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Arbeitsgemeinschaft: Limits of Structures

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Abstract. The goal of the Arbeitsgemeinschaft is to review current progress in the study of very large structures. The main emphasis is on the analytic approach that considers large structures as approximations of infinite analytic objects. This approach enables one to study graphs, hypergraphs, permutations, subsets of groups and many other fundamental structures.

Mathematics Subject Classification (2010): 05Cxx.

Introduction by the Organisers

Built on decades of research in ergodic theory, Szemerédi's regularity theory and statistical physics, a new subject is emerging that considers very large finite structures as approximations of infinite analytic objects. More precisely, one can introduce various convergence notions and limit objects for growing sequences of graphs, hypergraphs, permutations, and for several kinds of other important structures. Many properties of these structures are easier to study in the limiting setting since powerful tools from analysis become available. This approach creates new connections between analysis, combinatorics and group theory. The goal of the Arbeitsgemeinschaft is to present a landscape of beautiful ideas developed by researchers from diverse fields. The subject is very rich and many of its aspects are covered in the recent book [1] by L. Lovász.

The presentations at the workshop discussed a number of applications in extremal combinatorics, Fourier analysis (also in a higher order version of it), group theory, ergodic theory, topology and probability. The workshop was well attended with over 40 participants. It brought together researchers with backgrounds in Probability, Combinatorics, Ergodic theory, group theory and logic. Besides talks there was a problem session and an informal discussion of recent progress in random regular graphs.

References

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Nilmanifolds and nilspaces

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Recently Camarena and Szegedy have developed a beautiful theory of *nilspaces* - a certain generalization of nilmanifolds. The goal of this report is to present very succinctly, the elements of a new proof of the main case of a fundamental theorem appearing in [1], about the relation between nilmanifolds and nilspaces. I would like to thank Freddie Manners and Péter Varjú who worked with me on the new proof. I am grateful to Ben Green, Bernard Host and Balázs Szegedy for helpful discussions.

1. The prenilspace and k-step nilspace axioms

Define the functions $\rho_i : \{0,1\} \to \{0,1\}$, $i = 0,1,2,3$, by $\rho_0(x) \equiv 0\,, \rho_1(x) \equiv 1\,, \rho_2(x) = x\,, \rho_3(x) = 1-x$. Let $m,n \in \mathbb{N}$. A map $f : \{0,1\}^m \to \{0,1\}^n$ between *discrete cubes* is called a **discrete cube morphism** if for every $1 \le i \le n$ there exist $1 \le j \le m$ and $k \in \{0,1,2,3\}$ (depending on i) so that for any $(x_1,\dots,x_m) \in \{0,1\}^m$, $f(x_1,\dots,x_m)_{|i} = \rho_k(x_j)$. Observe that discrete cube morphisms are closed under composition. Let (X,d) be a compact metric space. Let $C^n(X) \subset X^{\{0,1\}^n}$, $n \in \mathbb{Z}_+$ be closed sets. The elements of $C^n(X)$ are referred to as the **(n)-cubes**. X is referred to as the **base space**. We define the following *axioms* $(n,k \in \mathbb{Z}_+)$: **n-Cube invariance** $(I)_n$: For any $m \in \mathbb{Z}_+$, $f \in C^m(\{0,1\}^n)$ and $c \in C^n(X)$ $c \circ f \in C^m(X)$. **k-Ergodicity** $(E)_k$: $C^k(X) = X^{\{0,1\}^k}$. **n-Completion** $(C)_n$: If $\tilde{f} : \{0,1\}^n \backslash \{\vec{1}\} \to X$ has the property that for every $1 \le i \le n$, $\tilde{f}_{|F_i} \in C^{n-1}(X)$ where $F_i = \{\vec{x} \in \{0,1\}^n |\, x_i = 0\}$, then there exists $c \in C^n(X)$ with $c^* := c_{|\{0,1\}^n \backslash \{\vec{1}\}} = \tilde{f}$. c is referred to as a *completion* of $\tilde{f}$. **n-Uniqueness** $(U)_n$: If $h,f \in C^n(X)$ and $h^* = f^*$ then $h = f$. Let $\mathcal{X} = (X, \{C^n(X)\}_{n=0}^{\infty})$. Define the following objects: **Prenilspace**: $[(I)_n$ for all $n \in \mathbb{Z}_+]$, $(E)_1$, $[(C)_n$ for all $n \in \mathbb{N}]$. **k-step Nilspace**: $[(I)_n$ for all $n \in \mathbb{Z}_+]$, $(E)_1$, $[(C)_n$ for all $n \in \mathbb{N}]$, $(U)_{k+1}$. A morphism between two prenilspaces $f : \mathcal{X} \to \mathcal{Y}$ consists of a continuous mapping $f : X \to Y$ such that $f(C^n(X)) \subset C^n(Y)$ for all $n \in \mathbb{N}$.

2. The structure of k-step nilspaces

Let $\mathcal{X}$ be a 1-step nilspace. Fix an arbitrary element $e \in X$ and let $a, b \in X$ arbitrary. Taking advantage of the fact that $\tilde{a} : \{0,1\}^2 \setminus \{\vec{1}\} \to X$ given by $\tilde{a}(0,0) = e$, $\tilde{a}(1,0) = a$, $\tilde{a}(0,1) = b$ has a unique completion, one obtains a continuous binary operation on X. It is not hard to show, using the axioms, this binary operation turn X into a compact Abelian group. For k-step nilspaces with $k > 1$ the situation is more complicated. Define a **principal bundle** to be a quadruple $\mathcal{E} = (E, B, \pi, G)$, where E, B are topological spaces, G is a topological group acting continuously on E and $\pi : E \to B$ a is continuous surjection such that, G preserves the fibers $\pi^{-1}(b)$, $b \in B$ and acts freely and transitively on each one of them. A (G)-**bundle map** $\phi : E \to E$ is a continuous G-equivariant map.

The Camarena–Szegedy Structure Theorem. Given a k-step nilspace $\mathcal{X}_k$, there is a finite series of finite-step nilspaces $\mathcal{X}_{k-1}, \ldots \mathcal{X}_0 = \{\bullet\}$ and compact Abelian groups $A_k, \ldots A_1$ as well as continuous prenilspace epimorphisms $\mathcal{X}_k \stackrel{\pi_k}{\to} \mathcal{X}_{k-1} \stackrel{\pi_{k-1}}{\to} \mathcal{X}_{k-2} \to \cdots \stackrel{\pi_1}{\to} \mathcal{X}_0$ such that $(X_j, X_{j-1}, \pi_j, A_j)$ is a A_j-principal bundle for $j = 1, \ldots, k$. □

$\mathcal{X}_k$ is said to be **toral** if all structure groups $A_1, \ldots A_k$ are tori (of various dimensions). Recall that a **nilmanifold** X is a quotient $X = G/\Gamma$ where G is a finite-step nilpotent Lie group and Γ a cocompact discrete subgroup. Our goal is to prove the following theorem:

Theorem 1. *The base space of a toral k-step nilspace is a nilmanifold.*

Proof. (Sketch.) The result is proven by induction. The base case $k = 1$: From the Camarena–Szegedy Structure Theorem it follows $X_1 = A_1$ is a torus. Assume the theorem has been established for $k - 1$. Let $\mathcal{X}_k = (X_k, \{C^n(X_k)\}_{n=0}^{\infty})$ be a k-step compact nilspace. We call a homeomorphism $\alpha : X_k \to X_k$ a **translation** if for any $c \in C^k(X_k)$, $[c, \alpha(c)] \in C^{k+1}(X_k)$ and $[c, \alpha^{-1}(c)] \in C^{k+1}(X_k)$ where the *concatenation* $[c_0, c_1] : \{0,1\}^{k+1} \to X_k$ is given by $[c_0, c_1](v, 0) = c_0(v)$ and $[c_0, c_1](v, 1) = c_1(v)$ for all $v \in \{0,1\}^k$. Note that translations are A_k-bundle maps. Let $\tilde{G}_k$ be the group of translations of $\mathcal{X}_k$ equipped with the supremum metric d_∞. Let G_k be the identity component of $\tilde{G}_k$. G_k will turn out to be the desired nilpotent Lie group for which $X_k = G_k/\Gamma_k$ (for suitable cocompact discrete Γ_k). Going through the proof of [1, Theorem 7] it is clear that the difficulty lies in establishing that the natural projection $\pi_k : G_k \to G_{k-1}$ is onto. Let $\alpha_{k-1} \in G_{k-1}$. By the inductive assumption $G_{\mathcal{X}_{k-1}}$ is a connected Lie group and therefore path connected. As a consequence one can find a (continuous) homotopy between Id and α_{k-1}, $H : X_{k-1} \times I \to X_{k-1}$. By *Gleason's Theorem* ([2, Theorem 3.3]), $(X_k, X_{k-1}, \pi_k, A_k)$ is a *fiber bundle*. Thus according to the *First Covering Homotopy Theorem* ([3, §11.3]), as X_{k-1} is compact, one can *lift* the homotopy H to a homotopy which is a bundle map. In particular there is a bundle map lift $h_k : X_k \to X_k$ of α_{k-1} ($\pi_k \circ h_k(x) = \alpha_{k-1} \circ \pi_k(x)$). However h_k may not be a translation. We associate to h_k the "cocycle" $\rho_k : C^k(X_k) \to A_k$, measuring its deviation from being a translation, defined by $\rho_k(c) = a$ iff $[c, (h_k(c^*), h_k(c(\vec{1})) +$

$a)] \in C^{k+1}(X_k)$, where $(h_k(c^*), h_k(c(\vec{1})) + a)$ is the configuration achieved from $h_k(c)$ by adding the element a to $h_k(c(\vec{1}))$. Using the $(C)_{k+1}$ and $(U)_{k+1}$ axioms, one can easily show that such an element a exists and that it is unique. This implies $\rho_k(c)$ is *continuous*. As ρ_k is constant on cubes with identical projection on $C^k(X_{k-1})$, one obtains a map $\rho_k : C^k(X_{k-1}) \to A_k$. It turns out that if $d_\infty(Id, h_k)$ is small enough (which can be assumed w.l.o.g) then there exists a continuous $g : X_{k-1} \to A_k$ such that the α_{k-1}-lift $\alpha_k := h_k + g : X_k \to X_k$ is a translation iff one can solve the equation $\rho_k(c) = \partial^k(g)(c) := \sum_{v \in \{0,1\}^k} g(c(v))(-1)^{\sum_i v_i}$ for all c. This equation is indeed solvable following the procedure in [1, Lemma 3.19] as one can explicitly write g as a certain average of ρ_k. Without getting into the details let us point out that the continuity of g is a consequence of the continuity of ρ_k.

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