



Homogeneous hypersurfaces

Michael Eastwood

[based on joint work with Vladimir Ezhov]

University of Adelaide

A classification problem

$$\Sigma \hookrightarrow G/H$$

where $G/H =$ your favourite homogeneous space

Examples of G/H

$$\mathbb{R}^n = (\mathrm{SO}(n) \ltimes \mathbb{R}^n) / \mathrm{SO}(n) = \text{Euclidean space}$$

$$\mathbb{R}^n = (\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n) / \mathrm{GL}(n, \mathbb{R}) = \text{affine space}$$

$$\mathbb{RP}_n = \mathrm{SL}(n+1, \mathbb{R}) / P = \text{real projective space}$$

$$S^n = \mathrm{SO}(n+1, 1) / P = \text{conformal space}$$

$$\mathbb{C}^n = (\mathrm{GL}(n, \mathbb{C}) \ltimes \mathbb{C}^n) / \mathrm{GL}(n, \mathbb{C}) = \text{complex affine space}$$

Today: real or complex (equi-)affine space

Curves in the real affine plane

- straight line
- circle (or ellipse)
- parabola
- $Y = X^\alpha$
- $r = e^{\alpha\theta}$ (spira mirabilis (Jacob Bernoulli))
- $Y = e^X$
- $Y = X \log X$

Normal forms $y = 0$ $y = x^2$ $y = x^2 \pm x^4 + bx^5 + \dots$

$$Y = X \log X \rightsquigarrow y = x^2 + x^4 + \frac{8}{5}x^5 + \dots$$

$$Y = e^X \rightsquigarrow y = x^2 - x^4 + \frac{2\sqrt{2}}{5}x^5 + \dots$$

$b^2 = \text{invariant}$

Non-degenerate equi-affine surfaces in \mathbb{C}^3

- $Z = XY$
 - $Z^2 = XY + 1$
 - $XYZ = 1$
 - $X^2(Z + Y^2)^3 = 1$
 - $Z = XY + X^3$ (Cayley surface)
- } Guggenheimer 1963
- $Z = XY + \log X$ Nomizu & Sasaki 1991

Equi-affine symmetries of the Cayley surface

$$\begin{array}{ll} X \mapsto X & X \mapsto X + s \\ Y \mapsto Y + t & Y \mapsto Y - 3sX - \frac{3}{2}s^2 \quad (\text{Abelian}) \\ Z \mapsto Z + tX & Z \mapsto Z + sY - \frac{3}{2}s^2X - \frac{1}{2}s^3 \end{array}$$

Infinitesimal symmetries

Cayley surface $z = xy + x^3$

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y + t \\ z &\mapsto z + tx \end{aligned} \quad x \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x &\mapsto x + s \\ y &\mapsto y - 3sx - \frac{3}{2}s^2 \\ z &\mapsto z + sy - \frac{3}{2}s^2x - \frac{1}{2}s^3 \end{aligned} \quad -3x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + \frac{\partial}{\partial x} \quad \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$z = xy + \dots = F(x, y)$$

$$\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, -1 \right] \left[M \begin{bmatrix} x \\ y \\ F(x, y) \end{bmatrix} + \beta \right] = 0$$

symmetry algebra $\subset \mathfrak{g}$

Normal forms

Non-degenerate

$$z = x^2 + y^2 + \dots \quad \text{or} \quad z = xy + \dots$$

Blaschke normal (Leichtweiß 1989)

$$z = g_{ij}x^i x^j + a_{ijk}x^i x^j x^k + \dots$$

$$x^i \mapsto x^i + r^i z \quad \rightsquigarrow \quad a^j_{jk} \mapsto a^j_{jk} + \frac{2(n+2)}{3} r_k \quad \rightsquigarrow \quad a_{ijk} \text{ trace-free}$$

Pick invariant

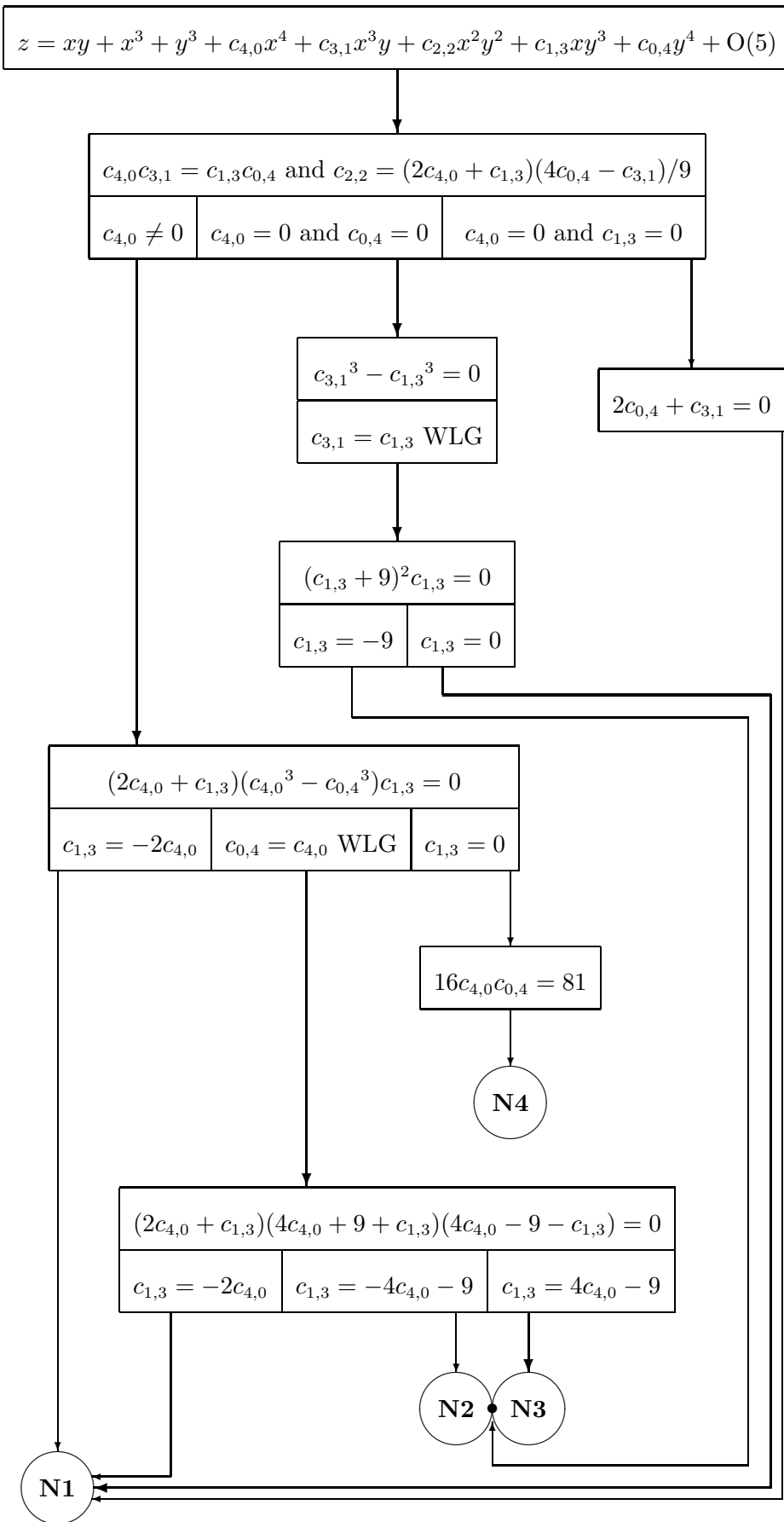
$$a_{ijk}a^{ijk} \quad \text{e.g.} \quad z = xy + ax^3 + by^3 + \dots \quad \rightsquigarrow \quad ab$$

$$\therefore \quad z = xy + x^3 + y^3 + \dots \quad \text{or} \quad z = xy + x^3 + \dots$$

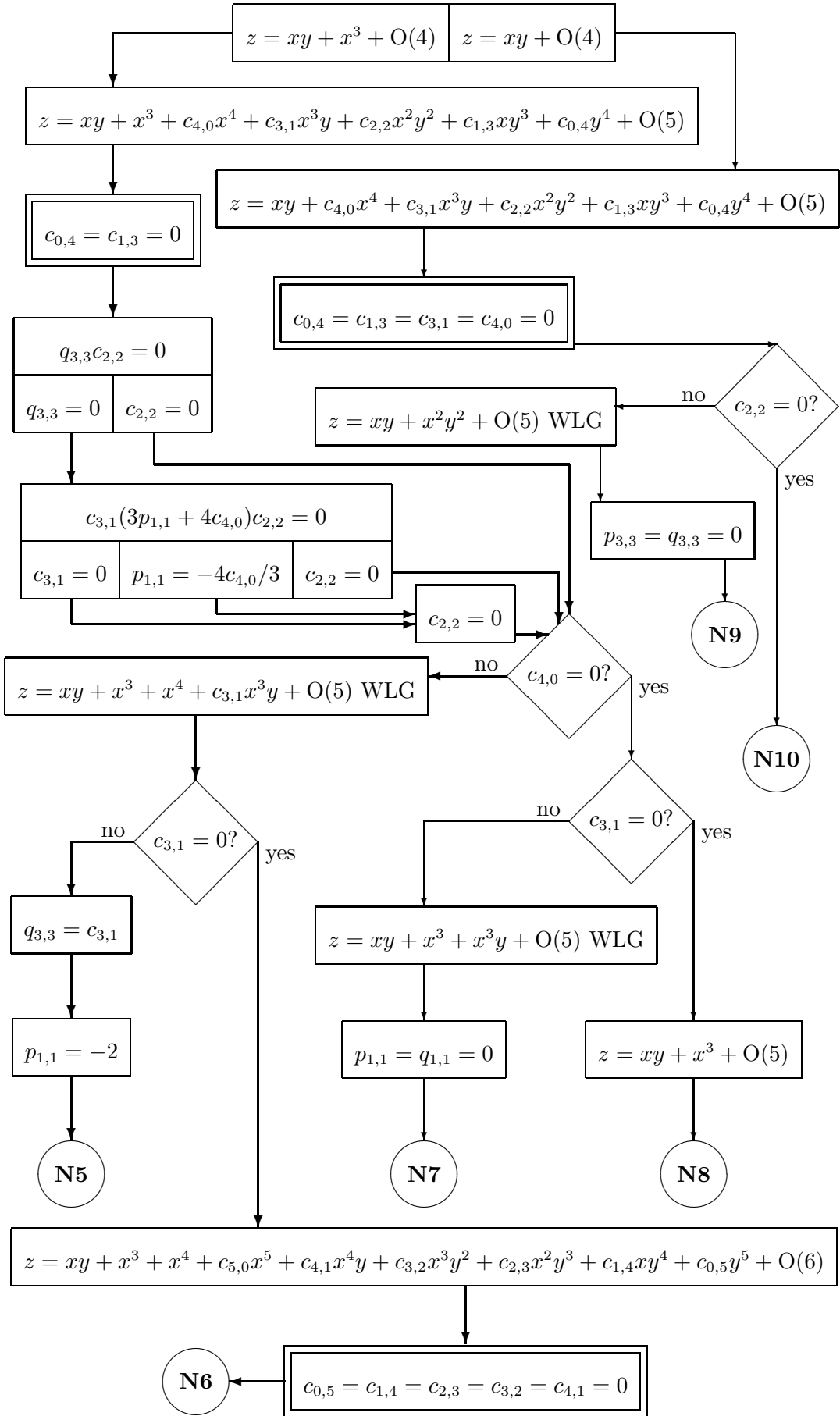
Affine homogeneous surfaces in \mathbb{C}^3

- N1 $z = xy + x^3 + y^3 + ax^4 - 2bx^3y - 2axy^3 + by^4 + O(5)$
- N2 $z = xy + x^3 + y^3$
 $+bx^4 - (4b+9)x^3y - \frac{1}{9}(2b+9)(8b+9)x^2y^2 - (4b+9)xy^3 + by^4 + O(5)$
- N3 $z = xy + x^3 + y^3$
 $+bx^4 + (4b-9)x^3y + (6b-9)x^2y^2 + (4b-9)xy^3 + by^4 + O(5)$
- N4 $z = xy + x^3 + y^3 + \frac{9}{4}bx^4 + \frac{9}{2}x^2y^2 + \frac{9}{4}(1/b)y^4 + O(5)$ where $b \neq 0$
- N5 $z = xy + x^3 + x^4 + bx^3y + O(5)$ where $b \neq 0$
- N6 $z = xy + x^3 + x^4 + bx^5 + O(6)$
- N7 $z = xy + x^3 + x^3y + O(5)$ (the Archimedean screw)
- N8 $z = xy + x^3 + O(5) = xy + x^3$ (the Cayley surface)
- N9 $z = xy + x^2y^2 + O(5) = (1 - \sqrt{1 - 4xy})/2$ (hyperboloid)
- N10 $z = xy + O(5) = xy$
- D1 $z = x^2 + x^2y + x^3y + x^2y^2 + x^5 + 3x^3y^2 + x^2y^3 + O(6)$ (\leftarrow twisted cubic)
- D2 $z = x^2 + x^4 + ax^5 + O(6) =$ a function of x alone (cylinder)
- D3 $z = x^2 + x^2y + x^2y^2 + x^5 + x^2y^3 + 4x^5y + x^2y^4 + ax^7 + 10x^5y^2 + x^2y^5 + O(8)$ (cone)
- D4 $z = x^2 + x^2y + x^2y^2 + x^2y^3 + O(6) = x^2/(1-y)$ (cone)
- D5 $z = x^2 + O(5) = x^2$ (cylinder)
- F $z = O(3) = 0$

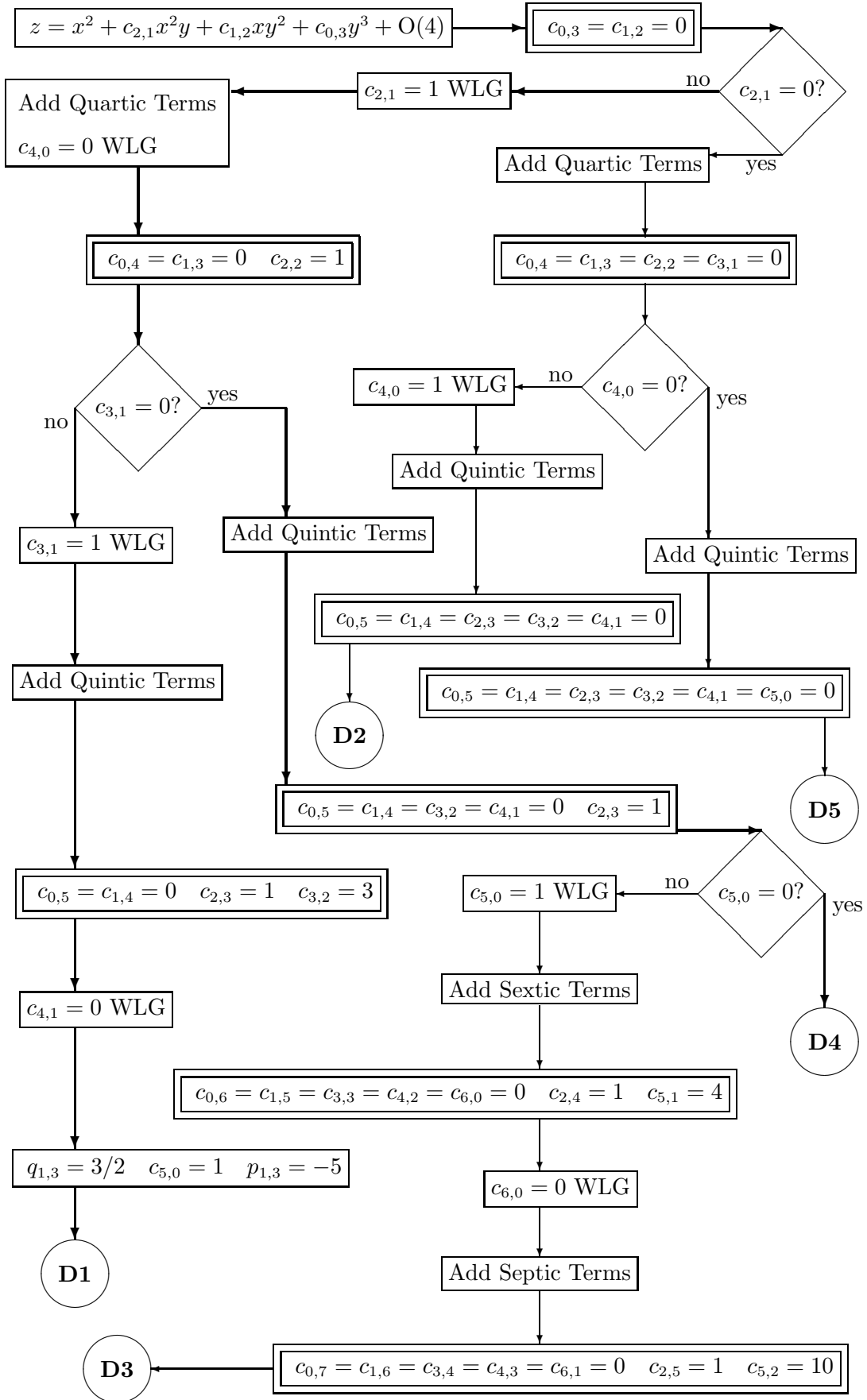
Non-degenerate with non-vanishing Pick invariant



Non-degenerate with vanishing Pick invariant



Degenerate



Non-degenerate equi-affine hypersurfaces in \mathbb{C}^4

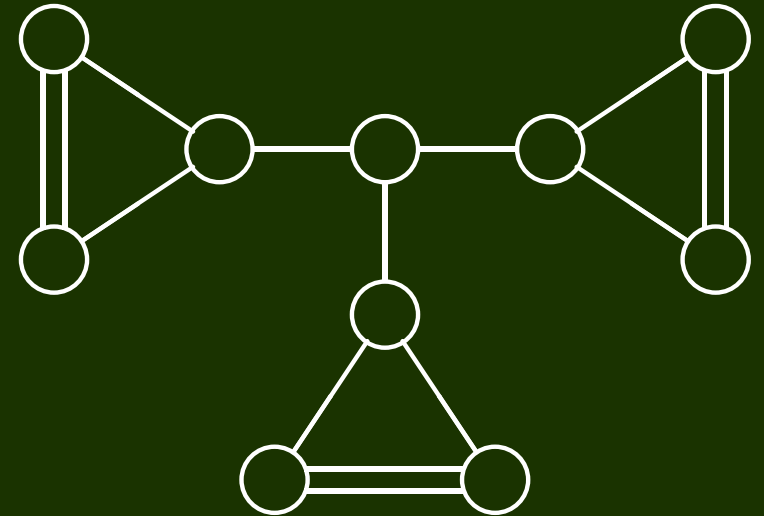
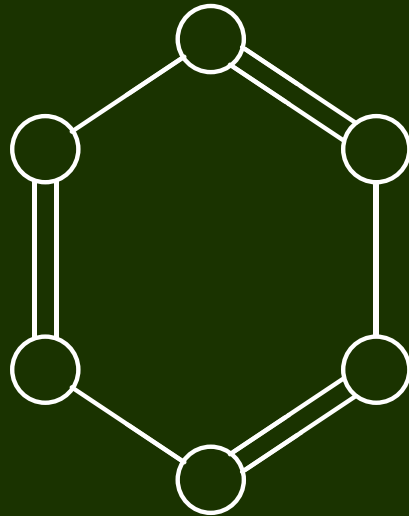
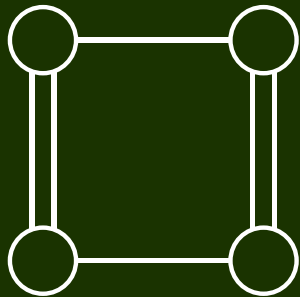
	Equation	Basepoint
I1	$W = XY + Z^2$	$(0, 0, 0, 0)$
I2	$W^2 = XY + Z^2 + 1$	$(1, 0, 0, 0)$
I3	$W = XY + Z^2 + X^3$	$(0, 0, 0, 0)$
I4	$W = XY + Z^2 + \log X$	$(0, 1, 0, 0)$
I5	$W(XY + Z)^2 = 1$	$(1, 0, 0, 1)$
I6	$W^2(XY + Z^2)^3 = 1$	$(1, 0, 0, 1)$
A1	$WXYZ = 1$	$(1, 1, 1, 1)$
A2	$WZ + WY^2 + X^2Z + X^2Y^2 = 1$	$(1, 0, 0, 1)$
A3	$6WXYZ - 4X^3Z - 3X^2Y^2 + W^2Z^2 + 4WY^3 = 1$	$(1, 0, 0, 1)$
A4	$W = XY + Z^2 + X^2Z + \alpha X^4$	$(0, 0, 0, 0)$

Non-degenerate equi-affine hypersurfaces in \mathbb{C}^4

	Equation	Basepoint
B1	$WZ^2 = Z + X^3 + XYZ$	$(1, 0, 0, 1)$
B2	$(W + XY + X^3)^2Z = 1$	$(1, 0, 0, 1)$
B3	$(WZ + Y^2 + X^2Z)^4 = Z$	$(1, 0, 0, 1)$
B4	$(W^2 + X^2)^2(Z + Y^2)^3 = 1$	$(1, 0, 0, 1)$
B5	$(W + X^2Z + XY)^2Z = 1$	$(1, 0, 0, 1)$
B6	$(W + XY)^2(Z + X^2) = 1$	$(1, 0, 0, 1)$
B7	$WZ^2 = Z + X^2 + XYZ$	$(1, 0, 0, 1)$
B8	$(W + YZ + X^2Z)^5 = Z$	$(1, 0, 0, 1)$
B9	$(WZ + X^2 + YZ^2)^5 = Z^4$	$(1, 0, 0, 1)$
B10	$W = XY + Z^2 + XZ^2$	$(0, 0, 0, 0)$
B11	$W^2 = XY + X^2Y + X^2Z$	$(1, 1, 0, 1)$

Invariants

Pick invariant = $a_{ijk}a^{ijk}$ $a_{ijk}a^{ijl}a_{mnl}a^{mnk}$...





THE END

THANK YOU