Conformal geodesics, and integrability

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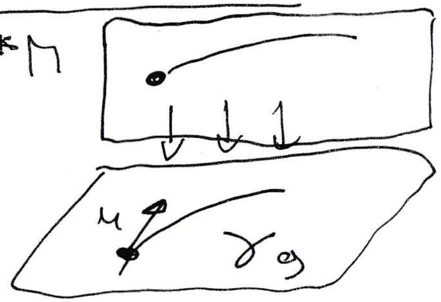
D - Paul Tod "conformal geodesics on gravitational instantons".
arXiv : 1906.08375

§1 Geodesics

T^*M



M

(M, g) (pseudo) Riemannian mfd.

$\omega = \sum_i dp_i \wedge dx^i \;;\; H = \frac{1}{2} \|p\|_g^2$

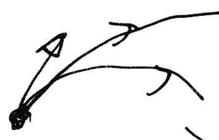
$\nabla_u u = 0 \;,\; g(u,u) = 1.$

First integrals = Killing vectors / Killing tensors ; $K : T^*M \to \mathbb{R}$

$K = K^{ij \dots k} p_i p_j \dots p_k \;;\; i,j = 1, \dots, 4 \;:\; \{H, K\} = 0.$

Integrable geodesic flow : 4 first integrals in involution

§ 2. Conformal geodesics ; $\hat{g} = f^2 g$; $f : M \to \mathbb{R}^+$ ②



γ_g geodesics do not map to geodesics.

$\gamma_{\hat{g}}$ unless $\|u\|_g^2 = 0$.

Conformal geodesics (CG) = preferred class of curves in conformal geometry (Yano 1950s, Penrose 1960s, ...)

Specify: position, unit tangent u, acceleration $a = \nabla_u u$

3rd order ODEs:

$$\boxed{\nabla_u a = -\left(\|a\|^2 + g(u, L(u))\right)u + L(u)} \quad (*)$$

Tod 2012.

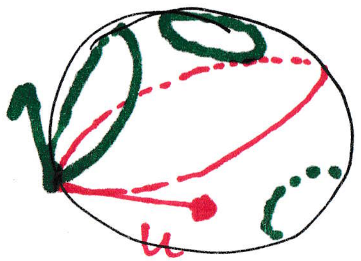
Schouten endomorphism $L^i{}_j = \frac{1}{2}\left(R^i{}_j - \frac{1}{6} R\, \delta^i{}_j\right)$

$R^i{}_j$: Ricci ; R: R. scalar

Assume (M, g) Einstein $R_{ij} = \frac{1}{4} R g_{ij}$ (3)

$(*) \longrightarrow \nabla_u a = -\|a\|^2 u \qquad (**) \begin{cases} \|a\|^2 \text{ const along} \\ g(u, a) = 0. \end{cases}$

Example 1 S^4



6-parameter family of great circles.

CG = all circles.

9 -''- of circles..

[a.k.a conformal circles].

Example (M, g) Kähler - Einstein ; $J : TM \to TM$, $J^2 = -\mathrm{Id}$.

$\underbrace{\nabla_u u}_{a} = \underbrace{e}_{\text{const}} J(u)$ "Magnetic" geodesics

$\nabla_u a = e\, J(J(u)) = -e^2 u, \qquad e = |a|$

Magnetic geodesics $\to CG$

§3. Integrability

Conformal Killing-Yano forms $Y \in \Lambda^2(M)$ [CKY]

$$\nabla_i Y_{jk} = \nabla_{[i} Y_{jk]} - 2 g_{i[j} K_{k]} \text{ for some } K \in \Lambda^1.$$

Tod 2012, Gover, Snell, Taghavi-Chabert 2018.

$$\mathcal{Y} = Y_{ij} u^i a^j - K_i u^i \quad \text{first integral of } (*).$$

Two non-conf flat integrable examples.

1) $\mathbb{CP}^2$, Fubini-Study ($\exists$ 5 CKYs. D-Tod 2010)
Adachi-Moede-Udagawa 1995

2) 1st (possibly only) non-symmetric example (D-Tod 2019)
Taub-Nut gravitational instanton

$$g = \left(1+\frac{m}{r}\right)\left(dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)\right) + \left(1+\frac{m}{r}\right)^{-1} (d\psi + m\cos\theta \, d\phi)^2$$

* regular at $r=0$ if $\psi \sim \psi + 4\pi m$.
* ALF (S^2 bundle over S^1 at ∞)
* Hyper-Kähler (so Ricci-flat)

Thm (D–Tod 2015) Conf. a flow (*) on Taub-Nut is completely integrable. Hamilton-Jacobi eq separable ~~be~~ elliptic functions.

$\exists\ I_\alpha : TM \to TM$; $\begin{cases} I_1^2 = I_2^2 = I_3^2 = -Id = I_1 I_2 I_3 \\ \nabla I_\alpha = 0, \quad \alpha = 1, 2, 3. \end{cases}$

$\Omega_\alpha(v, w) = g(v, I_\alpha(w))$. 3 1st integrals. Elliminate a

$$\nabla_u u = |a|\, J(u)\ ; \quad J = c_1 I_1 + c_2 I_2 + c_3 I_3, \quad |c| = 1.$$

(***)

Magnetic geodesic flow in self-dual magnetic field $F = g(J$

Need 4 1st integrals Poisson commuting w.r.t.

"changed" symplectic form $\omega = dP_i \wedge dq^i + |a|\, F$

$q^i = (r, \phi, \Theta, \Psi)$.

(b)

- $\underbrace{SU(2)}_{\text{rotates } I_\alpha} \times \underbrace{U(1)}_{\text{fixes } I_\alpha}$ isometry ; generates $K = \frac{\partial}{\partial \Psi}$; $\mathcal{L}_K F = 0$.

$\downarrow$ w.l.o.g.

set $F = \Omega_3$ preserved by $L = \left(\frac{\partial}{\partial \phi}\right)^2$. $\quad \Big\{ F = d\Phi$

- Hamiltonian $\mathcal{H} = \frac{1}{2}\|p\|_g^2$

- CKY. ; $Y = -\frac{(r+m)^3}{m}\left[d\left[\left(1+\frac{m}{r}\right)^{-1}(d\Psi + m\cos\theta\, d\phi)\right]\right]$

self-dual

$\tilde{\alpha}_1 = P_\Psi - |a| r\cos\theta$

$\tilde{\alpha}_2 = P_\phi - |a|\left(mr + \frac{1}{2}r^2 \sin^2\theta\right)$

$\tilde{\alpha}_3 = \frac{1}{2}\|P\|_g^2$

$\tilde{\alpha}_4 = |a|\, Y^{ij} F_j{}^k P_i P_k - 2\mathcal{H} K^i P_i$

$\{I_i, I_j\} = 0$.

§4 Separability

Magnetic Hamilton-Jacobi: $S : M \to \mathbb{R}$, $F = d\Phi$

$$\mu^2 = \|dS - |q|\,\Phi\|^2 \xrightarrow[\text{find } S]{} \dot{q}^i = g^{ij}\nabla_j S$$

separable in parabolic coordinates $\begin{cases} \eta = r(1+\cos\theta) \\ \zeta = r(1-\cos\theta) \end{cases}$

$$S = E \cdot \Psi + J\phi + A(\zeta) + B(\eta)\,; \quad E, J, \mu \text{ consts}$$

$$\int \frac{d\eta}{U(\eta, E, q)} = \int \frac{d\zeta}{U(\zeta, -E, -q)}\,; \quad q = \text{separation constant}$$

$$U(x, E, q) = \sqrt{u_0 + u_1 x + \dots + u_4 x^4}$$

$u_0, \dots, u_4$ consts

REFERENCES

- Bailey & Eastwood (1990) Conformal circles and parametrisations of curves in conformal manifolds Proc. Am. Math. Soc. 108.

- Tod (2012) Some examples of the behaviour of conformal geodesics. J. Geom. Phys 62.

- Gover, Snell & Taghavi-Chabert (2018) Distinguished curves and integrability ...; arXiv:1806.09830

- Gibbons & Ruback (1988) The hidden symmetries of multicentre metrics. CMP 115.

- Dunajski & Tod (2014) Self-Dual Conformal Gravity CMP 331.

- Adachi, Maeda, Udagawa (1995) Circles in a complex projective space Osaka J. Math. 32.