intersection property.

Let $n \in \mathbb{N}$ be large enough as specified later. Let $n' = \lfloor n/10 \rfloor$. We will construct a continuous $\mathbb{Z}^k$-equivariant mapping $\phi = \phi(n') : X \to \Phi$. Given $x \in X$ and $\mathbf{P} = \{P_i\}_{i=1}^M$ a partition of X let $P(x)$ denote the index of the element of $\mathbf{P}$ to which x belongs, i.e. $P(x) = j$ if and only if $x \in P_j$. Using the fact that the action is free, one can choose a clopen finite partition of X, $\mathbf{P} = \{P_i\}_{i=1}^M$, so that $P(x) = i$ implies that $P(T^g x) \neq i$ for all $g \in F_{n'}^* = F_{n'} \backslash \{\vec{0}\}$. Denote by $\alpha : X \to \{1, \ldots, M\}^{\mathbb{Z}^k}$ the factor mapping $\alpha(x) = (P(T^g x))_{g \in \mathbb{Z}^k}$ and let $S = \overline{\alpha(X)}$. To each $s \in S$ one can associate an n'-regular set of elements in $\mathbb{Z}^k$ that serve as Voronoi centers for an n'-regular Voronoi tiling. By [**Lig03**, Lemma 4.4] this can be done in a continuous $\mathbb{Z}^k$-invariant manner. In other words there exists a continuous $\mathbb{Z}^k$-equivariant mapping $\psi : S \to \mathcal{M}_{n'}$. Define $\phi = \psi \circ \alpha$. Let $x, y \in X$. Define $V_x = V_x(a) \in \phi(x)$ (a is the Voronoi center of V_x) to be one of the polytopes in $\phi(x)$ having the property $\vec{0} \in V_x$. By (2.1) we have $d(\vec{0}, a) \le n' + \theta_k$. This and a second application of (2.1) implies that $V_x \subset \overline{B}_{2(n'+\theta_k)}(\vec{0})$. Thus if n is large enough we have $V_x \in \phi(x)|_{B_{n/2}(\vec{0})}$. Denote $\mathscr{A} = \{W \in \phi(y) \mid W \cap V_x \neq \emptyset\}$ and $l = |\mathscr{A}|$. As for all $W \in \mathscr{A}$, $W \subset \overline{B}_{2(n'+\theta_k)}(V) \subset \overline{B}_{4(n'+\theta_k)}(\vec{0})$ one has that $l \cdot v(\overline{B}_{n'/2}(\vec{0})) \le \overline{B}_{4(n'+\theta_k)}(\vec{0})$, i.e. there exists a constant $C_1 = C_1(k) > 0$ (C_1 does not depend on n') such that $l \le C_1$. Choose $V_y \in \phi(y)$ any of the polytopes having the property $v(V_y \cap V_x) = \max_{Q \in \mathscr{A}} v(Q \cap V_x)$. Notice that $V_y \in \phi(y)|_{B_{5n'}(\vec{0})} \subset \phi(y)|_{B_{(1/2)n}(\vec{0})}$ and that

$$v(V_x \cap V_y) \ge \frac{1}{l} v(\overline{B}_{n'/2}(\vec{0})) = \frac{1}{l} C_2(k) n'^k.$$

By the discussion in the previous subsection we know that

$$|(V_x)_{-\theta_k} \cap (V_y)_{-\theta_k} \cap \mathbb{Z}^k| \ge v((V_x)_{-2\theta_k} \cap (V_y)_{-2\theta_k}).$$

Clearly

$$v((V_x)_{-2\theta_k} \cap (V_y)_{-2\theta_k}) \ge v(V_x \cap V_y) - v(B_{2\theta_k}(\partial V_x)) - v(B_{2\theta_k}(\partial V_y)).$$

It is easy to show that there exist $C_3(k) > 0$ and $m_0 \in \mathbb{N}$ so that if $m \ge m_0$ and $V \subset \mathbb{R}^k$ is a polytope with $B_m(\vec{0}) \subset V$ then $v(B_{\theta_k}(\partial V)) \le C\theta_k v(\partial V)$, where we let $v(\cdot)$ denote both $(k-1)$-dimensional and k-dimensional volume according to context. In particular for n large enough, $v(B_{2\theta_k}(\partial V_x)) \le \theta_k C_3(k) v(\partial V_x)$ and similarly for V_y. Conclude that

$$\begin{aligned} & v(V_x \cap V_y) - v(B_{2\theta_k}(\partial V_x)) - v(B_{2\theta_k}(\partial V_y)) \\ & \ge \frac{1}{l} C_2(k) n'^k - \theta_k C_3(k) \max\{v(\partial V_x), v(\partial V_y)\}. \end{aligned}$$

By the isodiametric inequality (this is a special case of [**Grü67**, Theorem 2, p. 417]) there exists a constant $C_4 = C_4(k)$ such that

$$v(\partial V_x) \le C_4(k)\, \mathrm{diam}(V_x)^{k-1} \le C_4(k)(4(n' + \theta_k))^{k-1}$$

(and similarly for V_y). Putting all these inequalities together we get

$$|(V_x)_{-\theta_k} \cap (V_y)_{-\theta_k} \cap \mathbb{Z}^k| \ge \frac{1}{l} C_2(k) n'^k - \theta_k(k) C_3(k) C_4(k)(4(n' + \theta_k))^{k-1}.$$

For n' large enough one can find $0 < \mu = \mu(k) < 1$ so that $|(V_x)_{-\theta_k} \cap (V_y)_{-\theta_k} \cap \mathbb{Z}^k| \ge \mu(2n+1)^k$ ($\mu < 1$ because $V_x,\ V_y \subset B_{(1/2)n}(\vec{0})$). □

2.4. *IP implies embedding.* We start by some definitions that will be used in what follows.

Definition 2.4.1. For $k = 1, 2, 3 \ldots$ define $o^k : \mathbb{R}^k \to \mathcal{P}(\mathbb{Z}^k)$, $s = (s_1, \ldots, s_k) \mapsto o_s^k$ inductively by

$$o_s^1 = (s - \lfloor s \rfloor)\delta_{\lceil s \rceil} + (1 - (s - \lfloor s \rfloor))\delta_{\lfloor s \rfloor}$$

and for $k \geq 2$ by

$$o_s^k = o_{s_1}^1 \times o_{s_2}^1 \times \cdots \times o_{s_k}^1.$$

When clear from the context the superscript k may be omitted.

Definition 2.4.2. For $s \in \mathbb{R}^k$ let

$$Q(s) = \{v \in \mathbb{Z}^k \mid \|s - v\|_\infty < 1\}.$$

$Q(s)$ consists of at most 2^k lattice points and is contained in a unit cube.

LEMMA 2.4.3. *o^k is continuous and $\mathbb{Z}^k$-equivariant and if $s \in \mathbb{R}^k$ then* $\operatorname{supp} o_s^k \subseteq Q(s)$.

Proof. For general k this follows from the case $k = 1$, which is verified directly. □

For a topological space Y denote by $\mathcal{P}(Y)$ the space of Borel probability measures on Y. Let $\nu \in \Phi$ be a tiling. For a tile $V \in \nu$ define its *barycenter* by $c(V) = \int_V \vec{x}\, dv$ where dv is the standard volume element in $\mathbb{R}^k$. Notice that as V is convex with non-empty interior one has $c(V) \in \overset{\circ}{V}$.

THEOREM 2.4.4. *There exists a continuous map*

$$\tau : \Phi \to \mathcal{P}(\mathbb{Z}^k),$$
$$\phi \mapsto \tau_\phi$$

such that if $V \in \phi$ and $B_{\theta_k}(\vec{0}) \subseteq V$ then

$$\operatorname{supp} \tau_\phi \subseteq Q(c(V))$$

and if additionally $s \in \mathbb{Z}^d$ and $B_{\theta_k}(\vec{0}) \subseteq V + s$, then $\tau_{T^s \phi} = T^s \tau_\phi$, where $T^s \tau_\phi(A) \triangleq \tau_\phi(A - s)$ for all Borel sets $A \subseteq \mathbb{R}^k$.

Proof. We start by constructing a continuous function $\rho : \Phi \to \mathcal{P}(\mathbb{R}^k)$ assigning to each tiling $\nu \in \Phi$ a measure $\rho_\nu(\cdot)$ supported on a finite set of $c(\nu) \triangleq \{c(V) \mid V \in \nu\}$ with the additional property that $B_{\theta_k}(\vec{0}) \subset V_s \in \nu$ implies that $\rho_\nu(s) = 1$ where $s = c(V_s)$ (here we use the convention $\rho_\nu(s) \triangleq \rho_\nu(\{s\})$).

Indeed fix $\nu \in \Phi$. To each $s \in c(\nu)$ associate $e(s) = v(V_s \cap B_{\theta_k}(\vec{0}))$. As ν is locally finite e is supported on a finite set of $c(\nu)$. As $\vec{0} \in V$ for some $V \in \nu$ we have $e(c(V)) > 0$. Define

$$\rho_\nu(s) = \frac{e(s)}{\sum_{s' \in c(\nu)} e(s')}.$$

It is easy to see ρ_ν has the desired properties mentioned above.

We now show that ρ is continuous at ν. Fix an open set $\mathcal{O}$ of ρ_ν, where

$$\mathcal{O} = \mathcal{O}(g, \epsilon) = \left\{ m \in \mathcal{P}(\mathbb{R}^k) \,\middle|\, \left| \int g\, dm - \int g\, d\rho_\nu \right| < \epsilon \right\}$$

for some $g \in C_c(\mathbb{R}^k)$ and $\epsilon > 0$. Let $\mathscr{C}$ be a finite collection of polytopes in ν so that $B_{\theta_k}(\vec{0}) \subset \bigcup \mathscr{C}$. This implies that ρ_ν is supported on the barycenters of the polytopes of $\mathscr{C}$. One chooses $\delta > 0$ so that for any

$$\tau \in \mathcal{U}(\nu, \delta) = \{\tau \in \Phi \mid \forall V \in \nu|_{B_{1/\delta}(\vec{0})} \ \exists V' \in \tau \ \text{ s.t. } d_H(V, V') < \delta\}$$

there exists a mapping $i_\tau : \mathscr{C} \to \tau$, so that both

$$|c(V) - c(i_\tau(V))| = \left| \int_V dv - \int_{i_\tau(V)} dv \right|$$

and

$$|e(V) - e(i_\tau(V))| = \left| \int_{V \cap B_{\theta_k}(\vec{0})} dv - \int_{i_\tau(V) \cap B_{\theta_k}(\vec{0})} dv \right|$$

are small enough for all $V \in \mathscr{C}$ so that $|\int g \, d\rho_\tau - \int g \, d\rho_\nu| < \epsilon$ holds. Indeed $\delta > 0$ is chosen in such a way that $d_H(V, i_\tau(V))$ is small enough (uniformly for all $\tau \in \mathcal{U}(\nu, \delta)$ and $V \in \mathscr{C}$) for this to hold.

Lastly, we define τ_ϕ as the distribution obtained by choosing $s \in \mathbb{R}^k$ according to ρ_ϕ and then $u \in \mathbb{Z}^k$ according to o_s. Formally,

$$\tau_\phi(A) = \int o_s(A) \, d\rho_\phi(s)$$

for all Borel sets $A \subset \mathbb{R}^k$. It is straightforward to deduce the properties of τ from those of ρ and o. □

We define several notions that will appear in the proof of the main theorem of this section. Suppose that α is an open finite cover of our compact metric space (X, d). Denote by $\text{mesh}(\alpha) = \max_{U \in \alpha} \text{diam}(U)$. Suppose further that a continuous mapping $h : X \to Y$ is given where Y is a topological space. h is said to be α *compatible* if for any $y \in Y$, $h^{-1}(y)$ is a subset of some $U \in \alpha$ and one writes $h \succ \alpha$. We are ready to present the proof of Theorem 1.7.2.

Proof. Let $(X, \mathbb{Z}^k)$ be a dynamical system admitting the intersection property with intersection ratio $0 < \mu < 1$. Write $M = \text{mdim}(X, \mathbb{Z}^k)$. Assume that $M < \infty$. We will embed $(X, \mathbb{Z}^k)$ in $(([0, 1]^d)^{\mathbb{Z}^k}, \text{shift})$ where $d = 1$ when $M = 0$, and $d = \lceil CM \rceil = \lceil ((2^{k+1} + 1)/\mu) M \rceil$ otherwise (for $M > 0$). This proof basically follows the strategy of [**Lin99**, proof of Theorem 5.1]. First we need to set up an abstract framework for applying the Baire category theorem. Suppose that $f \in C(X, [0, 1]^d)$. Recall the definition of I_f in §1.5. For I_f to be an embedding it is necessary that the inverse of every point in I_f is a point. Thus I_f must be compatible with every open cover α. Conversely if I_f is compatible with any sequence of open covers with mesh tending to zero then I_f will be an embedding. Write

$$\mathcal{F}_\alpha = \{f \in C(X, [0, 1]^d) \mid I_f \succ \alpha\}.$$ □

We adapt [**Lin99**, Lemma 5.3].

LEMMA 2.4.5. *Let $(X, \mathbb{Z}^k)$ be a dynamical system, $\alpha(i)$ be some sequence of open covers of X with* $\text{mesh}(\alpha(i)) \to 0$. *Then the set of all $f \in C(X, [0, 1])$ such that I_f is an embedding is equal to $\bigcap_{i=1}^\infty \mathcal{F}_{\alpha(i)}$, and every $\mathcal{F}_{\alpha(i)}$ is open in $C(X, [0, 1]^d)$ (in the uniform convergence topology).*

Continuation of proof of Theorem 1.7.2. Notice that if we can show that $\mathcal{F}_{\alpha(i)}$ is dense for all i, then by the Baire category theorem $\bigcap_{i=1}^{\infty} \mathcal{F}_i$ is dense in $C(X, ([0, 1]^d)^{\mathbb{Z}^k})$. This implies that one can $\mathbb{Z}^k$-equivariantly embed X in $([0, 1]^d)^{\mathbb{Z}^k}$ (in fact E^d is dense). It turns out we can prove $\mathcal{F}_\alpha$ is dense for any finite open cover α of X, so from now on we will drop the index i.

Suppose that $\tilde{f} : X \to [0, 1]^d$ is a continuous function and fix some $\epsilon > 0$. We will show that there exists a continuous function $f : X \to [0, 1]^d$ such that $\|f - \tilde{f}\|_\infty = \sup_{x \in X} |f(x) - \tilde{f}(x)| < \epsilon$ and I_f is α-compatible (as $\epsilon > 0$ is arbitrary this will imply that $\mathcal{F}_\alpha$ is dense). We start by a general construction and then relate it to $\tilde{f}$. Let $\beta \succ \alpha$ with $\max_{U \in \beta} \operatorname{diam}(\tilde{f}(U)) < \epsilon$. Clearly one can find $r_0 \in \mathbb{N}$ such that for all $n \geq r_0$ it holds that

$$\frac{1}{|F_r|} D(\beta^{F_n}) < M + 1 \quad \text{if } M > 0$$

and

$$\frac{1}{|F_n|} D(\beta^{F_n}) < \frac{\mu}{2^{k+2}} \quad \text{if } M = 0.$$

Using the intersection property for $(X, \mathbb{Z}^k)$ let $\phi : X \to \Phi$ be an (n, μ)-intersection function for some $n \geq r_0$, let $\gamma \succ \beta^{F_n}$ be an open cover so that $D(\beta^{F_n}) = \operatorname{ord}(\gamma)$. Let $(f_U)_{U \in \gamma}$ be a fixed vector, where $f_U \in [0, 1]^d$ for all $U \in \gamma$. Let $(\psi_U)_{U \in \gamma}$ be a partition of unity subordinate to γ. Define $H : X \to ([0, 1]^d)^{F_n}$ by

$$H(x) = \sum_{U \in \gamma} f_U \psi_U(x).$$

For brevity write $\tau_x = \tau_{\phi(x)}$. Let $p_n : \mathbb{Z}^k \to F_n = [-n, n]^k$ be given by $p_n(z_1, \ldots, z_k) = (z_1 \bmod(-n, n), \ldots, z_k \bmod (-n, n))$. Define (we use p_n because τ_x is not necessarily supported in F_n)

$$f(x) = \int (H(T^{-p_n(v)}x)_{p_n(v)})\, d\tau_x(v).$$

PROPOSITION 2.4.6. *f is continuous.*

Proof. Define

$$\nu : X \to \mathcal{P}(X \times \mathbb{Z}^k),$$
$$x \mapsto \nu_x = \delta_x \times \tau_x.$$

$x \mapsto \nu_x$ is continuous because $x \mapsto \delta_x$, $x \mapsto \tau_x$ are. Let $\widetilde{H}(y, u) = H(T^{-p_n(u)}y)_{p_n(u)}$. Then $\widetilde{H} : X \times \mathbb{Z}^k \to [0, 1]^d$ is continuous, so the map $R : \mathcal{P}(X \times \mathbb{Z}^k) \to \mathbb{R}$ defined by $R(\mu) = \int \widetilde{H}\, d\mu$ is continuous. Since $f(x) = R(\nu_x)$ is the composition of two continuous functions, f is continuous. □

For each $U \in \gamma$ choose $q_U \in U$ and define $v_U = (\tilde{f}(T^g q_U))_{g \in F_n}$. What remains to be shown is the following proposition.

PROPOSITION 2.4.7. *Given $\widetilde{f} : X \to [0, 1]^d$ and $\varepsilon > 0$, if we choose, for every $U \in \gamma$, a value $f_U \in ([0, 1]^d)^{F_n}$ such that $\|f_U - v_U\|_\infty < \epsilon$, then the following hold.*

(1) $\|f - \widetilde{f}\|_\infty < 2\varepsilon$.

(2) *For Lebesgue-almost every such choice, the corresponding f is α-compatible.*

We start by proving (2). We will need a definition and a proposition.

Definition 2.4.8. For $x \in X$ and $u \in \mathbb{Z}^k$ let $H^u(x) : F_n + u \to [0, 1]^d$ denote the function $(H^u(x))_v = H(T^{-u}x)_{u-v}$.

PROPOSITION 2.4.9. *For almost every choice of f_U the following holds. Let $x, y \in X$ and suppose that $V^x \in \phi(x)$, $V^y \in \phi(y)$ have the following property.*
(1) $V^x, V^y \subseteq B_{n/2}(\vec{0})$.
(2) *The set $S = V^x_{-\theta_k} \cap V^y_{-\theta_k}$ satisfies $|S| \geq \mu \cdot |F_n|$.*
Suppose that (attention: the integration is over vector-valued functions with values in $[0, 1]^d$)

$$\int H^u(x)|_S \, do_x(u) = \int H^u(y)|_S \, do_y(u) \tag{2.2}$$

where $o_x = o_{c(V_x)}$ and $o_y = o_{c(V_y)}$, then there exist $u, v \in F_n$ and $W \in \gamma$ such that $T^u x, T^v y \in W$.

In order not to interfere with the flow of the proof, we give the proof of the proposition below. Assume that $f(T^g x) = f(T^g y)$ for all $g \in \mathbb{Z}^k$ for some $x, y \in X$. Notice that by the intersection property we have $V^x \in \phi(x)$, $V^y \in \phi(y)$ which have properties (1) and (2) of the proposition. Let $s \in S$. Notice $\tau_{T^{-s}x}$ is supported in $Q(c(V^x) - s)$ and therefore

$$f(T^{-s}x) = \int (H(T^{-u}x)_u) \, d\tau_{T^{-s}x}(u) = \int (H(T^{-u}x)_{u-s}) \, do_x(u).$$

This can be written concisely as

$$f(T^{-s}x) = \int (H^u(x)_s) \, do_x(u).$$

Repeating the calculation for y and all $s \in S$ gives (2.2). We therefore conclude that there exist $u, v \in F_n$ and $W \in \gamma$ such that $T^u x, T^v y \in W$. As $\gamma \succ \beta^{F_n}$ we conclude that there exist some $V \in \beta$ such that $x, y \in V$. As $\beta \succ \alpha$ there exists $U \in \alpha$ such that $x, y \in U$. This proves (2).

In order to prove (1) notice that from the definition of f it is clear that

$$f(x) \in co\{H(T^{-p_n(s)}x)|_{p_n(s)} \mid s \in Q(c(\phi(x)))\}.$$

From the definition of H one concludes that, for all $\tilde{x} \in X$, $H(\tilde{x}) \in co\{H(q_U) \mid \tilde{x} \in U \in \gamma\}$. Combined with the previous equation one has

$$f(x) \in co\{H(q_U)|_l \mid l \in F_n, T^{-l}x \in U \in \gamma\}. \tag{2.3}$$

We claim that each element in the right side of equation (2.3) is within 2ϵ of $\tilde{f}$. Indeed,

$$\|H(q_U)|_l - \tilde{f}(x)\|_\infty \leq \|H(q_U)|_l - v_{U|l}\|_\infty + \|v_{U|l} - \tilde{f}(x)\|_\infty.$$

By the choice of f_U the first summand on the right-hand side is smaller than ϵ. As $v_U = (\tilde{f}(T^{l'}q_U))_{l' \in F_n}$ the second summand equals $\|\tilde{f}(T^l q_U) - \tilde{f}(x)\|_\infty$. Notice $x, T^l q_U \in T^l U$. We conclude that $\|\tilde{f}(T^l q_U) - \tilde{f}(x)\|_\infty \leq \operatorname{diam} \tilde{f}(T^l U)$. As $U \in \gamma \succ \beta^{F_n}$ we can find $V_l \in \beta$ so that $U \subset T^{-l}V_l$, i.e $T^l U \subset V_l$. We conclude that $\operatorname{diam} \tilde{f}(T^l U) \leq \operatorname{diam} \tilde{f}(V_l) \leq \epsilon$ which gives us the desired result. □

Proof of Proposition 2.4.9. For $x \in X$ define $\gamma_x = \{U \in \gamma \mid \psi_U(x) > 0\}$. Write

$$o_x = \sum_{j=1}^{2^k} \lambda_j \delta_{u_j} \quad \text{and} \quad o_y = \sum_{j=1}^{2^k} \eta_j \delta_{v_j},$$

where $u_j = u_x + \sigma_j$, $u_x \in F_n$ and $v_j = v_y + \sigma_j$, $v_y \in F_n$ with $u_j, v_j \in F_n$ and $\sigma_j \in \{0, 1\}^k$ for $j = 1, \ldots, 2^k$. We write (2.2) explicitly as:

$$\sum_{j=1}^{2^k} \sum_{U \in \gamma_{T^{-u_j}x}} \lambda_j \psi_U(T^{-u_j}x) f_U|_{u_j-g} - \sum_{j=1}^{2^k} \sum_{U \in \gamma_{T^{-v_j}y}} \lambda_j \psi_U(T^{-v_j}y) f_U|_{v_j-g} = 0, \quad g \in S. \tag{2.4}$$

Consider this equality as a symbolic equality in the (vector) symbols f_U, i.e. let $\gamma_{T^{-u_j}x} = \{U_1^j, \ldots, U_{m_j}^j\}$ and $\gamma_{T^{-v_j}y} = \{V_1^j, \ldots, V_{t_j}^j\}$ with $m_j, t_j \leq \mathrm{ord}(\gamma) + 1$ and represent $f_{U_i^j}$ as a vector composed of $|F_n|d$ symbols: $f_{U_i^j} = (f_{U_i^j}^{r,g})_{r \in \{1,\ldots,d\},\, g \in F_n}$. It is assumed that if two identical (symbolic) vectors appear in (2.4) then one regroups their coefficients so that the vector appears only once. This procedure is referred to as *cancelation*. We denote by $\mathscr{M}$ the matrix consisting of all different (symbolic) vectors appearing in (2.4) after cancelation, see an explicit example in the sequel. A specific substitution of values into the symbolic vectors will be called a *realization*. Our goal is to show that for almost all realizations $u_x = v_y$ and $\gamma_{T^{-u_i}x} = \gamma_{T^{-v_i}y}$ for $i = 1, \ldots, 2^k$. This implies that there is a $U \in \gamma$ such that both $T^{-u_i}x, T^{-v_i}y \in U$. We start by assuming that $u_x - v_y \neq \sigma_i - \sigma_j$ for all $i, j \in \{1, \ldots, 2^k\}$ and arrive at a contradiction. In this case $\mathscr{M}$ takes the form (through lack of space, two lines of vectors appear but actually we mean a one-dimensional array of vectors)

$$\mathscr{M} = \begin{matrix} [f_{U_1^1}|_{u_1-S}, \ldots, f_{U_{m_1}^1}|_{u_1-S}, f_{U_1^{2^k}}|_{u_{2^k}-S}, \ldots, f_{U_{m_{2^k}}^{2^k}}|_{u_{2^k}-S}, \ldots \\ f_{V_1^1}|_{v_1-S}, \ldots, f_{V_{m_1}^1}|_{v_1-S}, \ldots, f_{V_1^{2^k}}|_{v_{2^k}-S}, \ldots, f_{V_{m_{2^k}}^{2^k}}|_{v_{2^k}-S}]. \end{matrix}$$

The matrix has $d|S|$ rows. The number of columns of the matrix is bounded by $2(\mathrm{ord}(\gamma) + 1)2^k$. By assumption this implies that the number of rows is bigger or equal to the number of columns. Moreover each column contains distinct elements. The same is true for any particular row (here we use the fact that $u_x - v_y \neq \sigma_i - \sigma_j$ for all $i, j \in \{1, \ldots, 2^k\}$ to conclude that no cancelation took place). Recall that we are free to change the exact values of f_U as long as $\|f_U - v_U\|_\infty < \epsilon$ holds. We invoke [**Lin99**, Lemma 5.5] to conclude that for almost all realizations for the (column) vectors, these vectors are linearly independent. This constitutes the contradiction. Now assume that $u_x - v_y = \sigma_i - \sigma_j$ for some $i \neq j$. We claim that $\gamma_{T^{-u_j}x} = \gamma_{T^{-v_i}y}$. First notice that if $U \in \gamma_{T^{-u_j}x} \cap \gamma_{T^{-v_i}y}$, i.e. $U = U_j^k = V_i^{k'}$ for some k, k', then $v_g \triangleq f_{U_j^k}|_{u_j-g} = f_{V_i^{k'}}|_{v_i-g}$ and one can regroup the coefficients appearing before these vectors so that (v_g) appears only once. Assume that $\gamma_{T^{-u_j}x} \neq \gamma_{T^{-v_i}y}$. This means that in $\mathscr{M}$ we have representatives from the non-empty set $\gamma_{T^{-u_j}x} \triangle \gamma_{T^{-v_i}y}$. Notice that the number of rows in $\mathscr{M}$ is bigger or equal to the number of columns as cancelation decreases the number of columns. Each column contains distinct elements and this is also true for any particular row (here we use the fact that we performed cancelation). As before we arrive at a contradiction. In order to avoid the contradiction there must exist a bijective correspondence $i \to b(i)$ so that $\gamma_{T^{-u_i}x} = \gamma_{T^{-v_{b(i)}}y}$. Moreover the cancelation in (2.4) must result in the equation $0 = 0$, which corresponds to an empty $\mathscr{M}$. However if $u_x \neq v_y$ this is impossible. Indeed there must be some $i \in \{1, \ldots, 2^k\}$ so that for all j, $u_x - v_y \neq \sigma_i - \sigma_j$ (assume without loss of generality that $(u_x - v_y)_1 > 0$ and let i be such that $(\sigma_i)_1 = 0$). We conclude that $u_x = v_y$ and $b(i) = i$ for all i, i.e $\gamma_{T^{-u_i}x} = \gamma_{T^{-v_i}y}$ for $i = 1, \ldots, 2^k$. □

3. *Systems with TRP*

3.1. *SBP implies TRP.* In this section we prove Theorem 1.10.1.

LEMMA 3.1.1. *Let $\tilde{f} : X \to K$ be a Borel function into a finite subset $K \subset \mathbb{R}^k$. Denote by $D_{\tilde{f}}$ the set of discontinuity points of $\tilde{f}$ and assume that* $\mathrm{ocap}(D_{\tilde{f}}) = 0$. *Then, given $\epsilon > 0$, there exists a continuous function $f : X \to co(K) \subset \mathbb{R}^k$ so that* $\mathrm{ocap}(S_{f,\tilde{f}}) \leq \epsilon$ *where $S_{f,\tilde{f}} = \{x \in X \mid f(x) \neq \tilde{f}(x)\}$ and $co(K)$ denotes the convex hull of K.*

Proof. For any $\delta > 0$ define the compact set $A_\delta = \{x \in X \mid d(x, D_{\tilde{f}}) \geq \delta\}$. Notice $A_\delta^c = \{x \in X \mid d(x, D_{\tilde{f}}) < \delta\}$. By assumption $\mathrm{ocap}(D_{\tilde{f}}) = 0$. Moreover as K is finite $D_{\tilde{f}} = \bigcup_{k \in K} \partial \tilde{f}^{-1}(k)$ is closed. According to [**Lin99**, Lemma 6.3] a small enough fattening of a closed set of zero orbit-capacity has arbitrary small orbit-capacity, i.e. one can choose $\delta_0 > 0$ so that $\mathrm{ocap}(A_{2\delta_0}^c) \leq \epsilon$. On A_{δ_0}, $\tilde{f}$ is continuous. Therefore one can cover A_{δ_0} with the disjoint relatively open sets $\mathcal{C} = \{\tilde{f}|_{A_{\delta_0}}^{-1}(k) \mid k \in K\}$. Let $0 < \rho < \delta_0$ be a Lebesgue number for $\mathcal{C}$. Let $k : X \times X \to \mathbb{R}$ be the 'kernel function' given by $k(x, y) = \rho - d(x, y)$ for $x, y \in X$ with $d(x, y) \leq \rho$ and $k(x, y) = 0$ otherwise. Let μ be a measure of full support on X. Define

$$f(x) = \frac{\int k(x, y)\tilde{f}(y)\, d\mu(y)}{\int k(x, y)\, d\mu(y)}.$$

It is easy to see that f is well defined and continuous. Suppose that $x \in A_{2\delta_0}$. As $\rho < \delta_0$, $B_\rho(x) \subset A_{\delta_0}$. As ρ is a Lebesgue number for $\mathcal{C}$ we conclude that $\tilde{f}|_{B_\rho(x)}$ is constant. Therefore $\tilde{f}(x) = f(x)$, i.e. $S_{f,\tilde{f}} = \{x \in X \mid f(x) \neq \tilde{f}(x)\} \subset A_{2\delta_0}^c$. This implies that $\mathrm{ocap}(S_{f,\tilde{f}}) \leq \epsilon$. □

Proof of Theorem 1.10.1. From the definition of the topological Rokhlin property it is clear that if $(X, \mathbb{Z}^k)$ is an extension of a system having the topological Rokhlin property then $(X, \mathbb{Z}^k)$ has the topological Rokhlin property too. Therefore we can assume without loss of generality that $(X, \mathbb{Z}^k)$ is free and has the SBP. We will use a construction which is similar but not identical to the beginning of the proof of Theorem 1.7.3. Fix $n \in \mathbb{N}$. Let $\mathbf{P} = \{P_i\}_{i=1}^M$ be a partition so that $\mathrm{ocap}(\partial P_i) = 0$ for all i and so that $P(x) = i$ implies that $P(T^g x) \neq i$ for all $g \in F_n^* = F_n \backslash \{\vec{0}\}$ ($P(x)$ denotes the index of the element of $\mathbf{P}$ to which x belongs, i.e. $P(x) = j$ if and only if $x \in P_j$). Let $\alpha : X \to \{1, \ldots, M\}^{\mathbb{Z}^k}$ denote the ($\mathbb{Z}^k$-equivariant) factor mapping given by $\alpha(x) = (P(T^g x))_{g \in \mathbb{Z}^k}$ and let $S = \overline{\alpha(X)}$. Unlike the situation in Theorem 1.7.3, α is not necessarily continuous.

Lemma 4.4 of [**Lig03**] enables us to associate to each element in S (in an $\mathbb{Z}^k$-equivariant and continuous manner) an n-regular set of elements in $\mathbb{Z}^k$ that serves as Voronoi centers for an n-regular Voronoi tiling. Denote this mapping by $\psi : S \to 2^{\mathbb{Z}^k}$, $s \in S \mapsto \psi(s) \subset \mathbb{Z}^k$. We have that for any $v \in S$, $r \in \mathbb{N}$ and any sufficiently small neighborhood $\mathcal{U}$ of v, if $q \in \mathcal{U}$ then $\psi(v) \cap F_r = \psi(q) \cap F_r$ ($F_r = \{-r, \ldots, r\}^k$). In other words the associated Voronoi centers sets are identical on a (as large as we wish) finite portion of the lattice $\mathbb{Z}^k$. Define $\phi : S \to \mathcal{M}_n$ by $s \in S \mapsto \mathcal{V}(\psi(s)) \in \mathcal{M}_n$. This continuous and $\mathbb{Z}^k$-equivariant mapping associates to each set $\psi(s)$ its induced Voronoi tiling. Notice that for any $x \in X$, $r \in \mathbb{N}$ and any sufficiently small neighborhood $\mathcal{U}$ of x, if $y \in \mathcal{U}$ then $\phi(x) \cap B_r(\vec{0}) = \phi(y) \cap B_r(\vec{0})$.

We are ready to introduce our main object of study, namely the function $\tilde{f} : X \to F_{2n} \subset \mathbb{R}^k$ given by

$$x \mapsto \begin{cases} a \in \mathbb{Z}^k & \text{there exists } V(a) \in \phi(\alpha(x)) \text{ such that } \vec{0} \in \overset{\circ}{V}(a), \\ \vec{0} & \text{otherwise.} \end{cases}$$

This function associates to $x \in X$ the Voronoi center a of the Voronoi tile that contains the origin in its interior. In case the origin is contained in several tiles (i.e. it is on the boundary of several tiles) then $\tilde{f}(x) = \vec{0}$. Since, for all $x \in X$, $\phi(\alpha(x))$ is an n-regular Voronoi tiling, then for $V(a) \in \phi(\alpha(x))$ with $\vec{0} \in V(a)$ we do indeed have $a \in F_{2n}$. Notice that if the origin is far enough from the boundary of the tile that contains it, specifically if $\vec{0} \in V \in \phi(\alpha(x))$ and moreover $d(\vec{0}, \partial V) > 1$ then $\tilde{f}(T^{e_j}x) = \tilde{f}(x) + e_j$ for $j = 1, \ldots, k$. The last fact and similar techniques to the ones use in the proof of Theorem 1.7.3 make it possible to prove that for all $x \in X$, $\operatorname{ocap}(E_{\tilde{f}}) = o(1/n)$, where

$$E_{\tilde{f}} = \{x \in X \mid \exists 1 \le j \le k \ \tilde{f}(T^{e_j}x) \ne \tilde{f}(x) + e_j\}.$$

Up to now we have not proven $(X, \mathbb{Z}^k)$ has the topological Rokhlin property as $\tilde{f}$ is not necessarily continuous. However, we claim that the set of discontinuity points of $\tilde{f}$, which we denote by $D_{\tilde{f}}$, has orbit-capacity zero. Indeed by continuity arguments the value of $\tilde{f}(x)$ is determined by knowing a finite set of coordinates of $\alpha(x)$. In other words there exist $r \in \mathbb{N}$ and a (trivially) continuous function $h : \{1, \ldots, M\}^{F_r} \to F_{2n}$ so that $\tilde{f}(x) = h((P(T^g x))_{g \in F_r})$. From this representation it is clear that $D_{\tilde{f}}$ is contained in the discontinuity set of that mapping $x \to (P(T^g x))_{g \in F_r}$, which equals $\bigcup_{g \in F_r} \bigcup_{i=1}^{M} T^g \partial P_i$ (the discontinuity set of $x \to P(x)$ is $\bigcup_{i=1}^{M} \partial P_i$). By assumption $\bigcup_{i=1}^{M} \partial P_i$ has orbit capacity zero. As $\operatorname{ocap}(\cdot)$ is a sub-additive set function, $D_{\tilde{f}}$ has orbit-capacity zero. We can therefore invoke Lemma 3.1.1 and obtain $f : X \to R_r = co(F_r) \subset \mathbb{R}^k$ so that $\operatorname{ocap}(S_{f,\tilde{f}}) \le 1/n$ where $S_{f,\tilde{f}} = \{x \in X \mid f(x) \ne \tilde{f}(x)\}$. Let

$$E_f = \{x \in X \mid \exists 1 \le j \le k \ f(T^{e_j}x) \ne f(x) + e_j\}.$$

Notice that

$$E_f \subset E_{\tilde{f}} \cup S_{f,\tilde{f}} \cup \bigcup_{j=1}^{k} T^{-e_j} S_{f,\tilde{f}}.$$

We conclude that $\operatorname{ocap}(E_f) = O(1/n)$. This proves $(X, \mathbb{Z}^k)$ has the topological Rokhlin property. □

4. *Systems with SBP*

4.1. *Sufficient conditions for SBP.* In the previous section we introduced the small boundary property. In this section we give sufficient conditions for a system to have the small boundary property. The reader is advised to look up the definition of '$\mathrm{edim}(X, \mathbb{Z}^k) \leq l$ densely' in §1.5.

LEMMA 4.1.1. *If $(X, \mathbb{Z}^k)$ has the topological Rokhlin property then for every $\delta > 0$ there exists a continuous function $\tilde{f} : X \to \mathbb{R}^k$ so that $\mathrm{ocap}(\tilde{E}_{\tilde{f}}) < \delta$ where*

$$\tilde{E}_{\tilde{f}} = \{x \in X \mid \tilde{f}(x) \notin \mathbb{Z}^k \vee \exists 1 \leq j \leq k \ \tilde{f}(T^{e_j}x) \neq \tilde{f}(x) + e_j\}.$$

Proof. Without loss of generality we give the proof for the case $k = 1$. The general case is similar. Let $a \in (1 - (\delta/5), 1)$ be irrational. By unique ergodicity we find $M \in \mathbb{N}$ such that

$$\forall r \in \mathbb{R} \quad \frac{1}{M}\left|\left\{j \in \{0, 1 \ldots, M-1\} \,\middle|\, (r + ja) \bmod 1 \notin \left(\frac{\delta}{5}, 1 - \frac{\delta}{5}\right)\right\}\right| < \delta. \tag{4.1}$$

Choose $\epsilon > 0$ so that $M\epsilon < \delta$. Using the topological Rokhlin property we obtain $f : X \to \mathbb{R}^k$ so that the exceptional set $E_f = \{x \in X \mid f(Tx) \neq f(x) + 1\}$ is of orbit-capacity less than ϵ. This enables us to choose $N \in \mathbb{N}$ with $M|N$ such that,

$$\text{for all } x \in X \quad \frac{1}{N}\sum_{i=0}^{N-1} 1_{E_f}(T^i x) < \epsilon. \tag{4.2}$$

For $0 < \eta < 1$ define the 'η-floor-function' (recall $\{c\} = c - \lfloor c \rfloor$):

$$g_\eta(c) = \begin{cases} \lfloor c \rfloor & 0 \leq \{c\} \leq 1 - \eta, \\ \lfloor c \rfloor + \dfrac{\{c\} - (1-\eta)}{\eta} & \text{otherwise.} \end{cases}$$

We can now define $\tilde{f}(x) = g_{\delta/5}(af(x))$. Fix $x_0 \in X$. We call $\{j, j+1, \ldots, j+M-1\}$ a *good* (x_0, M)*-segment* if $\{T^j x_0, T^{j+1}x_0, \ldots, T^{j+M-1}x_0\} \cap E_f = \emptyset$, otherwise we call it a *bad* (x_0, M)*-segment*. We tile $\{0, 1, \ldots, N-1\}$ with the N/M disjoint segments $\{kM, kM+1, \ldots, (k+1)M-1\}$. By (4.2) at most $((\epsilon N)/(N/M)) = \epsilon M < \delta$ fraction of them are bad (x_0, M)-segments. By condition (4.1), in a good (x_0, M)-segment $S = \{j_0, j_0+1, \ldots, j_0+M-1\}$, for more than $1-\delta$ fraction of $j \in S$ one has (say $j = j_0 + l$ for some $l \in \mathbb{N}$)

$$\{af(T^j x_0)\} = \{af(T^{j_0}x_0) + al\} \in (\delta/5, 1 - (\delta/5)).$$

For such j this implies that

$$\tilde{f}(T^j x_0) = g_{\delta/5}(af(T^j x_0)) \in \mathbb{Z}$$

and

$$\tilde{f}(T^{j+1}x_0) = g_{\delta/5}(af(T^j x_0) + a) = \tilde{f}(T^j x_0) + 1$$

where we used $f(T^{j+1}x_0) = f(T^j x_0) + 1$. By discarding the elements of the bad (x_0, M)-segments we arrive at the conclusion,

$$\frac{1}{N}\,|\{j \in \{0, 1 \ldots, N-1\} \mid \tilde{f}(T^j x_0) \notin \mathbb{Z} \vee \tilde{f}(T^{j+1}x_0) \neq \tilde{f}(T^j x_0) + 1\}| < \delta + \delta.$$

From here it is easy to conclude that $\mathrm{ocap}(E_{\tilde{f}}) < 3\delta$. □

We prove Theorem 1.10.3.

Proof. By assumption $\operatorname{edim}(X, \mathbb{Z}^k) \le l$ densely which implies that there exists $s \le l$ such that for a dense G_δ subset A of all continuous $f : X \to ([0, 1]^s)^{\mathbb{Z}^k}$, I_f is an embedding. We follow the strategy of [**Lin99**, §6] by showing that there exists a dense G_δ subset B of functions f such that I_f has the small boundary property. As the intersection of A and B is not empty $(X, \mathbb{Z}^k)$ has the small boundary property. Notice that in [**Lin99**] the author (tacitly) assumes that $s = 1$. This assumption can be avoided as we will see in what follows. The idea of the proof is to show that for a dense G_δ subset of functions f, the intersection of $I_f(X)$ with the boundaries of the members of the basis consisting of the cylindrical sets,

$$\tilde{C}^{F_n}\left(\left\{\prod_{i=1}^{s}(b_g^i, q_g^i)\right\}_{g\in F_n}\right) = \left\{x \in ([0, 1]^s)^{\mathbb{Z}^k} \;\middle|\; \forall g \in F_n\ x_g \in \prod_{i=1}^{s}(b_g^i, q_g^i)\right\},$$

for all $n \in \mathbb{N}$ and rational $b_g^i < q_g^i$, $g \in F_n$, $i = 1, \ldots, s$ are small sets. Clearly it is sufficient to show that for any $t \in [0, 1] \cap \mathbb{Q}$ and $m \in \{1, \ldots, s\}$ there is a dense G_δ subset of functions f for which (we denote by $(y)_m$ the mth coordinate of $y \in [0, 1]^s$)

$$I_f(X) \cap \{x \in [0, 1]^{\mathbb{Z}^k} \mid (x_{\vec{0}})_m = t\}$$

is a small set.

Write

$$\mathcal{O}_{n,t}^m = \left\{f \;\middle|\; \operatorname{ocap}(\{(f(x))_m = t\}) < \frac{1}{n}\right\}.$$

Notice that

$$\{f | \{(f(x))_m = t\} \text{ is small}\} = \bigcap_{n=1}^{\infty} \mathcal{O}_{n,t}^m.$$

Using [**Lin99**, proof of Lemma 6.4] we see $\mathcal{O}_{n,t}^m$ are open for all n, $m \in \{1, \ldots, s\}$ and $t \in [0, 1]$. We complement this by showing that $\mathcal{O}_{n,t}^m$ are dense in $C(X, [0, 1])$ for all n, $m \in \{1, \ldots, s\}$ and $t \in [0, 1]$. We proceed in a similar fashion to Theorem 1.7.2. Fix $m \in \{1, \ldots, s\}$. Let $\tilde{f} \in C(X, [0, 1]^s)$, $t \in [0, 1]$ and $\epsilon > 0$. We will find $f \in C(X, [0, 1]^s)$ so that $\|f - \tilde{f}\|_\infty < \epsilon$ and $\operatorname{ocap}(\{x \in X \mid (f(x))_m = t\}) < \epsilon$. Let α be a finite open cover of X with $\operatorname{diam}(\tilde{f}(U)) < \epsilon/2$ for all $U \in \alpha$. Let n be large enough so that $D(\alpha^{F_n}) < (\epsilon|F_n|)/4$ and let $\beta \succ \alpha^{F_n}$ be a cover of X such that $\operatorname{ord}(\beta) = D(\alpha^{F_n})$. For each $U \in \beta$ choose $q_U \in U$ and define $v_U = (\tilde{f}(T^g q_U))_{g\in F_n}$. We quote [**Lin99**, Lemma 6.5, p. 28].

LEMMA 4.1.2. *Let β be a cover of X with $\operatorname{ord}(\beta) < \Delta$ for some $\Delta \in \mathbb{N}$. Suppose that $t \in [0, 1]$, and for every $U \in \beta$ we are given points $q_U \in U$ and $v_U \in [0, 1]^{F_n}$. Then it is possible to find a continuous function $H : X \to [0, 1]^{F_n}$ with the following properties.*

(1) $\|H(q_U) - v_U\|_\infty < \epsilon$.

(2) *For all $x \in X$, $H(x) \in co(\{H(x_U) \mid x \in U \in \beta\})$.*

(3) *For every $x \in X$, no more than Δ of the coordinates of $H(x)$ are equal to t.*

Continuation of proof of Theorem 1.10.3. Let $H : X \to [0, 1]^{F_n}$ be given by the lemma for $\Delta = (|F_n|\epsilon)/4$. By assumption $(X, \mathbb{Z}^k)$ has the topological Rokhlin property. Therefore by Lemma 4.1.1 one can find a continuous function $\psi : X \to \mathbb{R}^k$ such that

$$\tilde{E} = \{x \in X \mid \psi(x) \notin \mathbb{Z}^k \vee \exists 1 \le j \le k \ \ \psi(T^{e_j} x) \neq \psi(x) + e_j\}$$

satisfies $\text{ocap}(\tilde{E}) < \epsilon/(4|F_n|)$. Define

$$(f(x))_r = \begin{cases} \displaystyle\int H(T^{-p_n(v)}x)\, do_{\psi(x)}(v) & r = m, \\ (\tilde{f})_r & r \neq m. \end{cases}$$

- By Lemma 2.4.3, f is continuous.
- The proof that $\|f - \tilde{f}\|_\infty = |(f)_m - (\tilde{f})_m|_\infty < \epsilon$ is identical to the proof given in Theorem 1.7.2.
- The proof that $\text{ocap}\{x \in X \mid (f(x))_m = t\} < \epsilon$: apart from some obvious changes relating to the fact that we are dealing with higher dimensions, the proof is identical to the last part of the [**Lin99**, proof of Lemma 6.6]. □

5. *The $\mathbb{Z}^k$-symbolic extension entropy theorem*

5.1. *Principal extensions.* We start by introducing some definitions (if not stated otherwise all definitions, lemmas and theorems borrow heavily from [**Dow05**]). We denote by $\mathcal{P}_{\mathbb{Z}^k}(X)$ the space of $\mathbb{Z}^k$-invariant probability measures on X.

Definition 5.1.1. $\psi : (X, \mathbb{Z}^k) \to (Y, \mathbb{Z}^k)$ a topological factor map is a *principal extension* if the entropy $h(\mu) = h(\psi\mu)$ for all $\mu \in \mathcal{P}_{\mathbb{Z}^k}(X)$.

Definition 5.1.2. A partition $\mathcal{A}$ of X is *essential* if its elements have boundaries of measure zero for all invariant measures.

LEMMA 5.1.3. *Let $(X, \mathbb{Z}^k)$ be a dynamical system with* $\text{mdim}(X, \mathbb{Z}^k) = 0$*, then it admits a principal extension by a dynamical system with the small boundary property.*

Proof. Let $\mathcal{O} = \{0, 1\}^{\mathbb{N}}$ be the 2-adic odometer. Recall $\mathcal{O}$ is a topological abelian group, generated by the element $e = (1, 0, 0, \ldots)$; that is, $\mathcal{O} = \{e^k\}_{k \in \mathbb{Z}}$, where

$$e^k = \underbrace{e + e + \cdots + e}_{k \text{ times}}.$$

Define the natural $\mathbb{Z}$ action on $\mathcal{O}$ by $T^z x = x + e^k$, $z \in \mathbb{Z}$, $x \in \mathcal{O}$. Let $(\mathcal{O}_i, \mathbb{Z})$, $i = 1, 2, \ldots, k$, be k copies of $(\mathcal{O}, \mathbb{Z})$. Denote by the 'product 2-adic odometer' $(A, \mathbb{Z}^k) = \prod_{i=1}^k (\mathcal{O}_i, \mathbb{Z})$ the $\mathbb{Z}^k$-dynamical system (and topological group) constructed from the product of the $\mathcal{O}_i$. This system has topological entropy zero. We conclude that $\psi : X \times A \to X$ is a principal extension. Notice $\text{mdim}(X \times A, \mathbb{Z}^k) = 0$ and $(X \times A, \mathbb{Z}^k)$ is an extension of $(A, \mathbb{Z}^k)$ which is zero-dimensional and free. We conclude by the last part of Theorem 1.11.1 that $(X \times A, \mathbb{Z}^k)$ has the small boundary property. □

Using the construction of [**Dow05**, p. 73] we have the following lemma.

LEMMA 5.1.4. *If $(X, \mathbb{Z}^k)$ has the small boundary property then it has a zero-dimensional principal extension.*

The last two lemmas enable us to conclude a theorem interesting by its own right.

THEOREM 5.1.5. *Any dynamical system* $(X, \mathbb{Z}^k)$ *with* $\mathrm{mdim}(X, \mathbb{Z}^k) = 0$ *admits a zero-dimensional principal extension.*

5.2. *Entropy structure.* Let $\mu \in \mathcal{P}_{\mathbb{Z}^k}(X)$. Given a continuous function $f : X \to [0, 1]$ denote by

$$\mathcal{A}_f = \{\{(x, t) \in X \times [0, 1] \mid t \leq f(x)\}, \{(x, t) \in X \times [0, 1] \mid t > f(x)\}\}$$

the two-element partition of $X \times [0, 1]$. Let $\mathcal{F}, \mathcal{G}$ be finite families of functions, define

$$\mathcal{A}_{\mathcal{F}} = \bigvee_{f \in \mathcal{F}} \mathcal{A}_f, \quad \mathcal{A}_{\mathcal{G}} = \bigvee_{f \in \mathcal{G}} \mathcal{A}_f$$

and

$$H^{\mathrm{fun}}(\mu, \mathcal{F}) = H(\mu \times \gamma, \mathcal{A}_{\mathcal{F}}), \quad H^{\mathrm{fun}}(\mu, \mathcal{F}|\mathcal{G}) = H(\mu \times \gamma, \mathcal{A}_{\mathcal{F}}|\mathcal{A}_{\mathcal{G}}),$$

where H is the standard entropy function of a measure with respect to a partition and γ denotes the Lebesgue measure on the interval. Notice that both $H^{\mathrm{fun}}(\cdot, \mathcal{F}) : \mathcal{P}_{\mathbb{Z}^k}(X) \to \mathbb{R}$ and $H^{\mathrm{fun}}(\cdot, \mathcal{F}|\mathcal{G}) : \mathcal{P}_{\mathbb{Z}^k}(X) \to \mathbb{R}$ are continuous, where $\mathcal{G}$ is another finite family of functions. We recall the definition of h^{fun}, *entropy with respect to a family of functions*, as it appears in [**Dow05**, §6.2, p. 79] (where, $\mathcal{F}^{F_n} = \{f(T^g \cdot) : X \to [0, 1] \mid f \in \mathcal{F}, g \in F_n\}$):

$$h^{\mathrm{fun}}(\mu, \mathcal{F}) = \lim_{n\to\infty} \frac{1}{|F_n|} H^{\mathrm{fun}}(\mu, \mathcal{F}^{F_n}).$$

Two increasing sequences of functions on a compact domain $\mathcal{H} = (h_k)$ and $\mathcal{H}' = (h'_k)$ are called *uniformly equivalent* if for every index k and every $\gamma > 0$ there exists two indices m, m' so that $h'_{m'} > h_k - \gamma$ and (symmetrically) $h_m > h'_k - \gamma$. We can now present the definition of entropy structure such as it appears in [**Dow05**, p. 74].

Definition 5.2.1. An *entropy structure* of a finite topological entropy dynamical system $(X, \mathbb{Z}^k)$ is any increasing sequence $\mathcal{H} = (h_k)$ of functions defined on $\mathcal{P}_{\mathbb{Z}^k}(X)$ such that for any choice of a zero-dimensional principal extension $(X', \mathbb{Z}^k)$ and any choice of refining clopen partitions $\mathcal{C}'_k$ in X', $k \in \mathbb{N}$, the lift of $\mathcal{H}$ to $\mathcal{P}_{\mathbb{Z}^k}(X')$ is uniformly equivalent to the sequence of entropies with respect to partitions $\mathcal{H}^{\mathrm{ref}} = (h(\cdot, \mathcal{C}'_k))_{k\in\mathbb{N}}$. Notice that by Theorem 5.1.5 the condition in the definition is never void.

Definition 5.2.2. Let $\mathcal{A}, \mathcal{B}$ be partitions of X. Let $\mu \in \mathcal{P}_{\mathbb{Z}^k}(X)$. Define the $\mathbb{Z}^k$*-entropy of* $\mathcal{A}$ *with respect to* $\mathcal{B}$ as

$$h_\mu(\mathcal{A}|\mathcal{B}) = \lim_{n\to\infty} \frac{1}{|F_n|} H_\mu(\mathcal{A}^{F_n}|\mathcal{B}^{F_n}).$$

One can show that the limit always exists. Similarly one defines $h^{\mathrm{fun}}(\mu, \mathcal{F} \mid \mathcal{G})$ for two families of functions $\mathcal{F}, \mathcal{G}$. Let $P = \{z \in \mathbb{Z}^k \mid z < \vec{0}\}$, where $<$ is the lexicographic order. $\mathcal{A}^P = \bigvee_{z\in P} T^z\mathcal{A}$ is referred to as the *past σ-algebra* of $\mathcal{A}$. It is a well-known fact that

$$h_\mu(\mathcal{A}|\mathcal{B}) = H_\mu(\mathcal{A}|\mathcal{A}^P \vee \mathcal{B}^{\mathbb{Z}^k}).$$

In particular we conclude that $h^{\text{fun}}(\mu, \mathcal{F}|\mathcal{G})$ is an infimum of continuous functions and therefore upper semi-continuous. The following lemma is a generalization of [**Dow05**, Lemma 7.1.2, p. 89] (where it is assumed that $h_{\text{top}}(X) < \infty$) even in the case $k = 1$.

LEMMA 5.2.3. *Let $\mathcal{F}_k$ and $\mathcal{F}'_k$ be two refining increasing (by inclusion) sequences of finite families of continuous functions from X into* $[0, 1]$ *such that*

$$\lim_{k\to\infty} h^{\text{fun}}(\cdot, \mathcal{F}_k) = \lim_{k\to\infty} h^{\text{fun}}(\cdot, \mathcal{F}'_k).$$

Then the sequences $\mathcal{H} = (h^{\text{fun}}(\cdot, \mathcal{F}_k))$ and $\mathcal{H} = (h^{\text{fun}}(\cdot, \mathcal{F}'_k))$ are uniformly equivalent.

Proof. Fix k. Notice that, as $\mathcal{F}'_k$ is a refining sequence,

$$\lim_{k'\to\infty} h^{\text{fun}}(\cdot, \mathcal{F}_k|\mathcal{F}'_{k'}) = \lim_{k'\to\infty} H^{\text{fun}}(\cdot, \mathcal{F}_k|\mathcal{F}_k^P \vee \mathcal{F}'^{\mathbb{Z}^k}_{k'}) = 0.$$

We recognize a sequence of upper semi-continuous functions decreasing to a continuous limit. This implies that the limit is uniform. As

$$h^{\text{fun}}(\cdot, \mathcal{F}'_k) = h^{\text{fun}}(\cdot, \mathcal{F}'_k \cup \mathcal{F}_k) - h^{\text{fun}}(\cdot, \mathcal{F}_k|\mathcal{F}'_{k'})$$

we conclude that for any $\gamma > 0$ there exists k' such that

$$h^{\text{fun}}(\cdot, \mathcal{F}'_k) \geq h^{\text{fun}}(\cdot, \mathcal{F}'_k \cup \mathcal{F}_k) - \gamma \geq h^{\text{fun}}(\cdot, \mathcal{F}'_k) - \gamma.$$

The inequality where the roles of $\mathcal{F}_k$ and $\mathcal{F}'_k$ are reversed follows by symmetry. □

The following theorem's proof follows verbatim from the first paragraph of [**Dow05**, proof of Theorem 7.0.1, p. 95]. Notice one uses Lemma 5.2.3 in the proof.

THEOREM 5.2.4. *Let* $(X, \mathbb{Z}^k)$ *be a dynamical system with* $\text{mdim}(X, \mathbb{Z}^k) = 0$*, then it has an entropy structure.*

Notice that entropy structure is a topological invariant [**Dow05**, p. 75, Theorem 5.0.2]. We now prove Theorem 1.12.4.

Proof. Given that zero-dimensional principal $\mathbb{Z}^k$-extensions exist for finite entropy dynamical systems (Theorem 5.1.5), the proof is the same as that of [**Dow05**, Theorem 5.1.1]. The verification of this claim is straightforward and is left to the reader. □

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