(19.11.2021)

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ℓ-away ACM Bundles (joint work with F. Gawron)

The famous result of Horrock on v. bund. is

A v.b on $\mathbb{P}^n$ splits (sum of line b.) $\iff$ $H^i(\mathcal{E}(t))=0$ $0<i<n$ for all $t\in\mathbb{Z}$

Then we can give the defn

<u>Defn</u>: A v.b on $(X, \mathcal{O}_X(1))$ is ACM if
$$H^i(\mathcal{E}(t))=0 \quad \text{for } 0<i<\dim X \text{ and for } t\in\mathbb{Z}.$$

There is a parallel result for quadratic hyperp.

A v.b $\mathcal{E}$ on Q^n splits as a sum of line bundles and twisted spinor bundles

$\Updownarrow$ Knörrer (1987)

$\mathcal{E}$ is ACM

However, in general, there is no such splitting result for an ACM bundle on an arbitrary variety. But, these bundles got interest of many math. working on v.b since (due to above splitt. results) they are assumed to have simpler structure; and existence of them reflects simplicity of ambient variety in view of coh. (Ulrich bundles)

But, what about other v.b? For example $\Omega_{\mathbb{P}^2}$
(non-ACM bundles) $(h^1(\Omega_{\mathbb{P}^2})=1)$

Let's start:

<u>Defn</u>: A v.b $\mathcal{E}$ on a proj. var. (X,h) is called ℓ-away ACM if $H^i(\mathcal{E}(t))=0$ for $0<i<\dim X$ and for all $t\in\mathbb{Z}$ except ℓ-many integer,

[From now on, our v.b will be initialized; i.e., $H^0(\mathcal{E}(-1)) = 0$ but $H^0(\mathcal{E}) \neq 0$]

Defn: A smooth proj. var. X is called Fano if its antican. bundle $-K_X$ is ample. The index i_X of a Fano variety is the great. int. st $K_X = \mathcal{O}(-i_X h)$ for some apple line bundle h.
(Fano v. of dimension 2 is del Pezzo)

Defn: A v.b $\mathcal{E}$ is called special if if $c_1(\mathcal{E}) = ch$. for some integer c.

Q) What is the shape of coh. table of a special v.b on del Pezzo surfaces?

[For example is $H^1_*(X, \mathcal{E})$ is connected?
Does there exist v.b for any possible conf.?]

Thm: Let $\mathcal{E}$ be a special, b-away ACM bundle of rank 2 on a del Pezzo surface X. Let $t = k_0$ and $t = k_0 + s$ be smallest and largest int. st $H^1(\mathcal{E}(t)) \neq 0$. Then

(a) $c = -2k_0 - i_X - s$

(b) $k_0 \geq -s-2$

(c) $k_0 \leq -i_X + 1$

Proof: (a) Serre duality

(b) [X is non-sing ACM. So Koszul res (for a $\overset{2}{?}(x_1, x_2)$)

(*) $0 \to \mathcal{O}(-2) \to \mathcal{O}_X^2(-1) \to \mathcal{I}_Y \to 0$

If $\exists\, l$ st $h^2(\mathcal{E}(l-1)) = 0$ then $H^1(\mathcal{E}(l))$ generates $\bigoplus_{k \geq l} H^1(\mathcal{E}(k))$ as $S(X)$-module
($S(X)$: coord ring)

$$0 \to \mathcal{E}(k-1) \to \mathcal{E}(k) \to \mathcal{E}(k)|_H \to 0$$

$\Rightarrow H^2(\mathcal{E}(k)) = 0$ for $k \geq l-1$

$\overset{(*)}{\Rightarrow}$ $H^1(\mathcal{E}(k-1))^{\oplus 2} \to H^1(\mathcal{E}\otimes I_Y(k)) \to 0$ for $k \geq l+1$

$0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ $\Longrightarrow$ $H^1(\mathcal{E}\otimes I_Y(k)) \to H^1(\mathcal{E}(k)) \to 0$

$\Rightarrow$ $H^1(\mathcal{E}(k-1))^{\oplus 2} \to H^1(\mathcal{E}(k)) \to 0$ for $k \geq l+1$

$0 = H^0(\mathcal{E}(-1)) = H^2(\mathcal{E}(-c+1-i_X)) \Rightarrow$ take $l = 2-c-i_X$

$\Rightarrow H^1(2-c-i_X)$ gen . . .

$\Rightarrow H^1(\mathcal{E}(t)) = 0$ for all $t \geq 2-c-i_X$

$\Rightarrow 2-c-i_X > k_0 - 1$ (otherwise $\mathcal{E}$ is ACM)

$\Rightarrow$ combine with item (1)

(c) Assume to the contrary, $k_0 \geq -i_X + 2$

Then $\mathcal{E}(-c)$ is 0-regular (by former items)

$\Rightarrow \mathcal{E}(-c)$ is globally generated

Jose Carlos Sierra (2008)

$\mathcal{E}$ is glob. gen. of rank e

If $h^0(\mathcal{E}(-c_1(\mathcal{E}))) \neq 0$ then $\mathcal{E} \simeq \mathcal{O}_X^{e-1} \oplus \mathcal{O}(c_1(\mathcal{E}))$

$\Rightarrow$ Since $h^0(\mathcal{E}(-c)(c)) = h^0(\mathcal{E}) \neq 0$, $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(c)$ $\lightning$

So we can classify 1-away ACM on $\mathbb{P}^2$:

<u>Thm</u>: $\mathcal{E}$ is 1-away ACM on $\mathbb{P}^2$ of rank 2 $\Leftrightarrow$ $\mathcal{E} = \Omega_{\mathbb{P}^2}(2)$

<u>Proof</u>: 1-away $\Rightarrow s = 0$, $i_X = 3$ $\Big\} \Rightarrow -2 \leq k_0 \leq -2$, $c = 1$ $\Rightarrow$ $H^1(\mathcal{E}(t)) = 0$ for all $t \in \mathbb{Z}\setminus\{-2\}$

RRoch $\Rightarrow c_2 = 1$

Apply Beilinson

$\mathcal{O}$	$\Omega(1)$	$\mathcal{O}(1)$	
3	0	0	0
0	0	1	0
0	0	0	0
-4	-3	-2	-1

$h^2(\mathcal{E}(t)) = h^0(\mathcal{E}(-t-1-3))$

$-t-4 \leq -1$

$-3 \leq t$

$\Longrightarrow \quad \mathcal{E} = \Omega(2)$

Inverse is application of Bott's thm

When we come to l-away ACM for $l > 1$, there arise a question

Q) Is $H^1_*(\mathcal{E}(t))$ connected for rank 2 bundle on $\mathbb{P}^2$?

0	*	0	0	0	0
0	0	*	0	*	0
0	0	0	0	0	*
		-3	-2	-1	0

?

Thm: $H^1_*(\mathcal{E}(t))$ is connected

Proof: Assume to contrary $s \geq l > 1$

$\Longrightarrow -c+2-i_x > k_0 + s - 1$ $\leftrightarrow k_0$

$\Longrightarrow 2k_0 + i_x + s + 2 - i_x > k_0 + s - 1 \Longrightarrow k_0 \geq -2 \quad \Rightarrow k_0 = -2$
$(c = 1 - s)$

$s \in H^0(\mathcal{E}) \neq 0$

$\Longrightarrow \quad s_0 = M \cup Z$
$\quad\quad\quad\;\; \overset{\shortparallel}{mh}$

If Z is empt then $\mathcal{E}$ is extens. not pos.

$\Longrightarrow \quad 0 \to \mathcal{O} \to \mathcal{E}(-m) \to I_Z(-2m+1-s) \to 0 \quad (*)$

$\Longrightarrow$ (a) $h^1(\mathcal{E}(t)) \leq h^1(I_Z(t-m+1-s))$

Also (d) $h^1(\mathcal{E}(-1)) = h^1(I_Z(-m-s))$ $\quad\quad \underset{(*)}{\overset{\uparrow}{m-1 \geq 0}}$

Also $0 \to I_Z(t-m+1-s) \to \mathcal{O}(t-m+1-s) \to \mathcal{O}_Z \to 0 \quad (**)$

(b) $h^1(I_Z(t-m+1-s)) \leq h^0(\mathcal{O}_Z)$

Also (c) $h^0(\mathcal{O}_Z) = h^1(I_Z(-m-s))$ $\quad (**) + -m-s < 0$

$\Rightarrow h^1(\mathcal{E}(t)) \leq h^1(\mathcal{E}(-1)) = 0 \quad \forall t \Rightarrow \mathcal{E}$ is ACM ↯

Therefore $s = l - 1$, that is $H^1_*(\mathbb{P}^2, \mathcal{E})$ connected.

Thm: $\mathcal{E}$ is 2-away ACM bundle of rank 2 on $\mathbb{P}^2$

$$\Updownarrow$$

$$0 \to \mathcal{E} \to \mathcal{O}^4(2) \to \mathcal{O}^2(3) \to 0 \qquad (k_0=-3)$$

OR

$$0 \to \mathcal{E} \to \Omega(2) \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \qquad (k_0=-2)$$

Proof: Following former thm + beilinson thm $(k_0=-3,-2)$

Corollary: All indec. weakly Ulrich bundles of rank 2 on $\mathbb{P}^2$ are appropr. shift of 1-away and 2-away ACM bundles ($\Omega(2)$, $\mathcal{E}(-1)$ for the first item, $\mathcal{E}$ for the second item)

Weakly Ulrich bundle is a bundle whose Chow divisor on the Grassm. repr. by a map between vector bundles of same rank.

Cohomolog. on surfaces

*	*	*	0	0	0
0	0	*	*	0	0
0	0	0	*	*	*
-4	-3	-2	-1	0	1

Q) What about ℓ-away ACM of rank 2 for $\ell > 2$?

It is somehow general case (repr. as a monad)

Complete classification is not poss., but there always exists ℓ-away ACM of rank 2 for any ℓ.

Thm: The kernel bundle of repr.

$$0 \to \mathcal{E} \to \mathcal{O}(\ell)^{\ell+2} \to \mathcal{O}(\ell+1)^{\ell} \to 0 \qquad (k_0=-\ell-1)$$

is ℓ-away ACM bundle of rank 2 on $\mathbb{P}^2$. Also, it is μ-stable, and it corresponds to a smooth point of dim $\ell^2+2\ell-3$ in the mod. space of simple sh.

Corollary: The bundles constructed in former thm is supernatural. $(-\ell-2$ and $-1)$

Q) What about higher rank ℓ-away bundles?

<u>Thm</u>: There exists simple ℓ-away ACM bundle of any even rank $r = 2m$ on $\mathbb{P}^2$ for any $\ell \geq 1$ at which dimension of the moduli space $m^2(\ell^2 + 2\ell - 4) + 1$.

<u>Proof</u>: $\mathcal{E}$ is the bundle in the above thm, $\mathcal{E}$ is mapped to quiver with repr.

$$RHom(\mathcal{O}(2), \mathcal{E}) \Rrightarrow RHom(\Omega(3), \mathcal{E}) \Rrightarrow RHom(\mathcal{O}(1), \mathcal{E})$$

$$\begin{matrix} 0 & & 0 \\ \ell & & \ell+2 \\ 0 & & 0 \end{matrix}$$

$$\Rightarrow \quad \overset{\ell}{\bullet} \Rrightarrow \overset{\ell+2}{\bullet} \qquad K_3$$

$$\underline{d} = (d_1, d_2) = (\ell, \ell+2)$$

Note that $\chi(\underline{d}, \underline{d}) = d_1^2 + d_2^2 - 3d_1 d_2 = -\ell^2 - 2\ell + 4 < 0$

$\Rightarrow$ $\underline{d}$ is imaginary root $\Rightarrow$ $\underline{d}$ is Schur root

$\Rightarrow$ $m\underline{d}$ is schur root for every $m \geq 1$

$\Rightarrow$ Hence in $Rep(m\underline{d})$, the general element simple

$\Rightarrow$ Note that being supernatural is an open cond.

$\left.\begin{array}{l} \mathcal{E}^{\oplus m} \text{ supernat. } \ell\text{-away} \\ \mathcal{E}^{\oplus m} \text{ corresp. to repr. with dim vector } m\underline{d} \\ \text{being supernatural is open cond.} \end{array}\right\} \Rightarrow$ the general element in $Rep(m\underline{d})$ corresponds to simple (hence indec.) ℓ-away bundle

And $\dim = 1 - \chi(m\underline{d}, m\underline{d}) = 1 - m\chi(\underline{d}, \underline{d})$

$= m^2(\ell^2 + 2\ell - 4) + 1$

<u>Corollary</u>: There exists weakly Ulrich bundle of any even rank on $\mathbb{P}^2$.

<u>Corollary</u>: " supernatural bundle "

$\mathbb{P}^1 \times \mathbb{P}^1$: Results are parallel. The difference is line bundle case

Thm: $\mathcal{L}$ is a ℓ-away ACM line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Updownarrow$$

$$\mathcal{L} = \mathcal{O}((\ell+1)h_1) \quad \text{or} \quad \mathcal{L} = \mathcal{O}((\ell+1)h_2)$$

— o —

Someone, similarly, may classify special 1-away and 2-away ACM bundles of rank 2 on $\mathbb{P}^1 \times \mathbb{P}^1$

— o —

Similar to $\mathbb{P}^2$ case, we can show $H^1_*(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E})$ is connected for a special rank 2 bundle.

— o —

For any ℓ, we can construct special ℓ-away bundle of rank 2 on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\left(0 \to \mathcal{O}((\ell+1)h_1) \to \mathcal{E} \to \mathcal{O}((\ell+1)h_2) \to 0 \right)$$

which is supernatural,

— o —

We can construct simple ℓ-away bundle of any even rank $r = 2m$, for any ℓ, at which dim

$m^2(2\ell^2 + 4\ell - 2) + 1$

— o —

As a corollary, there exists weakly Ulrich (supernatural) bundle of any even rank on $\mathbb{P}^1 \times \mathbb{P}^1$

Further questions:

- Connectedness of $H^1_*(X, \mathcal{E})$ for a rank 2 bundle on other del Pezzo surfaces?
- What about the behaviour of ℓ-away bundles on Fano threefolds?
  (It is only known that $H^1_*(\mathbb{P}^3, \mathcal{E})$ is connected, for others I have no inf.)