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Contact projective structures and 5-dim para-CR manifolds with Levi form degenerate in 1-direction

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(1) 3^{rd} order ODEs modulo contact transformations.

In paper with Godliński arXiv 0902.4129 we produced EDSs describing geometry of 3^{rd} order ODE's, and in particular in the contact equivalence case we produced the following system:

$$
(GN)\qquad
\boxed{
\begin{aligned}
d\theta^1 &= \Omega_1\theta^1 + \theta^4\theta^2\\
d\theta^2 &= \Omega_2\theta^1 + \Omega_3\theta^2 + \theta^4\theta^3\\
\longrightarrow\ d\theta^3 &= \Omega_2\theta^2 + (2\Omega_3-\Omega_1)\theta^3 + A_2\theta^2\theta^1 + \boxed{\boxed{A_1}\,\theta^4\theta^1}\\
d\theta^4 &= \Omega_4\theta^1 + \Omega_5\theta^2 + (\Omega_1-\Omega_3)\theta^4\\
d\Omega_1 &= \Omega_6\theta^1 + \Omega_4\theta^2 - \Omega_2\theta^4\\
d\Omega_2 &= (\Omega_3-\Omega_1)\Omega_2 + \tfrac12\Omega_6\theta^2 + \Omega_4\theta^3 + A_3\theta^1\theta^2 + A_4\theta^1\theta^4\\
d\Omega_3 &= \tfrac12\Omega_6\theta^1 + \Omega_4\theta^2 + \Omega_5\theta^3 + A_5\theta^1\theta^2 + A_2\theta^1\theta^4\\
d\Omega_4 &= \Omega_5\Omega_2 + \Omega_4\Omega_3 + \tfrac12\Omega_6\theta^4 + (A_6+B_2)\theta^1\theta^2 + 2B_3\theta^1\theta^3\\
&\quad - A_3\theta^1\theta^4 + B_4\theta^2\theta^3\\
\longrightarrow\ d\Omega_5 &= (\Omega_1-2\Omega_3)\Omega_5 + \Omega_4\theta^4 + (A_7+B_3)\theta^1\theta^2 + B_4\theta^1\theta^3 - A_5\theta^1\theta^4\\
&\quad \boxed{+\boxed{B_1}\,\theta^2\theta^3}\\
d\Omega_6 &= \Omega_6\Omega_1 + 2\Omega_4\Omega_2 + C_1\theta^1\theta^2 + 2B_2\theta^1\theta^3 + A_8\theta^1\theta^4 + 2B_3\theta^2\theta^3
\end{aligned}
}
$$

These are curvature conditions $d\omega + \omega\wedge\omega = K_{ij}\theta^i\wedge\theta^j$ for the Cartan connection

$$
\boxed{
\omega = \begin{pmatrix}
\tfrac12\Omega_1 & \tfrac12\Omega_2 & -\tfrac12\Omega_4 & -\tfrac14\Omega_6\\
\theta^4 & \Omega_3-\tfrac12\Omega_1 & -\Omega_5 & -\tfrac12\Omega_4\\
\theta^2 & \theta^3 & \tfrac12\Omega_1-\Omega_3 & -\tfrac12\Omega_2\\
2\theta^1 & \theta^2 & -\theta^4 & -\tfrac12\Omega_1
\end{pmatrix}
}
\in \underline{sp}(4,\mathbb{R}) \simeq \underline{o}(2,3)
$$

The basic invariants for $y''' = F(x,y,y',y'')$, $y'=p$, $y''=q$ are: Wunschmann

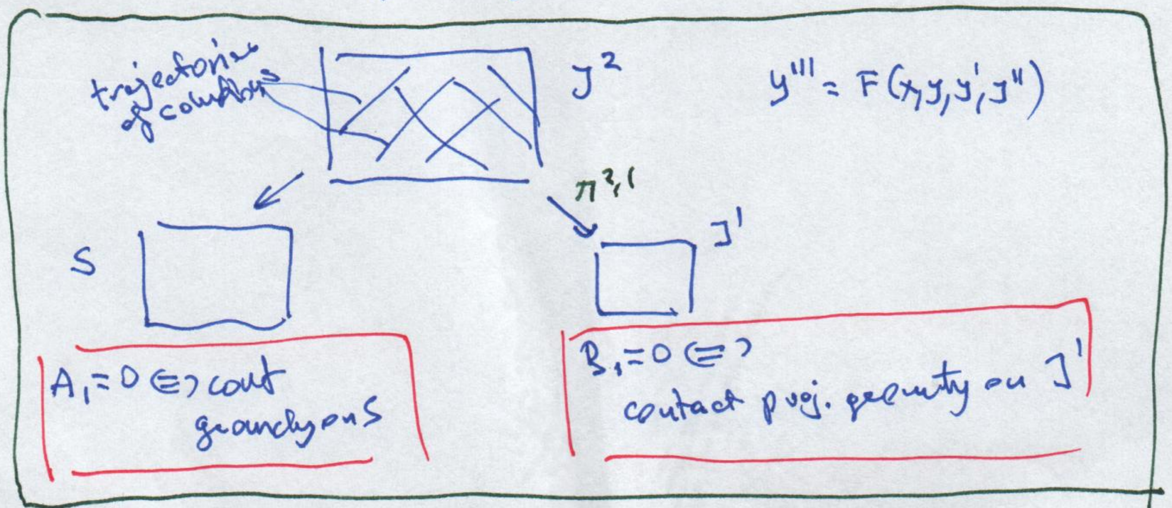
$$\boxed{A_1} = (\ )\cdot\left[9D^2F_q - 27DF_p - 18DF_qF_q + 18F_pF_q + 4F_q^3 + 54F_y\right]$$

$$\boxed{B_1} = (\ )\left[F_{pppp}\right] \leftarrow \text{chern}$$

All other A_i's are coframe derivatives of A_1; B_i's are coframe derivatives of B_1, and C_1 is a coframe derivative of A_1 and B_1.

If $\boxed{A_1 \equiv 0}$ the equation $y''' = F(x, y, y', y'')$ defines a Lorentzian conformal structure on the space of its solutions.

If $\boxed{B_1 \equiv 0}$ the equation $y''' = F(x, y, y', y'')$ defines a contact projective structure on the space of first jets.



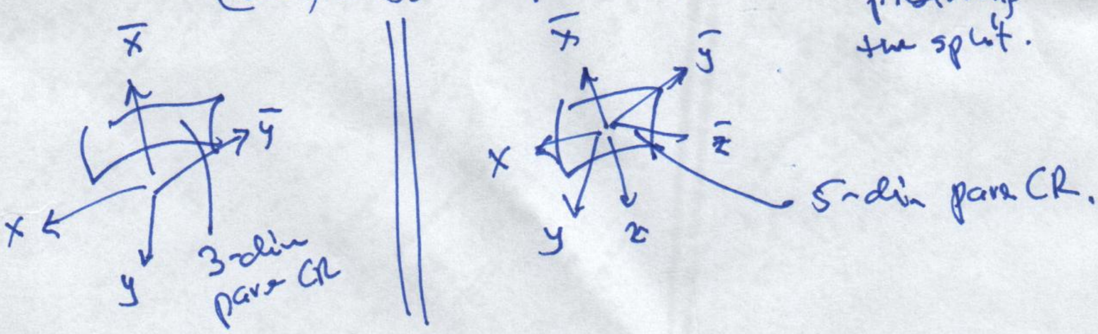
(2) Para-CR structures with Levi form degenerate in one dimension

$\rightsquigarrow$ $M_{2n-1} = \{ \Psi(x_1, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0 \quad (x, \bar{x}) \in \underbrace{\mathbb{R}^n}\times\underbrace{\mathbb{R}^n} \}$

given distinguished split of $\mathbb{R}^{2n}$.

$\uparrow$

(local) hypersurfaces like this considered modulo (local) diffeomorphisms $\varphi: \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\text{diff}} \mathbb{R}^n \times \mathbb{R}^n$ preserving the split.



Today $n=3$ $\rightsquigarrow$ 5-dim para-CR structures.

Such structure can be define as ~~~~ a graph

$$z = f(x, y; \bar{x}, \bar{y}, \bar{z})$$

and today we will only consider para-CR structures for which such z's are GENERAL solutions of a system of PDE's on the plane.

We want that such a GENERAL solution depends PRECISELY on 3 integration constants $\bar{x}, \bar{y}, \bar{z}$.

Example

consider $\boxed{(x-\bar{x})^2 + (y-\bar{y})(z-\bar{z}) = 0.}$

$$z = -\frac{(x-\bar{x})^2}{y-\bar{y}} + \bar{z}$$

$$\Downarrow$$

$$\boxed{z_{xxx} = 0} \qquad \left.\begin{aligned} z_y &= \frac{(x-\bar{x})^2}{(y-\bar{y})^2} \\ z_x &= -\frac{2(x-\bar{x})}{(y-\bar{y})} \end{aligned}\right\} \Rightarrow \boxed{z_y = \frac{1}{4} z_x^2} \quad \checkmark$$

?

$$\boxed{z_y = \frac{1}{4} z_x^2 \quad \& \quad z_{xxx} = 0}$$

$$\Big(\; z = \alpha(y)\,x^2 + \beta(y)\,x + \gamma(y)$$

$$z_y = \frac{1}{4} z_x^2$$

$$\alpha' = \alpha^2 \Rightarrow \alpha = \frac{1}{-y+\bar{y}}$$

$$\beta' = \frac{\beta}{-y+\bar{y}} \Rightarrow \beta = \frac{-2\bar{x}}{-y+\bar{y}}$$

$$\gamma' = +\frac{\bar{x}^2}{(-y+\bar{y})^2} \Rightarrow \gamma = -\frac{\bar{x}^2}{y-\bar{y}} + \bar{z}$$

$$z = -\frac{\bar{x}^2}{y-\bar{y}} + \frac{x^2}{-y+\bar{y}} - \frac{2x\bar{x}}{-y+\bar{y}} + \bar{z}$$

$$z - \bar{z} = \frac{(x-\bar{x})^2}{-y+\bar{y}} \rightsquigarrow \boxed{(x-\bar{x})^2 + (y-\bar{y})(z-\bar{z}) = 0} \quad \checkmark \; \Big)$$

— * —

In general we will consider ~~eq~~ the system of PDEs:

$$\boxed{z_y = G(x, y, z, z_x, z_{xx}) \quad \& \quad z_{xxx} = H(x, y, z, z_x, z_{xx})} \quad (S)$$

for $z = z(x,y)$.

Fact The general solution of (S) is of the form
$$z = z(x, y; \bar{x}, \bar{y}, \bar{z})$$
if and only if

(I.C.) $\boxed{D^3 G = \Delta H}$ where

$\boxed{D = \partial_x + p\partial_z + r\partial_p + H\partial_r}$, $\boxed{\Delta = \partial_y + G\partial_z + DG\partial_p + D^2G\partial_r}$

$\boxed{p = z_x,}$ $\boxed{r = z_{xx}}$

Solutions of such system naturally produce examples of 5-dim para-CR structures.

We can either describe such structures in terms of $z = z(x, y; \bar{x}, \bar{y}, \bar{z})$, or directly in terms of a geometry of the system (S).

The system (S) defines:

(a) contact form $\omega^1 = dz - p\,dx - G\,dy$ on $M^5 \ni (x, y, z, p, r)$

but (b) to represent a para-CR structure it should have TWO INTEGRABLE rank 2-distributions (tangent to intersections of $M^5 \in \mathbb{R}^3 \times \mathbb{R}^3$ with $(\bar{x}, \bar{y}, \bar{z})$=const and (x, y, z)= const).

These are defined to be annihilators of

$$(1) \quad \underbrace{\begin{pmatrix} \omega^1 = dz - p\,dx - G\,dy \\ \omega^2 = dp - r\,dx - DG\,dy \\ \omega^3 = dr - H\,dx - D^2G\,dy \end{pmatrix}^{\perp}}_{= D_1} \quad \text{and} \quad \underbrace{\begin{pmatrix} \omega^1 = dz - p\,dx - G\,dy \\ \omega^4 = dx \\ \omega^5 = dy \end{pmatrix}^{\perp}}_{= D_2}$$

D_2 – integrable

D_1 ← integrable on the ground of (I.C.)

In this context, a para-CR structure it is a structure given in terms of a contact distribution $D_0 = (\omega^1)^\perp$ and 2-integrable distributions D_1 and D_2.

In other words this para-CR structure is a G structure for a coframe $(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ given up to transformations

$$\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix} \to \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & g e^{\varphi} & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \bar{f}_2 & 0 & 0 & g e^{-\varphi} & \bar{f}_4 \\ \bar{f}_5 & 0 & 0 & \bar{f}_6 & \bar{f}_7 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix} \qquad (2)$$

~~One can easily observe that the condition~~

<u>Define</u> L_{ij} via

$$\left(d\omega^1 - L_{11}\omega^2\wedge\omega^4 - L_{12}\omega^2\wedge\omega^5 - L_{21}\omega^3\wedge\omega^4 - L_{22}\omega^3\wedge\omega^5\right)\wedge\omega^1 = 0$$

One can check that the signature of

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$ is an invariant property

for the para-CR structure as defined through (1) and (2).

Hence $\det(L) = 0$ or $\det(L) \neq 0$ is an invariant.

L - is called Levi-form for M^5-para-CR.

We are interested for para-CR's with $L \neq 0$ but such that $\boxed{\det(L) \equiv 0}$

$$\Longleftrightarrow$$

$$\boxed{G_r \equiv 0}$$

We also want to avoid situation when our para-CR manifold is locally equivalent to (3-dim para-CR) $\times$ ($\mathbb{R}^1 \times \mathbb{R}^1$) $\Big\} \Longleftrightarrow \boxed{G_{pp} \neq 0}$.

③ Subject proper of this seminar

System of PDEs

$$\boxed{z_y = G(x,y,z,p)}\,, \quad \boxed{z_{xxx} = H(x,y,z,p,r)}$$

$$p = z_x,\quad r = z_{xx} \qquad (G_r \equiv 0)$$

(∗) $$\boxed{D^3 G = \Delta H},\quad \boxed{G_{pp} \neq 0}$$

$$D = \partial_x + p\partial_z + r\partial_p + H\partial_r, \quad \Delta = \partial_y + G\partial_z + DG\,\partial_p + D^2G\,\partial_r$$

considered modulo POINT transformations of variables

$\Updownarrow$

coframe

$$\begin{cases} \omega^1 = dz - p\,dx - G\,dy \\ \omega^2 = dp - r\,dx - DG\,dy \\ \omega^3 = dr - H\,dx - D^2G\,dy \\ \omega^4 = dx \\ \omega^5 = dy \end{cases}$$

$D^3G = \Delta H$, $G_{pp} \neq 0$, $G_r = 0$

modulo

(∗∗) $$\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix} \to \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 \\ f_2 & \rho e^{\varphi} & f_4 & 0 & 0 \\ f_5 & f_6 & f_7 & 0 & 0 \\ \bar f_2 & 0 & 0 & \rho e^{-\varphi} & \bar f_4 \\ \bar f_1 & 0 & 0 & \bar f_6 & \bar f_7 \end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix}$$

Symmetries X - vector field on $M^5 \ni (x,y,z,p,r)$

is a symmetry of the para-CR structure as in (∗)

$$\iff \quad (\mathcal{L}_X \omega^1)\wedge\omega^1 = 0$$
$$(\mathcal{L}_X \omega^2)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0$$
$$(\mathcal{L}_X \omega^3)\wedge\omega^1\wedge\omega^2\wedge\omega^3 = 0$$
$$(\mathcal{L}_X \omega^4)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0$$
and $$(\mathcal{L}_X \omega^5)\wedge\omega^1\wedge\omega^4\wedge\omega^5 = 0$$

④ $\boxed{\text{Goal}}$ find all homogeneous models, i.e.

find M^5 as in (∗) such that it has at least $X_1, X_2, \dots, X_5$ s.t. $X_1\wedge X_2\wedge X_3\wedge X_4\wedge X_5 \neq 0$

and X_i is a symmetry.

First step – Determine basic (relative) invariants.

Thm Given a para-CR structure represented by the forms

$$\omega^1 = dz - p\,dx - G\,dy, \quad \omega^2 = dp - r\,dx - DG\,dy, \quad \omega^3 = dr - H\,dx - D^2G\,dy$$
$$\omega^4 = dx, \quad \omega^5 = dy, \quad G_r = 0,\ G_{pp} \neq 0, \quad D^3G = \Delta H$$

it is always possible to force the lifted coframe

$$\theta^1 = f_1\omega^1, \quad \theta^2 = f_2\omega^1 + g e^{\varphi}\omega^2 + f_4\omega^3, \quad \theta^3 = f_5\omega^1 + f_6\omega^2 + f_7\omega^3$$
$$\theta^4 = \bar f_2\omega^1 + g e^{-\varphi}\omega^4 + \bar f_4\omega^5, \quad \theta^5 = \bar f_5\omega^1 + \bar f_6\omega^4 + \bar f_7\omega^5$$

to satisfy the following EDS:

(PCR-EDS)

$$d\theta^1 = -\theta^1\Omega_1 + \theta^2\theta^4$$
$$d\theta^2 = \theta^2(\Omega_2 - \tfrac{1}{2}\Omega_1) - \theta^1\Omega_3 + \theta^3\theta^4$$
$$d\theta^3 = 2\theta^3\Omega_2 - \theta^2\Omega_3 + Q\,\theta^1\theta^3 + \frac{e^{3\varphi}}{27g^3}\boxed{A}\,\theta^1\theta^4 + \frac{e^{-\varphi}}{3g}\boxed{C}\,\theta^2\theta^3$$
$$d\theta^4 = -\theta^2\theta^5 - \theta^4(\tfrac{1}{2}\Omega_1 + \Omega_2) - \theta^1\Omega_4$$
$$d\theta^5 = -2\theta^5\Omega_2 + \theta^4\Omega_2 + \frac{e^{-3\varphi}}{27g^3}\boxed{B}\,\theta^1\theta^2 + Q\,\theta^1\theta^5 + \frac{e^{\varphi}}{3g}\tilde{C}\,\theta^4\theta^5$$

$$\boxed{A} = -\frac{1}{2}\left[9D^2H_r - 27DH_p - 18DH_r H_r + 18H_pH_r + 4H_r^3 + 54H_z\right]$$ Wunschmann!!!

$$\boxed{B} = \frac{1}{2F_{pp}^3}\left[40G_{ppp}^3 - 45G_{pp}G_{ppp}G_{pppp} + 9G_{pp}^2G_{ppppp}\right]$$ Monge!!!

$$\boxed{C} = \frac{1}{F_{pp}}\left[2F_{ppp} + F_{pp}H_{rr}\right]$$

($\tilde{C}$ is zero if $C \equiv 0$)

and (vanishing or not) of each A, B, C is an invariant property of the para-CR structure.

Remark We were unable to make normalizations to make the EDS into the curvature of some Cartan connection. (BUT we did not tried hard)

Flat model $A = B = C = 0$

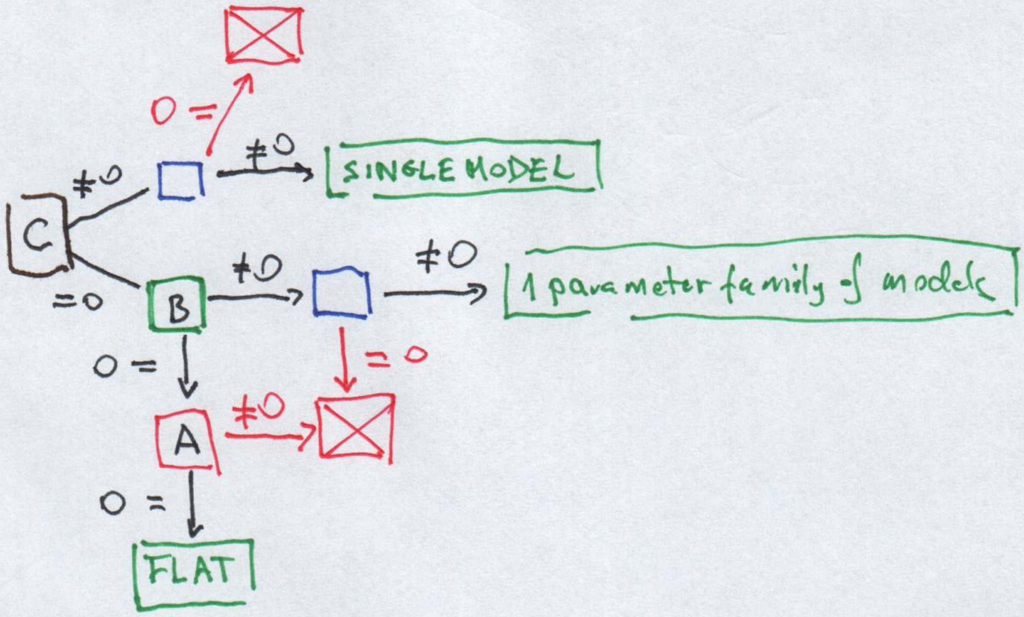
$$(x-\bar x)^2 + (y-\bar y)(z-\bar z) = 0 \iff \left(z_{xxx} = 0,\ z_y = \tfrac{1}{4}z_x^2\right)$$

Symmetry group $SO(2,3)$

(5) Homogeneous models

Method? Cartan's reduction procedure applied to the system (pCR-EDS),

(required quite a gymnastics!!!)



Cartan's reduction produces eventually the homogeneous models in terms of a Maurer-Cartan system for the maximal symmetry group of the model:

FLAT

$$d\theta^1=\theta^2\theta^4-\theta^1\Omega_1$$
$$d\theta^2=\theta^3\theta^4+\theta^2(\Omega_2-\tfrac{1}{2}\Omega_1)-\theta^1\Omega_3$$
$$d\theta^3=2\theta^3\Omega_2-\theta^2\Omega_3$$
$$d\theta^4=-\theta^2\theta^5-\theta^4(\tfrac{1}{2}\Omega_1+\Omega_2)-\theta^1\Omega_4$$
$$d\theta^5=-2\theta^5\Omega_2+\theta^4\Omega_4$$
$$d\Omega_1=-\theta^4\Omega_3+\theta^2\Omega_4-\theta^1\Omega_5$$
$$d\Omega_2=-\theta^3\theta^5-\tfrac{1}{2}\theta^4\Omega_3-\tfrac{1}{2}\theta^2\Omega_4$$
$$d\Omega_3=-(\tfrac{1}{2}\Omega_1+\Omega_2)\Omega_3+\theta^3\Omega_4-\tfrac{1}{2}\theta^2\Omega_5$$
$$d\Omega_4=(\Omega_2-\tfrac{1}{2}\Omega_1)\Omega_4+\theta^5\Omega_3-\tfrac{1}{2}\theta^4\Omega_5$$
$$d\Omega_5=-\Omega_1\Omega_5+2\Omega_3\Omega_4$$

$\dim G=\boxed{10}$

$G=\boxed{\begin{matrix}SO(2,3)\\ Sp(4,\mathbb{R})\end{matrix}}$

SINGLE $\boxed{\varepsilon=\pm1}$

$$d\theta^1=\theta^2\theta^4+\varepsilon(-6\theta^1\theta^3+\tfrac{1}{2}\theta^1\theta^4-\tfrac{3}{2}\theta^1\theta^5)$$
$$d\theta^2=\varepsilon(-\tfrac{1}{16}\theta^1\theta^2-2\theta^2\theta^3+\tfrac{1}{2}\theta^2\theta^4-\theta^3\theta^5)-\theta^1\theta^3+\tfrac{1}{32}\theta^1\theta^4-\tfrac{1}{8}\theta^1\theta^5+\theta^3\theta^4$$
$$d\theta^3=\varepsilon(-\tfrac{3}{16}\theta^1\theta^3+\tfrac{1}{2}\theta^3\theta^4-\tfrac{1}{2}\theta^3\theta^5)+\tfrac{1}{32}\theta^2\theta^4-\tfrac{1}{8}\theta^2\theta^5$$
$$d\theta^4=\varepsilon(-\tfrac{1}{8}\theta^1\theta^4+\tfrac{1}{4}\theta^1\theta^5+4\theta^3\theta^4-\tfrac{1}{2}\theta^4\theta^5)-\theta^2\theta^5$$
$$d\theta^5=\varepsilon(-\tfrac{1}{16}\theta^1\theta^5+2\theta^3\theta^5-\tfrac{1}{4}\theta^4\theta^5)$$

$\dim G=\boxed{5}$

1-PAR. FAMILY. $\boxed{\begin{matrix}\varepsilon=\pm1\\ s\in\mathbb{R}\end{matrix}}$

$$d\theta^1=\theta^2\theta^4-\varepsilon(\theta^1\theta^3+\theta^1\theta^5)$$
$$d\theta^2=\varepsilon(s\theta^1\theta^2-\theta^2\theta^5)-s\theta^1\theta^4+\theta^3\theta^4$$
$$d\theta^3=\varepsilon(\theta^1\theta^4-\theta^3\theta^5)-\theta^1\theta^2-s\theta^2\theta^4$$
$$d\theta^4=\varepsilon(-s\theta^1\theta^4+\theta^3\theta^4)+s\theta^1\theta^2-\theta^2\theta^5$$
$$d\theta^5=\varepsilon(-\theta^1\theta^4+\theta^3\theta^5)+\theta^1\theta^2+s\theta^2\theta^4$$

$\dim G=\boxed{5}$

- **Question:** Can these be realized?

Worry: Mike's talk ~~on~~ every affine surface which is homogeneous give rise to a para CR, and looking at Mike's classification (DEGENERATE ones) one has (1) flat model, ~~(2)~~ single models and (2) 1-par. families.

Actually the homogeneous affine surfaces (with the degeneracy ~~for~~ matching ~~the~~ our degeneracy assumption for para $C(k)$) are:

- $\Sigma = \{\mathbb{R}^3 : xy + z^2 = 0\}$ our flat model

$$\boxed{z_y = \tfrac{1}{4}(z_x)^2, \quad z_{xxx} = 0.}$$

- $\Sigma = \{\mathbb{R}^3 : \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + r\begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, r, t \in \mathbb{R}\}$ our single model

$$\boxed{z_y = \tfrac{1}{4}(z_x)^2, \quad z_{xxx} = (z_{xx})^2}$$

a) $\Sigma_\alpha = \{\mathbb{R}^3 : \begin{pmatrix} r \\ re^t \\ re^{\alpha t} \end{pmatrix}, r, t \in \mathbb{R}\}, \alpha > 2$

b) $\Sigma = \{\mathbb{R}^3 : \begin{pmatrix} r \\ rt \\ re^t \end{pmatrix} r, t \in \mathbb{R}\}$

c) $\Sigma_\omega = \{\mathbb{R}^3 : \begin{pmatrix} r\cos t \\ r\sin t \\ re^{\omega t} \end{pmatrix}, r, b \in \mathbb{R}\}, \omega > 0$

our 1-parameter family parametrized by s.

$$\boxed{z_y = \tfrac{1}{4}(z_{xx})^b, \quad z_{xxx} = (2-b)\frac{(z_{xx})^2}{z_x}} \quad a), b)$$

Ad a) $s < -3(2)^{-5/3}$ $\Updownarrow$ $1 < b < 2$

$$s = -\frac{3}{2}\frac{1-b+b^2}{[(b-2)(2b-1)]^{2/3}}$$

Ad b) $s = -3(2)^{-5/3}$ $\quad b = 1$

Ad c) $s > -3(2)^{-5/3}$ $\Updownarrow$ $b > 0$

and a rather complicated system of PDEs:

$$\boxed{z_y = f(z_x) \qquad z_{xxx} = h(z_x)(z_{xx})^2} \quad c)$$

and f and h are related via:

$$(z_x^2 + f(z_x)^2)\exp\left[2b\arctan\frac{bz_x - f(z_x)}{z_x + bf(z_x)}\right] = 1 + b^2$$

$$h(z_x) = \frac{(b^2-3)z_x - 4bf(z_x)}{(f(z_x) - bz_x)^2}$$

$$s = -\frac{3}{2}\frac{b^2-3}{[2b(9+b^2)]^{2/3}}$$

- **Lemma** Given a para CR structure associated with the system $\boxed{z_y = G(x,y,z,p),\ z_{xxx} = H(x,y,z,p,r),\ G_{pp} \neq 0,\ D^3 G = \Delta H,}$ given modulo point transformation of variables, there always exist 1-forms $\tilde{\omega}^1, \tilde{\omega}^2, \tilde{\omega}^3, \tilde{\omega}^4$ ~~such that~~ and forms $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ from the equivalence class of forms
$[\omega^1 = dz - p\,dx - G\,dy,\ \omega^2 = dp - r\,dx - DG\,dy,\ \omega^3 = dr - H\,dx - D^2G\,dx,$
$\omega^4 = dx,\ \omega^5 = dy]$ and that

$$\left[\begin{aligned}
d\omega^1 &= -\omega^1\wedge\tilde{\omega}_1 + \omega^2\wedge\omega^4\\
d\omega^2 &= -\omega^1\wedge\tilde{\omega}_3 + \omega^2\wedge(\tilde{\omega}_2 - \tfrac{1}{2}\tilde{\omega}_1) + \omega^3\wedge\omega^4\\
d\omega^3 &= -\omega^2\wedge\tilde{\omega}_3 + 2\omega^3\wedge\tilde{\omega}_2 + \tfrac{1}{8}(2I_{34} + I_{352})\omega^1\wedge\omega^3 +\\
&\quad + I_1\,\omega^1\wedge\omega^4 + I_3\,\omega_2\wedge\omega_3\\
d\omega^4 &= -\omega^1\wedge\tilde{\omega}_4 - \omega^4\wedge(\tilde{\omega}_2 + \tfrac{1}{2}\tilde{\omega}_1) - \omega^2\wedge\omega^5\\
d\omega^5 &= \omega^4\wedge\tilde{\omega}_4 - 2\omega^5\wedge\tilde{\omega}_2 + I_2\,\omega^1\wedge\omega^2 + \tfrac{1}{8}(2I_{34} + I_{352})\omega^1\wedge\omega^5\\
&\quad - \tfrac{1}{2} I_{35}\,\omega^4\wedge\omega^5,\\
&\dots
\end{aligned}\right]$$

Moreover, vanishing or not of each I_1, I_2, I_3 is an invariant property of the para-CR structure. (Proposition 4.1, formula (4.3)-(4.4) in the cervix paper).

It follows that:

Wunschmann !!!

$$I_1 \sim [9D^2H_r - 27DH_p - 18DH_r H_r + 18H_pH_r + 4H_r^3 + 54H_z]$$

$$I_2 \sim [40G^3_{ppp} - 45G_{pp}G_{ppp}G_{pppp} + 9G_p^2 G_{ppppp}] \leftarrow \text{Monge!}$$

$$I_3 \sim [2G_{ppp} + G_{pp}H_{rr}] \quad \dots$$

- $\boxed{\text{Crazy question}}$:

Can I, only using para-CR ~~stransformations~~ transformations (**) move the forms $(\omega^1, \dots, \omega^5)$ of this Lemma to be $\theta^1, \theta^2, \theta^3, \theta^4, \theta^5$ of the EDS for 3rd order ODEs?

not so stupid as in both cases flat models are governed by the $\mathfrak{o}(3,2)$ flat Cartan connection.

- $\boxed{\text{Answer}}$:

two possibilities:

(A) use $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$	(B) use $\omega^1, \omega^2 \leftrightarrow \omega^4, \omega^3 \leftrightarrow \omega^5$

and: ALMOST POSSIBLE

has to extend possible transformation for ω^5 enabling $\omega^5 \to \alpha\omega^1 + \underbrace{\beta\omega^2 + \gamma\omega^3}_{} + \delta\omega^4 + \varepsilon\omega^5$

In such case one transforms $(\omega^1, \omega^2, \omega^3, \omega^4, \omega^5)$ to satisfy EDS (GN) with.

$\boxed{A_1} \sim I_1$ Wunschmann	$\boxed{A_1} \sim I_2$ Monge
$\boxed{B_1} \sim I_{333}$	$\boxed{B_1} \sim I_{3555}$

If we want that transformation of the para-CR type can move everything to (GN) one needs:

$\boxed{I_{33} \equiv 0}$ each of them $\boxed{I_{355} \equiv 0}$

$$\boxed{I_3 \equiv 0.}$$

$I_{333} = 0.$ $\rightsquigarrow$ $I_{3555} \Rightarrow$

• <u>Theorem</u>

Every para-CR structure with $I_3 = 0$.
defines <u>two</u> contact projective structures in dim 3;
one of them having Wunschmann as the basic invariant,
and the other Monge as the basic invariant.

$$A_1 \sim I_1 \text{ - Wunschm} \quad B_1 \equiv 0 \qquad \Bigg| \qquad A_1 \sim I_2 \text{ - Monge.} \quad B_1 \equiv 0.$$

Assuming $I_3 \equiv 0$, I can write down the explicit para-CR transformations bringing the para-CR in question to either (GN) with Wunschmann or (GN) with Monge.

Kaup - Fells

Surfaces

$$\Sigma = \left\{ \mathbb{R}^3 \ni \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}, \quad t, r \in \mathbb{R} \right\}$$

$$z = 3xy - 2y^3 + 2\,(y^2 - x)^{3/2}$$

(3)

$$\boxed{z_y = -\tfrac{1}{3}(z_x)^2, \qquad z_{xxx} = \tfrac{2}{9}\,(z_{xx})^3}$$

S

ours $\boxed{z_y = \tfrac{1}{4}(z_x)^2 \;\; \& \;\; z_{xxx} = (z_{xx})^3}$

$$\Sigma = \left\{ \mathbb{R}^3 \ni \begin{pmatrix} r \\ rt \\ re^t \end{pmatrix}, \; t, r \in \mathbb{R} \right\}.$$

$\cancel{z = x e^{\frac{y}{x}}}$ $\quad \cancel{t = \frac{y}{z}}$ $\quad \cancel{x = z e^{\frac{y}{z}}}$

$$\begin{cases} x = r \\ y = rt \\ z = re^t \end{cases}$$

$$\to z = e^{\frac{y}{x}} \cdot x$$

$$\Downarrow$$

$$\boxed{z_y = \tfrac{1}{2} z_x, \quad z_{xxx} = -\frac{4(z_{xx})^2}{z_x}}$$

Ours $\boxed{z_y = \tfrac{1}{4} z_x, \quad z_{xxx} = \frac{(z_{xx})^2}{z_x}}$

Actually the homogeneous affine surfaces with an appropriate degeneracy are: (Kaup & Fels)

flat cone

$\boxed{x_1 x_2 + x_3^2 = 0}$

our flatmodel

$\Sigma = \left\{ \mathbb{R}^3 : \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \right\}$

our single model

$\Sigma = \left\{ \mathbb{R}^3 : \begin{pmatrix} r \\ rt \\ re^t \end{pmatrix} \right\}$

our $s = -3 \cdot (2)^{-5/3}$

$\Sigma = \left\{ \mathbb{R}^3 : \begin{matrix} r \\ re^t \\ re^{\theta t} \end{matrix} \right\}$

$\theta > 2$

our $s < -3(2)^{-5/3}$

$\Sigma = \left\{ \mathbb{R}^3 : \begin{pmatrix} r\cos t \\ r\sin t \\ e^{\omega t} \end{pmatrix} \right\}$

$\boxed{\omega > 0}$

our $s > -3(2)^{-5/3}$