Secondary Calculus

Luca Vitagliano

University of Salerno
INFN, Gruppo Collegato di Salerno

XXII IWDGMTM
Będlewo, August 19–26, 2007

Introduction

Secondary Calculus generalizes standard calculus on manifolds to the (functional) space of solutions of a given PDE by using only *differential geometry* and *homological algebra* in the environment of an *infinite jet manifold*. In a sense

Secondary = Functional, Variational

Thus, Secondary Calculus has relevant applications to Physics and, in particular, Field Theory. As an example the geometry of the *Covariant Phase Space* may be formalized within Secondary Calculus.

Outline: Part I

Outline: Part II

Differential Calculus may be formalized over any associative, unitary, (graded) commutative algebra A over a ring R. In the case $A = C^\infty(M)$ the theory reduces to standard calculus on manifolds

Example

Let $P, Q \in \mathbf{Mod}_A$. A k–th order, Q–valued, differential operator over P is defined to be any R–linear operator $\square : P \longrightarrow Q$ such that

$$[a_0, [a_1, [\cdots [a_k, \square] \cdots]]] = 0, \quad \forall a_0, a_1, \ldots, a_k \in A.$$

$$Diff_k(P, Q) = \{\square : P \longrightarrow Q \mid \square \text{ is a } k\text{–th order differential operator}\}$$

Remark

$Diff_k(P, Q)$ is an A–module.

Remark

$$\mathbf{Mod}_A \ni Q \mapsto \mathit{Diff}_k(P, Q) \in \mathbf{Mod}_A$$

is a functor and a representable one. I.e., for any Q there is a canonical isomorphism of modules

$$\mathit{Diff}_k(P, Q) \simeq \mathit{Hom}_A(\mathcal{J}^k(P), Q)$$

with $\mathcal{J}^k(P) = \{k - \text{jets of elements in } P\}$.

A number of functors may be similarly introduced in $\mathbf{Mod}_A$. An *object of differential calculus* is any among these functors, their repr. objects, etc.

Example

The de Rham complex $0 \to A \xrightarrow{d} \Lambda^1(A) \xrightarrow{d} \cdots \to \Lambda^q(A) \xrightarrow{d} \cdots$ is an object of differential calculus.

Remark

The solution space $\boldsymbol{M}$ of a PDE may be understood as the space of maximal integral submanifold of a distribution.

$$
\begin{array}{rcl}
(\ldots, x^\mu, \ldots, u^j_\sigma, \ldots) \in & J^k\pi \\
& \downarrow \\
(\ldots, x^\mu, \ldots, u^j, \ldots) \in & E \\
& \downarrow{\scriptstyle \pi} \\
(\ldots, x^\mu, \ldots) \in & M
\end{array}
$$

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "i_{\mathcal{E}}"] \arrow[d] & J^\infty\pi \arrow[r, "F"] \arrow[d] & J^\infty\nu \arrow[d] \\
\mathcal{E}_0 \arrow[r, hook] & J^k\pi \arrow[r, "F_0"] \arrow[dr, "\pi_k"'] & V \arrow[d, "\nu"] \\
& & M
\end{tikzcd}
$$

The Cartan distribution over $J^\infty\pi$ restricts to $\mathcal{E}$

$$\mathcal{C} = \langle \ldots, D^{\mathcal{E}}_\mu, \ldots \rangle,\ D^{\mathcal{E}}_\mu = \left(\frac{\partial}{\partial x^\mu} + u^j_{\sigma+\mu}\frac{\partial}{\partial u^j_\sigma}\right)\Big|_{\mathcal{E}}$$

Remark

$\{\text{solutions to } \mathcal{E}\} = \{\text{maximal integral submanifolds of } (\mathcal{E}, \mathcal{C})\}$!!!

The tangent bundle to a differential equation splits $T\mathcal{E} \simeq \mathcal{C} \oplus V\mathcal{E}$.

The Variational Bi–Complex and the $\mathcal{C}$–Spectral Sequence

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \uparrow d^V & & \uparrow d^V & & \\
\cdots & \longrightarrow & \Lambda^{q,p+1}(\mathcal{E}) & \xrightarrow{\overline{d}} & \Lambda^{q+1,p+1}(\mathcal{E}) & \xrightarrow{\overline{d}} & \cdots \\
 & & \uparrow d^V & & \uparrow d^V & & \\
\cdots & \longrightarrow & \Lambda^{q,p}(\mathcal{E}) & \xrightarrow{\overline{d}} & \Lambda^{q+1,p}(\mathcal{E}) & \xrightarrow{\overline{d}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & &
\end{array}$$

$$\{(\mathcal{C}E_r(\mathcal{E}), d_r)\}_r$$

$$\begin{array}{c}(\mathcal{C}E_0(\mathcal{E}), d_0)\\ \wr\| \\ (\Lambda(\mathcal{E}), \overline{d})\end{array}$$

$$\mathcal{C}E_0^{p,q}(\mathcal{E}) \simeq \Lambda^{q,p}(\mathcal{E}) \simeq \mathcal{C}^p\Lambda^p(\mathcal{E}) \otimes \overline{\Lambda}^q(\mathcal{E})$$
$$\mathcal{C}^p\Lambda^p(\mathcal{E}) = \langle \ldots, i_{\mathcal{E}}^*(\omega_{\sigma_1}^{j_1} \wedge \cdots \wedge \omega_{\sigma_p}^{j_p}), \ldots \rangle, \quad \omega_\sigma^j = du_\sigma^j - u_{\sigma+\mu}^j dx^\mu$$
$$\overline{\Lambda}^q(\mathcal{E}) = \langle \ldots, i_{\mathcal{E}}^*(\overline{d}x^{\mu_1} \wedge \cdots \wedge \overline{d}x^{\mu_q}), \ldots \rangle, \quad \overline{d}x^\mu = dx^\mu.$$

$\mathcal{C}$ determines a "horizontal" differential calculus on $\mathcal{E}$.

Definition

Let $P, Q \in \mathbf{Mod}_{\mathcal{F}(J^\infty\pi)}$ be modules of sections of vector bundles over $J^\infty\pi$. A linear $\mathcal{C}$–diff. operator $\square : P \longrightarrow Q$ is one locally in the form

$$\square(p) = \square^\sigma_{aA}(D_\sigma p^a)\varepsilon^A, \quad p = p^a e_a, \quad D_\sigma = D_1^{\sigma_1} \circ \cdots \circ D_n^{\sigma_n}, \quad \square^\sigma_{aA} \in \mathcal{F}(\mathcal{E})$$

$\ldots, e_a, \ldots$ a local basis of P; $\quad\ldots, \varepsilon^A, \ldots$ a local basis of Q.

Any $\mathcal{C}$–differential operator $\square$ restricts to $\mathcal{E}$: $\square \mapsto \square^{\mathcal{E}}$.

Horizontal jet–spaces may be also defined.
P module of sections of a vector bundle over $\mathcal{E}$ $\Rightarrow$ $\overline{\mathcal{J}}^k(P)$ module of horizontal jets of elements in P, $k \leq \infty$.

$$\square : P \longrightarrow Q \quad \Rightarrow \quad \overline{\Psi}_\square : \overline{\mathcal{J}}^k(P) \longrightarrow Q \quad \Rightarrow \quad \overline{\Psi}^\infty_\square : \overline{\mathcal{J}}^\infty(P) \longrightarrow \overline{\mathcal{J}}^\infty(Q)$$

Adjoint Operators

P – module of sections of a vector bundle over $\mathcal{E}$.
$P^* = Hom(P, \mathcal{F}(\mathcal{E}))$ – dual module.
$\widehat{P} = P^* \otimes \overline{\Lambda}^n(\mathcal{E})$ ($n = \dim M$) – adjoint module.

$\square : P \to Q$ a $\mathcal{C}$–diff. operator $\Rightarrow \widehat{\square} : \widehat{Q} \to \widehat{P}$ the adjoint operator, i.e.,

$$\widehat{\square}(\widehat{q}) = (-1)^{|\sigma|} D_\sigma(\square^\sigma_{aA} \widehat{q}^A)(e^a \otimes \overline{d}^n x)$$
$$\widehat{q} = \widehat{q}^A(\varepsilon_A \otimes \overline{d}^n x), \quad \overline{d}^n x = \overline{d}x^1 \wedge \cdots \wedge \overline{d}x^n$$

$\ldots, e^a, \ldots$ and $\ldots, \varepsilon_A, \ldots$ dual bases of $\ldots, e_a, \ldots$ and $\ldots, \varepsilon^A, \ldots$

For $\Delta : P \to Q$ and $\nabla : Q \to R$ $\mathcal{C}$–differential operators,

$$\widehat{\widehat{\Delta}} = \Delta \text{ and } \widehat{\nabla \circ \Delta} = \widehat{\Delta} \circ \widehat{\nabla}.$$

P – module of sections of a vector bundle $\mathcal{O} \to \mathcal{E}$ over $\mathcal{E}$

Algebraic Definition

A $\mathcal{C}$–connection in P is an $\mathcal{F}(\mathcal{E})$–linear corresp. $\nabla : \mathcal{C} \supset X \mapsto \nabla_X$, such that $\nabla_X : P \to P$ is a der–operator (covariant derivative) over X, i.e.

$$\nabla_X(fp) = f\nabla_X p + X(f)p, \quad f \in \mathcal{F}(\mathcal{E}),\ p \in P.$$

A ∇ is *flat* if $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$, $X, Y \subset \mathcal{C}$

Geometric Definition

A $\mathcal{C}$–connection in $\mathcal{O}$ is an n–dimensional hor. distribution $\mathcal{C}(\mathcal{O})$ over $\mathcal{O}$ which projects isomorphically onto $\mathcal{C}$. $\mathcal{C}(\mathcal{O})$ is *flat* if it is involutive.

Definition

The module P of sections of a vector bundle over $\mathcal{E}$ endowed with a flat $\mathcal{C}$–connection is called a $\mathcal{C}$–module.

Example 1

the module of vertical vector fields $VD(\mathcal{E})$:

$$\nabla_X Z = [X, Z]^V, \quad X \subset \mathcal{C},\ Z \in VD(\mathcal{E}).$$

Example 2

the module of Cartan p–forms $\mathcal{C}^p\Lambda^p(\mathcal{E})$

$$\nabla_X \omega = L_X \omega, \quad X \subset \mathcal{C},\ \omega \in \mathcal{C}^p\Lambda^p(\mathcal{E}).$$

Example 3

$\Delta : P \longrightarrow P_1$ a $\mathcal{C}$–differential operator and $\overline{\Psi}^\infty_\Delta : \overline{\mathcal{J}}^\infty(P) \longrightarrow \overline{\mathcal{J}}^\infty(P_1)$ the associated horizontal jet prolongation. $R_\Delta = \ker \overline{\Psi}^\infty_\Delta \subset \overline{\mathcal{J}}^\infty(P)$:

$$\nabla_X (f \cdot \bar{j}^\infty(p)) = X(f) \cdot \bar{j}^\infty(p), \quad X \subset \mathcal{C},\ f \in \mathcal{F}(\mathcal{E}),\ p \in P.$$

There is a de Rham–like complex associated with a $\mathcal{C}$–module P

$$\cdots \longrightarrow P \otimes \overline{\Lambda}^q(\mathcal{E}) \xrightarrow{\overline{d}_P} P \otimes \overline{\Lambda}^{q+1}(\mathcal{E}) \xrightarrow{\overline{d}_P} \cdots \qquad (\star)$$

Definition

The graded cohomology vector space of $(\star)$, $\overline{H}^\bullet(P)$, is called the horizontal cohomology space of P. $\overline{d}_P$ is a $\mathcal{C}$–differential operator.

Example

$P = \mathcal{C}^p\Lambda^p(\mathcal{E}) \Rightarrow \overline{d}_P = \overline{d}$, $(\star)$ is the p–th row of the variational bi–complex and $\overline{H}^\bullet(\mathcal{C}^p\Lambda^p(\mathcal{E})) = \mathcal{C}E_1^{p,\bullet}(\mathcal{E})$.

A connection in a bundle π over a manifold M, $P = \Gamma(\pi)$ may be used to integrate a (suitably supported) element $p \in P \otimes \Lambda^q(M)$ over a q–fold $\gamma \subset M$. Similarly, the flat $\mathcal{C}$–connection in a $\mathcal{C}$–module P may be used to integrate an element $p \in P \otimes \overline{\Lambda}^q(\mathcal{E})$ over an integral q–fold of $\mathcal{C}$.

Horizontal Calculus on a PDE: Summary

On the infinite prolongation $\mathcal{E}$ of a differential equation $\mathcal{E}_0$, the Cartan distribution $\mathcal{C}$ determines

- the space $\boldsymbol{M}$ of solutions of $\mathcal{E}_0$;
- the class of $\mathcal{C}$–modules, P,
- the class of associated de Rham–like complexes, $(P \otimes \overline{\Lambda}^\bullet(\mathcal{E}), \overline{d}_P)$,
- the class of horizontal cohomology spaces, $\overline{H}^\bullet(P)$.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, hook, "i_{\mathcal{E}}"] \arrow[d] & J^\infty\pi \arrow[r, "F"] \arrow[d] & J^\infty\nu \arrow[d] \\
\mathcal{E}_0 \arrow[r, hook] & J^k\pi \arrow[r, "F_0"] \arrow[dr, "\pi_k"'] & V \arrow[d, "\nu"] \\
& & M
\end{tikzcd}$$

$$(\ldots, x^\mu, \ldots, v_\alpha, \ldots) \in V$$
$$v_\alpha = F_\alpha(\ldots, x^\mu, \ldots, u^j_\sigma, \ldots) = F_\alpha[x, u]$$
$$F_\alpha[x, u] = 0$$

$$\ell^{\mathcal{E}}_F(\chi) = \left(\frac{\partial F_\alpha}{\partial u^j_\sigma} D_\sigma \chi^j\right)[x, u] = 0$$
$$\chi = (\ldots, \chi^j[x, u], \ldots)$$

$$\begin{tikzcd}
\mathcal{E} \times_M E \arrow[r] \arrow[d, shift right, "\pi^{\mathcal{E}}"'] & E \arrow[d, "\pi"] \\
\mathcal{E} \arrow[r] \arrow[u, shift right, bend right, "\chi"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E} \times_M V \arrow[r] \arrow[d, "\nu^{\mathcal{E}}"'] & E \arrow[d, "\nu"] \\
\mathcal{E} \arrow[r] & M
\end{tikzcd}$$

$$\varkappa = \{\text{sect. of } \pi^{\mathcal{E}}\} \qquad P_1 = \{\text{sect. of } \nu^{\mathcal{E}}\}$$

$$\ell^{\mathcal{E}}_F : \varkappa \longrightarrow P_1$$
$\ell^{\mathcal{E}}_F$ is a $\mathcal{C}$–diff. oper.

$$\ker \ell^{\mathcal{E}}_F = \{\text{symm. of } \mathcal{E}_0\}$$

Proposition

$\overline{H}^0(VD(\mathcal{E})) \simeq \ker \ell^{\mathcal{E}}_F$.

$\overline{H}^0(VD(\mathcal{E}))$ is a space of (local) vector fields on $\boldsymbol{M}$.

Definition

$\boldsymbol{D}(\boldsymbol{M}) = \overline{H}^{\bullet}(VD(\mathcal{E})) = \{\text{secondary vector fields on } \boldsymbol{M}\}$.

Another example: elements in $\overline{H}^n(\mathcal{C}^p\Lambda^p(\mathcal{E})) \simeq \mathcal{C}E_1^{p,n}(\mathcal{E})$ identify with variational p–forms on $\mathcal{E}$. For $p = 0$, $[\omega] \in \overline{H}^q(\mathcal{F}(\mathcal{E})) \simeq \mathcal{C}E_1^{0,q}(\mathcal{E})$ identifies with the functional

$$\boldsymbol{M} \ni \boldsymbol{x} \mapsto \int_\gamma j^\infty(\boldsymbol{x})^*(\omega), \quad \gamma \text{ a suitable } q\text{–fold in } M.$$

$\overline{H}^q(\mathcal{C}^p\Lambda^p(\mathcal{E}))$ is a space of (local) differential p–forms on $\boldsymbol{M}$.

Definition

$\boldsymbol{\Lambda}^p(\boldsymbol{M}) = \overline{H}^{\bullet}(\mathcal{C}^p\Lambda^p(\mathcal{E})) \simeq \mathcal{C}E_1^{p,\bullet}(\mathcal{E}) = \{\text{secondary } p\text{–forms on } \boldsymbol{M}\}$.

All standard operations with vector fields and forms have secondary analogues, defined in purely algebraic (and homological) way!

So far I defined

- $\boldsymbol{M} = \{\text{secondary points}\} = \{\text{solutions } \boldsymbol{x} \text{ of } \mathcal{E}_0\}$,
- $\boldsymbol{D}(\boldsymbol{M}) = \{\text{secondary vector fields}\} = \overline{H}^\bullet(VD(\mathcal{E}))$,
- $\boldsymbol{\Lambda}^\bullet(\boldsymbol{M}) = \{\text{secondary differential forms}\} = \overline{H}^\bullet(\mathcal{C}^\bullet\Lambda^\bullet(\mathcal{E}))$.

Secondarization Principle

The secondary version $\boldsymbol{\Phi}(\boldsymbol{M})$ of an object Φ of differential calculus is the horizontal cohomology of the $\mathcal{C}$–"module" of vertical analogues of "elements" in Φ.

The following *Secondarization Scheme* may be used to define $\boldsymbol{\Phi}(\boldsymbol{M})$:

Secondarization Scheme

1. Define a vertical version $V\Phi(\mathcal{E})$ of Φ over $\mathcal{E}$,
2. Note that $V\Phi(\mathcal{E})$ has got a canonical $\mathcal{C}$–"module" structure,
3. Put $\boldsymbol{\Phi}(\boldsymbol{M}) =$ "$\overline{H}^\bullet$"$(V\Phi(\mathcal{E}))$.

Apply the Secondarization Scheme to the de Rham Complex.

1. The vertical version of the de Rham complex over $\mathcal{E}$ is the vertical de Rham complex $\cdots \to \mathcal{C}^p\Lambda^p(\mathcal{E}) \stackrel{d^V}{\to} \mathcal{C}^{p+1}\Lambda^{p+1}(\mathcal{E}) \stackrel{d^V}{\to} \cdots$
2. $\mathcal{C}^\bullet\Lambda^\bullet(\mathcal{E})$ has a $\mathcal{C}$–module structure and d^V is compatible with it, i.e., it extends to the var. bi–complex $(\mathcal{C}^\bullet\Lambda^\bullet(\mathcal{E}) \otimes \overline{\Lambda}^\bullet(\mathcal{E}), \overline{d}, d^V)$.
3. Put $\mathbf{\Lambda}^p(\boldsymbol{M}) = \overline{H}^\bullet(\mathcal{C}^p\Lambda^p(\mathcal{E})) \simeq \mathcal{C}E_1^{p,\bullet}(\mathcal{E})$. moreover d^V passes in horizontal cohomology, giving a complex (secondary de Rham complex) $\cdots \to \mathbf{\Lambda}^p(\boldsymbol{M}) \stackrel{\boldsymbol{d}}{\to} \mathbf{\Lambda}^{p+1}(\boldsymbol{M}) \stackrel{\boldsymbol{d}}{\to} \cdots$.

$$(\mathbf{\Lambda}^\bullet(\boldsymbol{M}), \boldsymbol{d}) = (\mathcal{C}E_1^{\bullet,\bullet}(\mathcal{E}), d_1^{\bullet,\bullet}).$$

Calculus of variations is an aspect of Secondary Calculus. Put $\mathcal{E} = J^\infty\pi$. Then $\boldsymbol{M} = \{\text{sections of } \pi\}$. $\boldsymbol{S} = [L] \in \overline{H}^n(\mathcal{F}(\mathcal{E})) \subset \boldsymbol{C}^\infty(\boldsymbol{M})$ is an action functional: $L = \mathscr{L}[x,u]\overline{d}^n x$, $\boldsymbol{S} \simeq \int \mathscr{L}[x,u]d^n x$ and

$$\boldsymbol{dS} = [d^V L] \equiv (\ldots, (-1)^\sigma D_\sigma(\tfrac{\partial \mathscr{L}}{\partial u^j_\sigma}), \ldots) \in \mathbf{\Lambda}^1(\boldsymbol{M}).$$

Theorem [Goldschmidt]

Let $\Delta : P \longrightarrow P_1$ be a $\mathcal{C}$–differential operator. There exists a *formal resolution* of $\ker \Delta$, i.e. a formally exact complex (*compatibility complex*) of $\mathcal{C}$–diff. operators $P \xrightarrow{\Delta} P_1 \xrightarrow{\Delta_1} \cdots \longrightarrow P_q \xrightarrow{\Delta_q} \cdots$, i.e. such that the sequence $\overline{\mathcal{J}}^\infty(P) \xrightarrow{\overline{\Psi}^\infty_\Delta} \overline{\mathcal{J}}^\infty(P_1) \xrightarrow{\overline{\Psi}^\infty_{\Delta_1}} \cdots \longrightarrow \overline{\mathcal{J}}^\infty(P_q) \xrightarrow{\overline{\Psi}^\infty_{\Delta_q}} \cdots$ is exact.

Theorem [Spencer]

Horizontal cohomologies of $R_\Delta = \ker \overline{\Psi}^\infty_\Delta$ are isomorphic to cohomologies of any compatibility complex of Δ.

Corollary

Horizontal cohomologies of R^*_Δ are isomorphic to homologies of any adj. complex $\widehat{P} \xleftarrow{\widehat{\Delta}} \widehat{P}_1 \longleftarrow \cdots \xleftarrow{\widehat{\Delta}_{q-1}} \widehat{P}_q \longleftarrow \cdots$ of a compat. complex of Δ.

The lenght of a compatibility complex of Δ measures the "degree of overdeterminacy" of the equation $\Delta(p) = 0$.

Proposition

$VD(\mathcal{E}) \simeq R_{\ell_F^{\mathcal{E}}}$ and therefore hor. cohom. of $VD(\mathcal{E})$ is isomorphic to cohom. of a compatibility complex $\varkappa \xrightarrow{\ell_F^{\mathcal{E}}} P_1 \xrightarrow{\Delta_1} \cdots \to P_q \xrightarrow{\Delta_q} \cdots$.

Then, if equation $\ell_F^{\mathcal{E}}(\chi) = 0$ is not overdetermined

$$\overline{H}^q(VD) \simeq \left| \begin{array}{ll} \ker \ell_F^{\mathcal{E}} & \text{if } q = 0 \\ \operatorname{coker} \ell_F^{\mathcal{E}} & \text{if } q = 1 \\ 0 & \text{if } q > 1 \end{array} \right. , \quad \overline{H}^q(C^1\Lambda^1) \simeq \left| \begin{array}{ll} \operatorname{coker} \widehat{\ell_F^{\mathcal{E}}} & \text{if } q = n \\ \ker \widehat{\ell_F^{\mathcal{E}}} & \text{if } q = n-1 \\ 0 & \text{if } q < n-1 \end{array} \right.$$

If $\mathcal{E} = J^\infty\pi$, then $\boldsymbol{D}(\boldsymbol{M}) \simeq \varkappa$ and $\boldsymbol{\Lambda}^1(\boldsymbol{M}) \simeq \widehat{\varkappa}$.

Present Limits of Secondary Calculus

At the moment Secondary Calculus only deals with local functionals!

Example

Multilocal functionals are not represented is Secondary Calculus

$$\int \cdots \int \mathscr{L}[x_{[1]}, \ldots, x_{[r]}, u_{[1]}, \ldots, u_{[r]}] d^n x_{[1]} \cdots d^n x_{[r]}$$

Example

Feynman–like functionals are not represented in Secondary Calculus

$$\exp i \int \mathscr{L}[x, u] d^n x$$

The space of secondary functions is still too small!

A *Lagrangian field theory* is a bundle $\pi : E \longrightarrow M$ togheter with an action $\boldsymbol{S} = [L] \in \overline{H}^n(\mathcal{F}(J^\infty\pi))$, $L \in \overline{\Lambda}^n(J^\infty\pi)$ is a *lagrangian density* and $\mathcal{E}_0 : \boldsymbol{dS} = 0$ the associated Euler–Lagrange equations of motion.

Definition

The *Covariant Phase Space* $\boldsymbol{P}$ of the lagrangian field theory $(\pi, \boldsymbol{S})$ is the space of solutions of $\mathcal{E}_0$, i.e., $\boldsymbol{P} = \{\text{max. int. subman. of } (\mathcal{E}, \mathcal{C})\}$.

Proposition [Zuckerman]

There exists a canonical, closed (secondary) 2–form on $\boldsymbol{P}$.

Proof. $\boldsymbol{dS} = [d^V L] \in \widehat{\varkappa} \hookrightarrow \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^n$. Thus, $d^V L - \boldsymbol{dS} = \overline{d}\theta$, for some $\theta \in \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^{n-1}$. Put $\omega = i^*_{\mathcal{E}}(d^V\theta) \in \mathcal{C}^2\Lambda^2(\mathcal{E}) \otimes \overline{\Lambda}^{n-1}(\mathcal{E})$, and note that

$$\overline{d}\omega = 0 \quad \Rightarrow \quad \boldsymbol{\omega} = [\omega] \in \overline{H}^{n-1}(\mathcal{C}^2\Lambda^2(\mathcal{E})) \subset \boldsymbol{\Lambda}^2(\boldsymbol{P})$$

does only depend on $\boldsymbol{S}$. Moreover, $\boldsymbol{d\omega} = [d^V\omega] = 0$.

The first Noether theorem has a formulation in terms of $(\boldsymbol{P}, \boldsymbol{\omega})$!

Let $\chi \in \varkappa$ be a Noether symmetry of $(\pi, \boldsymbol{S})$, i.e., $\boldsymbol{L}_\chi \boldsymbol{S} = 0$. χ is, in particular, an infinitesimal symmetry of $\mathcal{E}_0$, i.e.,

$$\boldsymbol{X} = \chi|_{\mathcal{E}} \in \ker \ell^{\mathcal{E}}_{\boldsymbol{dS}} \simeq \overline{H}^0(VD(\mathcal{E})) \quad \Rightarrow \quad \boldsymbol{X} \in \boldsymbol{D}(\boldsymbol{P})$$

According to first Noether theorem there exists an associated conservation law

$$\boldsymbol{f} \in \overline{H}^{n-1}(\mathcal{F}(\mathcal{E})) \quad \Rightarrow \quad \boldsymbol{f} \in \boldsymbol{C}^\infty(\boldsymbol{P})$$

Proposition [LV]

$\boldsymbol{df} = -\boldsymbol{i}_{\boldsymbol{X}}\boldsymbol{\omega}$.

Similar to hamiltonian mechanics!

The second Noether theorem has a formulation in terms of $(\boldsymbol{P}, \boldsymbol{\omega})$.
Define $\qquad \boldsymbol{\Gamma} : \boldsymbol{D}(\boldsymbol{P}) \ni \boldsymbol{X} \mapsto \boldsymbol{i}_{\boldsymbol{X}}\boldsymbol{\omega} \in \boldsymbol{\Lambda}^1(\boldsymbol{P})$

In hamilt. mech.: degeneracy distrib. of presympl. form $=$ $\langle$gauge symm.$\rangle$

$\ker \boldsymbol{\Gamma}$ ($=$ degeneracy distribution of $\boldsymbol{\omega}$) $=$ $\langle$gauge symmetries of $(\pi, \boldsymbol{S})\rangle$?

Standard Definition

A *local* (or gauge) *symmetry* of $(\pi, \boldsymbol{S})$ is a $\mathcal{C}$–differential operator $G : P \longrightarrow \varkappa$, such that $G(p)$ is a Noether symmetry for any $p \in P$.

Remark

$\mathrm{im}\, G \subset \ker \boldsymbol{\Gamma}$.

(Natural) Definition

A *gauge symmetry* of $(\pi, \boldsymbol{S})$ is an element in $\ker \boldsymbol{\Gamma}$.

In hamilt. mechanics: gauge symmetries $\Leftrightarrow$ first class constraints

$\ell^{\mathcal{E}}_{\boldsymbol{dS}} = \widehat{\ell^{\mathcal{E}}_{\boldsymbol{dS}}} \Rightarrow$ If the eq. $\ell^{\mathcal{E}}_{\boldsymbol{dS}}(\chi) = 0$ is overdetermined then it is also underdetermined (i.e. constrained). Let

$$\varkappa \xrightarrow{\ell^{\mathcal{E}}_{\boldsymbol{dS}}} \widehat{\varkappa} \xrightarrow{\Delta_1} P_2 \xrightarrow{\Delta_2} \cdots$$

$$\widehat{\varkappa} \xleftarrow{\ell^{\mathcal{E}}_{\boldsymbol{dS}}} \varkappa \xleftarrow{\widehat{\Delta}_1} \widehat{P}_2 \longleftarrow \cdots$$

be a non trivial compat. complex and the adjoint complex.

$$\ell^{\mathcal{E}}_{\boldsymbol{dS}} \circ \widehat{\Delta}_1 = 0.$$

Theorem [LV]

$\ker \boldsymbol{\Gamma} = \operatorname{im} \widehat{\Delta}_1$.

Corollary [LV]

$\boldsymbol{\omega}$ is non–degenerate iff $\ell^{\mathcal{E}}_{\boldsymbol{dS}}(\chi) = 0$ is a non–constrained eq.

Suppose that ω is non–degenerate $\Rightarrow$ (as in hamiltonian mechanics) there are brackets $\{\cdot,\cdot\}$ on the space of secondary functions on $\boldsymbol{P}$.

$$\text{Let } \boldsymbol{f} \in \boldsymbol{C}^\infty(\boldsymbol{P}). \text{ Put } \boldsymbol{X_f} = \Gamma^{-1}(\boldsymbol{df}) \in \boldsymbol{D}(\boldsymbol{P})$$

Let $\boldsymbol{f},\boldsymbol{g} \in \overline{H}^{n-1}(\mathcal{F}(\mathcal{E})) \subset \boldsymbol{C}^\infty(\boldsymbol{P})$ and $\boldsymbol{H} \in \overline{H}^{n}(\mathcal{F}(\mathcal{E})) \subset \boldsymbol{C}^\infty(\boldsymbol{P})$. Put

$$\{\boldsymbol{f},\boldsymbol{g}\}_0 = -\boldsymbol{L}_{\boldsymbol{X_f}}\boldsymbol{g}, \qquad \{\boldsymbol{f},\boldsymbol{H}\}_1 = -\boldsymbol{L}_{\boldsymbol{X_f}}\boldsymbol{H}.$$

Proposition [LV]

$(\overline{H}^{n-1}(\mathcal{F}(\mathcal{E})),\{\cdot,\cdot\}_0)$ is a Lie algebra and $(\overline{H}^{n}(\mathcal{F}(\mathcal{E})),\{\cdot,\cdot\}_1)$ an its representation.

Theorem [Barnich–Henneaux–Schomblond]

$\{\cdot,\cdot\}_0$ coincide with the Peierls bracket between conservation laws.

In hamilt. mech. gauges are quotiented out via symplectic reduction.
How to define a secondary symplectic reduction?

Geometric Definition of Degeneracy Distribution

Let $\boldsymbol{X} = [X] \in \overline{H}^0(VD(\mathcal{E})) \subset \boldsymbol{D}(\boldsymbol{P})$ be a gauge symmetry. $X \in VD(\mathcal{E})$ is a standard vector field over $\mathcal{E}$. Put $\mathcal{G} = \langle X \mid [X] \in \ker \boldsymbol{\Gamma} \rangle$ and $\widetilde{\mathcal{C}} = \mathcal{C} + \mathcal{G}$.

Conjecture 1

$(\mathcal{E}, \widetilde{\mathcal{C}})$ is (locally) isomorphic to the infinite prolongation of a PDE $\widetilde{\mathcal{E}}_0$.

Put $\widetilde{\boldsymbol{P}} = \{\text{solutions of } \widetilde{\mathcal{E}}_0\}$. There is a morphism $\boldsymbol{p}^* : \boldsymbol{\Lambda}^\bullet(\widetilde{\boldsymbol{P}}) \to \boldsymbol{\Lambda}^\bullet(\boldsymbol{P})$.

Conjecture 2 on Secondary Symplectic Reduction

There exists a unique secondary 2–form $\widetilde{\boldsymbol{\omega}}$ on $\widetilde{\boldsymbol{P}}$ such that $\boldsymbol{p}^*(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{\omega}$ and $\widetilde{\boldsymbol{\omega}}$ has zero degeneracy distribution.

Bibliography

Secondary Calculus

- A. M. Vinogradov, *Introduction to Secondary Calculus*, in M. Henneaux, I. Krasil'shchik, and A. Vinogradov (Eds.), Secondary Calculus and Cohomological Physics, Contemp. Math. **219**, AMS, 1998.
- J. Krasil'shchik and A. M. Verbovetsky, *Homological Methods in Equations of Mathematical Physics*, Adv. Texts in Math., Open Education & Sciences (Opava) 1998.
- A. M. Vinogradov, *Cohomological Analysis of Partial Differential Equations and Secondary Calculus*, AMS "Transl. of Math. Monogr." series **204**, 2001.

Bibliography

Geometry of the Covariant Phase Space

G. J. Zuckerman, *Action Principles and Global Geometry*, in S. T. Yau (Ed.), Mathematical Aspects of String Theory (San Diego, 1986), World Scientific (Singapore) 1987.

J. Lee and R. Wald, *Local Symmetries and Constraints*, Journ. Math. Phys. **31** (1990) 725.

G. Barnich, M. Henneaux and S. Schomblond, *Covariant Description of the Canonical Formalism*, Phys. Rev. **D44** (1991) R939.

E. G. Reyes, *On Covariant Phase Space and the Variational Bicomplex*, Int. Journ. Theor. Phys. **43**, n. 5 (2004) 1267.