Multidimensional consistency of the (discrete) Hirota equation

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Outline

1. Multidimensional consistency
2. Desargues maps
   - Non-commutative Hirota equation and its 4D consistency
   - The Zamolodchikov equation
   - The normalization map and the Veblen map
   - The ten-term relation
3. Non-commutative map and its quantum reduction
   - Normalization map and its ultralocal/quantum reduction
   - The Veblen map and its quantum reduction
   - Quantum map with Zamolodchikov's property

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Dispersioness Hirota (Veronese web) equation

$$(\lambda_i - \lambda_j) f_{,k} f_{,ij} + (\lambda_k - \lambda_i) f_{,j} f_{,ki} + (\lambda_j - \lambda_k) f_{,i} f_{,jk} = 0$$

$$f = f(x_1, x_2, \dots, x_N), \quad f_{,i} = \frac{\partial f}{\partial x_i}, \qquad \lambda_i = \lambda_i(x_i)$$

- Veronose webs, bi-Hamiltonian systems *[Gelfand, Zakharevich 1991, 2000]*
- HR-integrable Lagrangean systems *[Ferapontov, Khusnutdinova , Tsarev 2006]*
- Einstein–Weyl geometry *[Dunajski, Kryński 2014]*
- $N > 3$ *[Kryński 2016]*

Fact

For any quadruple (i, j, k, ℓ) of distinct indices three dHVw equations in triplets (i, j, ℓ), (i, k, ℓ) and (j, k, ℓ) imply the forth equation in the triplet (i, j, k)

Variables x_ℓ, for $\ell > 3$, can be interpreted as parameters of commuting symmetries of the dHVw equation in variables (x_1, x_2, x_3). Such property of an equation (symmetric with respect to all independent variables) is called its MULTIDIMENSIONAL CONSISTENCY

Example (rather trivial)

$$u_{,i} = u_{,j}, \qquad u(x_1, x_2, x_3, \dots) = U(x_1 + x_2 + x_3 + \dots)$$

Hirota's discrete KP equation (1981)

$$\tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} + \tau_{(k)}\tau_{(ij)} = 0, \qquad i < j < k$$

$$\tau = \tau(n_1, n_2, n_3, \dots), \quad \tau_{(i)}(n_1, \dots, n_i, \dots) = \tau(n_1, \dots, n_i + 1, \dots)$$

Kadomtsev–Petviashvili equation (1970)

$$(-4v_{,t} + v_{,xxx} + 6vv_{,x})_{,x} + 3v_{,yy} = 0$$

$\boldsymbol{t} = (t_1, t_2, t_3, \dots)$ infinite sequence of its commuting symmetries (time variables), where $t_1 = x$, $t_2 = y$, and $t_3 = t$

Theorem (Miwa 1982)

For $\lambda \in \mathbb{C}$ define the collective shift $\boldsymbol{t} \mapsto \boldsymbol{t} + [\lambda]$ in all time variables by $t_k \mapsto t_k + \frac{\lambda^k}{k}$. If $v(\boldsymbol{t})$ is a solution of the KP hierarchy, then the function $\tau(\boldsymbol{t})$ defined by $v = -2(\log\tau)_{,xx}$ satisfies

$$\lambda_i(\lambda_j - \lambda_k)\tau(\boldsymbol{t} + [\lambda_i])\tau(\boldsymbol{t} + [\lambda_j] + [\lambda_k]) + \lambda_j(\lambda_k - \lambda_i)\tau(\boldsymbol{t} + [\lambda_k])\tau(\boldsymbol{t} + [\lambda_i] + [\lambda_k]) +$$

$$+ \lambda_k(\lambda_i - \lambda_j)\tau(\boldsymbol{t} + [\lambda_k])\tau(\boldsymbol{t} + [\lambda_i] + [\lambda_j]) = 0 \qquad \lambda_i, \lambda_j, \lambda_k \in \mathbb{C}$$

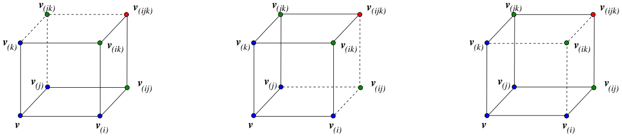
EXAMPLE

[Nijhoff 2002], [Adler, Bobenko, Suris 2003]

The discrete modified Korteweg – de Vries equation is multidimensionally consistent

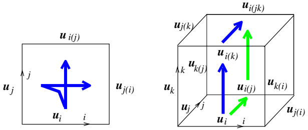
$$v_{(ij)} = v\frac{\lambda_j v_{(i)} - \lambda_i v_{(j)}}{\lambda_j v_{(j)} - \lambda_i v_{(i)}}, \qquad v = v(n_1, n_2, \ldots), \qquad \lambda_i = \lambda_i(n_i), \qquad i \neq j$$

i.e. three ways to calculate $v_{(ijk)}$ from the initial data $v, v_{(i)}, v_{(j)}, v_{(k)}$ give the same result



On the level of the edge-valued fields $u_i = \frac{v_{(i)}}{v}$ we have the system

$$u_{i(j)} = \frac{1}{u_j}\frac{\lambda_j u_i - \lambda_i u_j}{\lambda_j u_j - \lambda_i u_i}$$

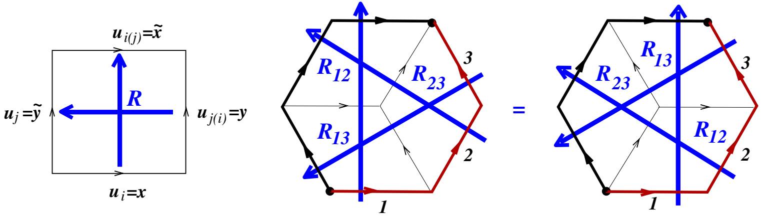


3D consistent equations and Yang–Baxter maps II

A map $\mathcal{R}\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ satisfying the relation

$$\mathcal{R}_{12}\circ\mathcal{R}_{13}\circ\mathcal{R}_{23}=\mathcal{R}_{23}\circ\mathcal{R}_{13}\circ\mathcal{R}_{12}, \qquad \text{in} \quad \mathcal{X}\times\mathcal{X}\times\mathcal{X}$$

is called Yang–Baxter map *[Drinfeld 1992]*



Observation

The *companion map* $\mathcal{R}(x,y)=(\tilde{x},\tilde{y})$ to a three dimensionally consistent edge-map is a Yang–Baxter map *[Adler, Bobenko, Suris 2004]*

$\mathrm{R}\in\mathrm{End}(\mathbb{V}\otimes\mathbb{V})$ is a solution of the quantum Yang–Baxter equation if

$$\mathrm{R}_{12}\mathrm{R}_{13}\mathrm{R}_{23}=\mathrm{R}_{23}\mathrm{R}_{13}\mathrm{R}_{12}, \qquad \text{in} \quad \mathbb{V}\otimes\mathbb{V}\otimes\mathbb{V}$$

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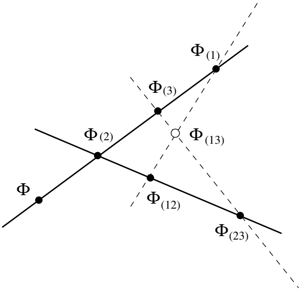
Maps $\phi : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D})$, such that the points $\phi(n)$, $\phi_{(i)}(n)$ and $\phi_{(j)}(n)$ are collinear, for all $n \in \mathbb{Z}^N$, $i \neq j$; here $\mathbb{D}$ is a division ring

NOTATION: $\phi_{(i)}(n_1, \dots, n_i, \dots, n_N) = \phi(n_1, \dots, n_i + 1, \dots, n_N)$



In homogeneous coordinates $\boldsymbol{\Phi} : \mathbb{Z}^N \to \mathbb{D}_*^{M+1}$

$$\boldsymbol{\Phi} + \boldsymbol{\Phi}_{(i)} A_{ij} + \boldsymbol{\Phi}_{(j)} A_{ji} = 0, \qquad i \neq j, \qquad \text{where} \quad A_{ij} : \mathbb{Z}^N \to \mathbb{D}_*$$

The compatibility condition of the above linear system reads

$$A_{ij}^{-1} A_{ik} + A_{kj}^{-1} A_{ki} = 1,$$
$$A_{ik(j)} A_{jk} = A_{jk(i)} A_{ik}, \qquad i, j, k \quad \text{distinct}$$

In homogeneous coordinates of the projective space, and in a suitable gauge

$$\boldsymbol{\Phi}_{(i)} - \boldsymbol{\Phi}_{(j)} = \boldsymbol{\Phi} U_{ij}, \qquad 1 \leq i \neq j,$$

whose compatibility is

$$U_{ij} + U_{ji} = 0, \quad U_{ij} + U_{jk} + U_{ki} = 0, \quad U_{ki} U_{kj(i)} = U_{kj} U_{ki(j)} \quad i, j, k \text{ distinct}$$

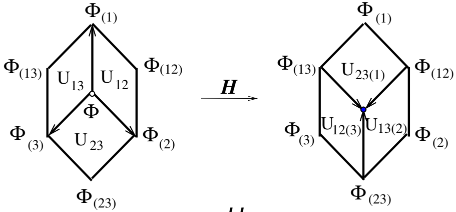
[Nimmo 2006]

When $\mathbb{D}$ is commutative then the functions U_{ij} can be parametrized in terms of a single potential $\tau : \mathbb{Z}^N \to \mathbb{D}$

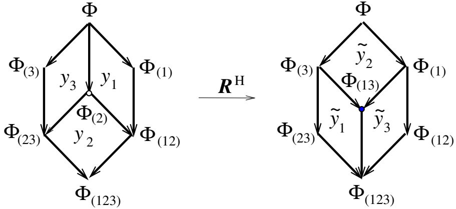
$$U_{ij} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}}, \qquad 1 \leq i < j \leq N$$

and the nonlinear system reads *[Hirota 1981], [Miwa 1982]*

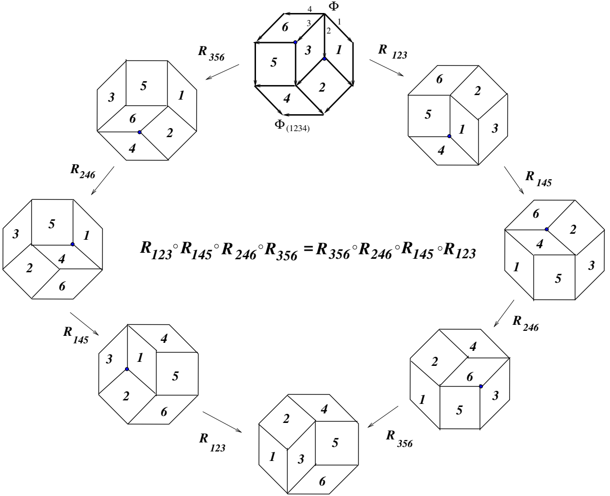
$$\tau_{(i)} \tau_{(jk)} - \tau_{(j)} \tau_{(ik)} + \tau_{(k)} \tau_{(ij)} = 0, \qquad 1 \leq i < j < k$$



$$y_1 = U_{12}, \qquad y_2 = U_{13(2)}, \qquad y_3 = U_{23},$$
$$\tilde{y}_1 = U_{12(3)}, \qquad \tilde{y}_2 = U_{13}, \qquad \tilde{y}_3 = U_{23(1)}.$$



$$\tilde{y}_1 = (y_1 + y_3)^{-1} y_1 y_2,$$
$$\tilde{y}_2 = y_1 + y_3,$$
$$\tilde{y}_3 = (y_1 + y_3)^{-1} y_3 y_2.$$

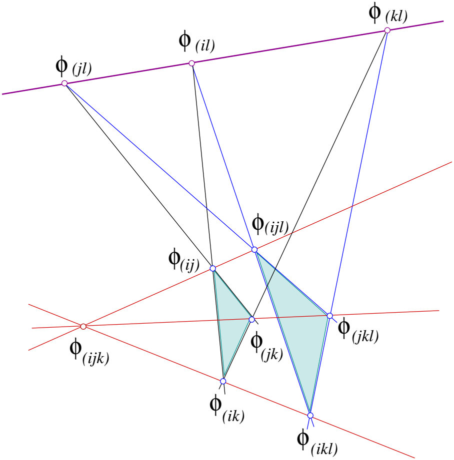


$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123}$$

PROPOSITION *[Kashaev, Korepanov, Sergeev 1998], [AD, Kashaev]*

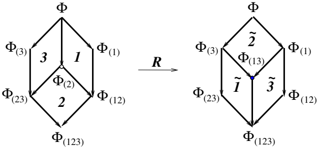
The companion to the Hirota map satisfies the functional Zamolodchikov equation

Desargues (10_3) configuration



4D consistency of Desargues maps is a consequence of the Desargues theorem

$$\begin{aligned} \mathbf{\Phi}_{(2)} - \mathbf{\Phi}_{(1)} &= (\mathbf{\Phi} - \mathbf{\Phi}_{(1)})x_1, \\ \mathbf{\Phi}_{(23)} - \mathbf{\Phi}_{(12)} &= (\mathbf{\Phi}_{(2)} - \mathbf{\Phi}_{(12)})x_2, \\ \mathbf{\Phi}_{(3)} - \mathbf{\Phi}_{(2)} &= (\mathbf{\Phi} - \mathbf{\Phi}_{(2)})x_3 \end{aligned} \qquad \begin{aligned} \mathbf{\Phi}_{(23)} - \mathbf{\Phi}_{(13)} &= (\mathbf{\Phi}_{(3)} - \mathbf{\Phi}_{(13)})\tilde{x}_1, \\ \mathbf{\Phi}_{(3)} - \mathbf{\Phi}_{(1)} &= (\mathbf{\Phi} - \mathbf{\Phi}_{(1)})\tilde{x}_2, \\ \mathbf{\Phi}_{(13)} - \mathbf{\Phi}_{(12)} &= (\mathbf{\Phi}_{(1)} - \mathbf{\Phi}_{(12)})\tilde{x}_3 \end{aligned}$$



Proposition *[AD, Kashaev]*

The birational map $R^A\colon (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$

$$\tilde{x}_1 = [x_3 + x_1(1 - x_3)]^{-1} x_1 x_2, \qquad \tilde{x}_2 = x_3 + x_1(1 - x_3),$$
$$\tilde{x}_3 = 1 + (x_2 - 1)\left[(1 - x_1)x_3 + x_1(1 - x_2)\right]^{-1}(x_3 + x_1(1 - x_3)),$$

satisfies Zamolodchikov's condition

$$R^A_{123} \circ R^A_{145} \circ R^A_{246} \circ R^A_{356} = R^A_{356} \circ R^A_{246} \circ R^A_{145} \circ R^A_{123},$$

Decomposition into pentagonal maps

The birational map $R^A\colon (x_1, x_2, x_3) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ can be decomposed as

$$R^A = P_{23} \circ V_{12} \circ N_{13}$$

where

- the (affine) normalization map $N\colon (x_1, x_2) \to (x_1', x_2')$

$$x_1' = (x_2 + x_1 - x_1 x_2)^{-1} x_1, \qquad x_2' = x_2 + x_1 - x_1 x_2,$$

N satisfies the pentagonal condition

$$N_{12} \circ N_{13} \circ N_{23} = N_{23} \circ N_{12}$$

- the Veblen map $V\colon (x_1, x_2) \to (\bar{x}_1, \bar{x}_2)$

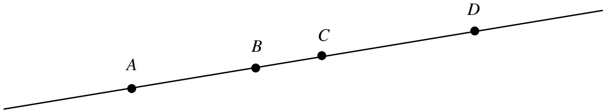
$$\bar{x}_1 = x_1 x_2, \qquad \bar{x}_2 = (1 - x_1) x_2 (1 - x_1 x_2)^{-1},$$

V satisfies the reversed pentagonal condition

$$V_{23} \circ V_{13} \circ V_{12} = V_{12} \circ V_{23}$$

- transposition $P\colon (x_1, x_2) \to (x_2, x_1)$

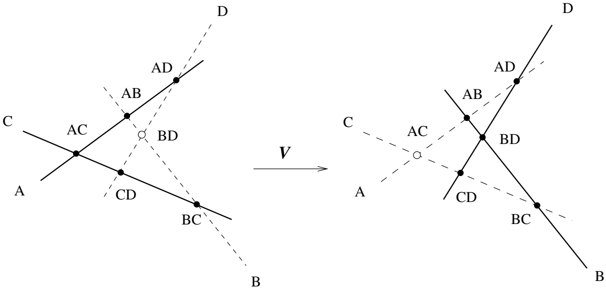
Given four collinear points *A*, *B*, *C* and *D*, consider two pairs of linear relations between their *non-homogeneous* coordinates



$$\begin{aligned} \phi_A - \phi_B &= (\phi_C - \phi_B)x_1 \\ \phi_D - \phi_A &= (\phi_C - \phi_A)x_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_A - \phi_B &= (\phi_D - \phi_B)x_1' \\ \phi_D - \phi_B &= (\phi_C - \phi_B)x_2'. \end{aligned}$$

The (affine) normalization map is a consequence of that change

$$x_1' = (x_2 + x_1 - x_1 x_2)^{-1} x_1, \qquad x_2' = x_2 + x_1 - x_1 x_2,$$



The Veblen $(6_2, 4_3)$ configuration

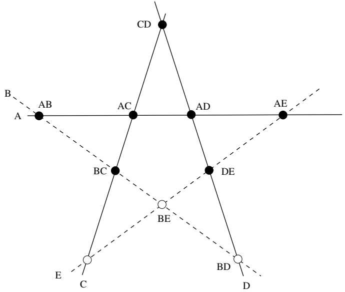
$$\begin{aligned} \phi_{AC} - \phi_{AD} &= (\phi_{AB} - \phi_{AD}) x_1 \\ \phi_{BC} - \phi_{CD} &= (\phi_{AC} - \phi_{CD}) x_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \phi_{BC} - \phi_{BD} &= (\phi_{AB} - \phi_{BD}) \bar{x}_1 \\ \phi_{BD} - \phi_{CD} &= (\phi_{AD} - \phi_{CD}) \bar{x}_2 \end{aligned}$$

$$\bar{x}_1 = x_1 x_2, \qquad \bar{x}_2 = (1 - x_1) x_2 (1 - x_1 x_2)^{-1},$$

THEOREM *[Kashaev, Sergeev 1998]*

Given a solution N of the functional pentagon equation, and given a solution V of the reversed functional pentagon equation on the same set $\mathcal{X}$, then the map $R = P_{23} \circ V_{12} \circ N_{13}$ satisfies the Zamolodchikov equation, provided

$$V_{13} \circ N_{12} \circ V_{14} \circ N_{34} \circ V_{24} = N_{34} \circ V_{24} \circ N_{14} \circ V_{13} \circ N_{12}.$$



Start from seven points (black circles) of the star configuration $(10_2, 5_4)$ AND FOUR CORRESPONDING LINEAR RELATIONS there are two distinct ways to complete the configuration using the normalization and Veblen flips

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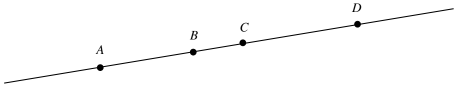
The normalization map in homogeneous coordinates



$$\begin{aligned} \phi_A &= \phi_C x_1 + \phi_B y_1, \\ \phi_D &= \phi_C x_2 + \phi_A y_2, \end{aligned} \qquad \begin{aligned} \phi_A &= \phi_D x_1' + \phi_B y_1' \\ \phi_D &= \phi_C x_2' + \phi_B y_2' \end{aligned}$$

Proposition *[AD, Sergeev 2014]*

The normalization map $N\colon [(x_1, y_1), (x_2, y_2)] \to [(x_1', y_1'), (x_2', y_2')]$

$$\begin{aligned} x_1' &= (x_2 + x_1 y_2)^{-1} x_1, \\ x_2' &= x_2 + x_1 y_2, \end{aligned} \qquad \begin{aligned} y_1' &= y_1 x_1^{-1} x_2 (x_2 + x_1 y_2)^{-1} x_1 \\ y_2' &= y_1 y_2 \end{aligned}$$

satisfies the pentagon relation

$$N_{12} \circ N_{13} \circ N_{23} = N_{23} \circ N_{12}$$

OBSERVATION

In the commutative case the normalization map provides a Poisson automorphism of the field of rational functions $\Bbbk(x_1, y_1, x_2, y_2)$ equipped with the Poisson structure

$$\{x_i, y_i\} = x_i y_i, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = 0, \qquad i \neq j$$

PROPOSITION

[AD, Sergeev 2014]

Assume that the normalization map preserves the ultra-locality conditions

$$x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i, \qquad x_i y_j = y_j x_i, \qquad i \neq j$$

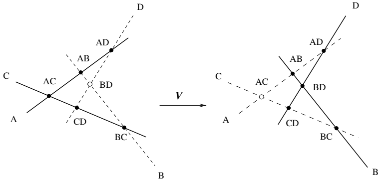
then (under certain general position conditions) there exists a central non-zero element q such that

$$y_i x_i = q x_i y_i, \qquad y_i' x_i' = q x_i' y_i' \qquad i = 1, 2$$

REMARK Ultralocality in the case of the normalization map implies Weyl commutation relations

The full Veblen map

$$\begin{aligned}\phi_{AC} &= \phi_{AB} x_1 + \phi_{AD} y_1 \\ \phi_{BC} &= \phi_{AC} x_2 + \phi_{CD} y_2\end{aligned} \quad \text{and} \quad \begin{aligned}\phi_{BC} &= \phi_{AB} \bar{x}_1 + \phi_{BD} \bar{y}_1 \\ \phi_{BD} &= \phi_{AD} \bar{x}_2 + \phi_{CD} \bar{y}_2\end{aligned}$$



Due to the gauge freedom in rescaling ϕ_{BD} the coefficient $\bar{y}_1 = \lambda$ is free

Proposition

[AD, Sergeev 2014]

The Veblen map $V^\lambda \colon [(x_1, y_1), (x_2, y_2)] \to [(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)]$

$$\begin{aligned}\bar{x}_1 &= x_1 x_2, & \bar{y}_1 &= \lambda \\ \bar{x}_2 &= y_1 x_2 \lambda^{-1}, & \bar{y}_2 &= y_2 \lambda^{-1}\end{aligned}$$

satisfies the pentagon relation $V^C_{23} \circ V^B_{13} \circ V^A_{12} = V^H_{12} \circ V^G_{23}$ provided $B = H$ and $G = CB$

When the gauge factors are functions of the corresponding arguments then the above relations between the factors become complicated functional equations

OBSERVATION

In the commutative case the Veblen map V^λ with the gauge function $\lambda = \alpha y_1 + \beta x_1 y_2$, where $\alpha, \beta \in \Bbbk$ are parameters, provides a Poisson automorphism of the field of rational functions $\Bbbk(x_1, y_1, x_2, y_2)$ equipped with the Poisson structure

$$\{x_i, y_i\} = x_i y_i, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = 0, \qquad i \neq j$$

Moreover, the map satifies the pentagon relation $V_{23}^C \circ V_{13}^B \circ V_{12}^A = V_{12}^H \circ V_{23}^G$ provided

$$\alpha_G = \alpha_B \alpha_C, \quad \alpha_H = \alpha_A \alpha_B, \quad \beta_A = \alpha_C \beta_H, \quad \beta_B = \beta_G \beta_H, \quad \beta_C = \alpha_A \beta_G$$

PROPOSITION

With the same form of the gauge function the Veblen map preserves the ultralocal Weyl commutation relations

$$y_i x_i = q x_i y_i, \qquad x_i x_j = x_j x_i, \qquad y_i y_j = y_j y_i, \qquad x_i y_j = y_j x_i, \qquad i \neq j$$

Moreover such map satisfies pentagonal condition with the same relations as above between the parameters

Ultralocal/quantum map with Zamolodchikov's property

The map $R^{\lambda} = P_{23} \circ V^{\lambda}_{12} \circ N_{13}$ given explicitly by

$$\tilde{x}_1 = (x_3 + x_1 y_3)^{-1} x_1 x_2 \qquad \tilde{y}_1 = (\alpha y_1 x_3 + \beta x_1 y_2)(x_3 + x_1 y_3)^{-1}$$
$$\tilde{x}_2 = x_3 + x_1 y_3 \qquad \tilde{y}_2 = y_1 y_3$$
$$\tilde{x}_3 = y_1 x_2 x_3 (\alpha y_1 x_3 + \beta x_1 y_2)^{-1} \qquad \tilde{y}_3 = y_2 (x_3 + x_1 y_3)(\alpha y_1 x_3 + \beta x_1 y_2)^{-1}$$

preserves ultralocal Weyl commutation relations, and satisfies Zamolodchikov's condition

$$R^{H}_{123} \circ R^{G}_{145} \circ R^{F}_{246} \circ R^{E}_{356} = R^{D}_{356} \circ R^{C}_{246} \circ R^{B}_{145} \circ R^{A}_{123},$$

provided the parameters of the gauge functions satisfy

$$\beta_B = \beta_F \beta_H, \quad \alpha_C = \alpha_E \alpha_G, \quad \alpha_F = \alpha_B \alpha_D, \quad \beta_G = \beta_A \beta_C$$

$$\alpha_H \beta_F = \beta_D \alpha_B, \quad \beta_E \alpha_G = \beta_C \alpha_A, \quad \alpha_A \alpha_B = \alpha_G \alpha_H, \quad \beta_A \alpha_E = \alpha_D \beta_H$$

Remark

The above Zamolodchikov-type map can be realized by a unitary operator (in appropriate Hilbert space) whose definition involves the non-compact quantum dilogarithm function.

CONCLUSION

- The possibility of adding to an integrable equation arbitrary number of its copies involving additional independent variables provides new meaning to its commuting symmetries *[AD, Santini 1997], [AD, Santini, Mañas 2000]*
- Multidimensional consistency of 2D discrete equations can serve as integrability detector *[Nijhoff 2002], [Adler, Bobenko, Suris 2003]*
- Zamolodchikov's equation as multidimensional generalization of the Yang–Baxter equation *[Zamolodchokov 1981], [Kashaev, Korepanov, Sergeev 1998]*
- The Hirota equation in integrable systems theory *[Hirota 1981], [Miwa 1982], ...*
- The non-commutative Hirota equation, its geometric meaning, and its A_N-type affine Weyl group symmetry *[Nimmo 2006], [AD 2010], [AD 2011]*
- Pentagonal maps in construction of solutions to the Zamolodchikov equation (classical, non-commutative, quantum) *[Kashaev, Sergeev 1998], [AD, Sergeev 2014]*