Deformations of the Veronese embedding and Finsler 2-spheres of constant curvature

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Path geometries

Setup: M connected oriented smooth surface

Path geometry: Prescription of a path on M for each direction in every tangent space (e.g. geodesics of a Finsler metric, geodesics of a projective structure)

Projective circle bundle

$$\pi : \mathbf{S}M := \left(TM \setminus \{0_M\}\right)/\mathbf{R}^+ \to M$$

Contact structure

$$\tau_{[v]} = \left\{\xi \in T_{[v]}\mathbf{S}M : \pi'(\xi) \wedge v = 0\right\}$$

Immersed curve $\gamma : (a, b) \to M$ lifts s.t. $\dot{\delta}(t)$ lies in τ

$$\delta := [\dot{\gamma}] : (a, b) \to \mathbf{S}M$$

Path geometry: 1-dim distribution $P \to \mathbf{S}M$ so that $P + \ker \pi' = \tau$.

Paths: Integral curves of P projected to M

The dual of a path geometry

Definition (Bryant). A **generalised path geometry** is a 3-manifold N together with an ordered pair (P, L) of transverse 1-dim distributions spanning a contact structure.

Path geometry:

$$N = \mathbf{S}M, \quad P = \text{“path bundle”}, \quad L = \text{vertical bundle of projection } \mathbf{S}M \to M$$

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ \text{surface } \; N/\mathcal{L} & & \text{space of paths } \; N/\mathcal{P} \end{tikzcd}$$

Definition. The **dual** of a generalised path geometry (N, P, L) is the generalised path geometry (N, L, P).

Question. Are there (non-trivial global) examples where the dual of a path geometry is again a path geometry?

Projective structures

Affine connection: connection ∇ on TM, assume ∇ is torsion-free

Geodesic: immersed curve $\gamma : I \to M$ s.t.

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0.$$

Projective equivalence: $\nabla \sim \nabla'$ iff ∇ and ∇' have the same geodesics up to parametrisation.

Projective structure: Equivalence class $\mathfrak{p}$ of connections

Lemma (Cartan, Eisenhart, Weyl). $\nabla \sim \nabla'$ *iff* $\exists\ \beta \in \Omega^1(M)$ *such that*

$$\nabla_X Y - \nabla'_X Y = \beta(X)Y + \beta(Y)X.$$

Projective surface $(M, \mathfrak{p})$ is called **flat** if it is locally diffeomorphic to S^2 so that geodesics are mapped onto (segments of) great circles.

Finsler metrics

A **Finsler norm** is a continuous function $F : TM \to [0, \infty)$ which is smooth away from the zero section and so that

- $F(\lambda v) = \lambda F(v)$ for $\lambda \geqslant 0$
- $F(v) > 0$ unless $v = 0$
- the symmetric bilinear form
$$g_v(X, Y) = \frac{1}{2} \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} \left[F(v + sX + tY)^2 \right]$$
is positive definite.

F is called **reversible** is $F(v) = F(-v)$ for all $v \in TM$

Length of immersed curve $\gamma : [a, b] \to M$, $L(\gamma) := \int_a^b F(\dot{\gamma}(t))\, \mathrm{d}t$ is invariant under orientation preserving reparametrisations

Locally length minimising curves are the **geodesics** of F.

Finsler norm is determined by its **unit tangent bundle**

$$UM := \{v \in TM : F(v) = 1\}.$$

Zermelo deformation: Construct new Finsler metric by translating each fibre of *UM* with a vector of small enough length.

Cartan: *UM* is equipped with a coframing (χ, η, ν) which satisfies the structure equations

$$\mathrm{d}\chi = -\eta \wedge \nu, \qquad \mathrm{d}\eta = -\nu \wedge (\chi - I\eta), \qquad \mathrm{d}\nu = -(K\chi - J\nu) \wedge \eta,$$

for $I, J, K \in C^\infty(UM)$.

Riemannian case: (M,g) choose **isothermal coordinates** (x, y)

$$g = \mathrm{e}^{2u(x,y)}(\mathrm{d}x^2 + \mathrm{d}y^2)$$

Coframing

$$\chi = \mathrm{e}^u\left(\cos\alpha\ \mathrm{d}x + \sin\alpha\ \mathrm{d}y\right), \quad \eta = \mathrm{e}^u\left(-\sin\alpha\ \mathrm{d}x + \cos\alpha\ \mathrm{d}y\right), \quad \nu = \mathrm{d}\alpha + \star\,\mathrm{d}u,$$

where α is the **angle coordinate** on the unit tangent bundle.

Riemannian Finsler metric: $I \equiv J \equiv 0$ and K is (the pullback to UM of) the **Gauss curvature** K_g.

K is the **Finsler–Gauss curvature** or flag curvature.

Theorem (Akbar-Zadeh, 1988). *If a Finsler metric on a compact surface has constant negative curvature, then it is Riemannian, and, if it has zero curvature, then it is locally Minkowskian.*

Theorem (Bryant, 2006). *If a reversible Finsler metric on a compact surface has constant positive curvature, then it is Riemannian.*

Fact: A Zermelo deformation of a constant curvature Finsler metric by a Killing vector field has again constant curvature.

Example. *(Katok) First example of non-Riemannian $K \equiv 1$ Finsler metric on S^2 via Zermelo deformation of constant curvature metric.*

Theorem (Bryant, 1997). Classification of $K \equiv 1$ Finsler 2-spheres that are projectively flat.

(Generalised) thermostats

Dual vector fields (X, H, V) to (χ, η, ν)

$$[V, X] = H, \quad [V, H] = -X, \quad [X, H] = K_g V$$

Tautological bundle $\tau = \{\eta = 0\}$, **vertical bundle** $\{\chi = \eta = 0\}$

Thermostat: flow ϕ generated by $X + \lambda V$ for $\lambda \in C^\infty(UM)$

Choice of metric g identifies path geometry P with thermostat.

$\lambda = \lambda(x, y, \alpha)$, 2π-periodic in α, **Fourier-decomposition** in α

Volume form: $\Theta = \chi \wedge \eta \wedge \nu$ and **inner product:**

$$\langle u, v \rangle = \int_{UM} u\bar{v}\, \Theta,$$

Densely defined operator $-\mathrm{i}V$ is self-adjoint

$$L^2(UM) = \bigoplus_{m \in \mathbf{Z}} \mathcal{H}_m, \quad \mathcal{H}_m = \ker(m\mathrm{Id} + \mathrm{i}V)$$

Examples of thermostats

Example. $\alpha \in \Omega^2(M)$, $g \in \mathrm{Riem}(M)$. *Consider flow of Hamiltonian vector field* X_η *on* $(T^*M, \Omega_0 + \nu^*\alpha)$ *generated by Hamiltonian* $\eta(\xi) = \frac{1}{2}|\xi|^2_{g^\sharp}$.

Magnetic flows correspond to thermostats of degree 0, i.e. $V\lambda = 0$

$$\pi^*\alpha = \lambda\chi \wedge \eta.$$

1-forms $\lambda \in C^\infty(UM) \cap (\mathcal{H}_{-1} \oplus \mathcal{H}_1) \leftrightarrow \Omega^1(M)$

To $\theta \in \Omega^1(M)$ – thought of as a function $\theta : UM \to \mathbf{R}$ – we associate the thermostat ϕ generated by the vector field

$$F = X - V(\theta)V.$$

Orbits of ϕ – when projected to M – are reparametrisations of the geodesics of the **Weyl connection** defined by (g, θ).

Weyl connections

Weyl connection: Affine torsion-free connection ∇ preserving a conformal structure $[g]$, i.e. parallel transport maps of ∇ are **angle preserving** w.r.t. $[g]$,

$$\nabla g = 2\theta \otimes g,$$

Weyl connections are of the form

$$^{(g,\theta)}\nabla = {}^{g}\nabla + g \otimes \theta^{\sharp} - \theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta$$

with $g \in [g]$ and $\theta \in \Omega^1(M)$.

Weyl structure is an equivalence class $[(g, \theta)]$ where

$$(g, \theta) \sim (\hat{g}, \hat{\theta}) \quad \iff \quad \hat{g} = \mathrm{e}^{2u} g \;\text{ and }\; \hat{\theta} = \theta + \mathrm{d}u, u \in C^{\infty}(M)$$

Weyl structures are in one-to-one correspondence with Weyl connections

$$[(g, \theta)] \mapsto {}^{g}\nabla + g \otimes \theta^{\sharp} - \theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta$$

Weyl connections with θ **exact** correspond to Levi-Civita connections

A Weyl structure $[(g,\theta)]$ is called **positive** if $\mathrm{Sym}(\mathrm{Ric}({}^{(g,\theta)}\nabla))$ is positive definite

On oriented surface M

$$[(g,\theta)] \;\text{ is positive} \qquad \Longleftrightarrow \qquad (K_g - \delta_g\theta)dA_g > 0$$

Lemma. *For a positve Weyl structure $[(g,\theta)]$ there exists a unique gauge (g,θ) – henceforth called the **natural gauge** – so that $K_g - \delta_g\theta \equiv 1$.*

Lemma. *Let $[(g,\theta)]$ be a positve Weyl structure with natural gauge (g,θ) and let $\pi : UM \to M$ denote the unit tangent bundle of g with coframing (χ,η,ν).* Then the forms

$$\hat{\chi} := \pi^*(\star_g\theta) - \nu, \qquad \hat{\eta} := -\eta, \qquad \hat{\nu} := -\chi$$

satisfy the structure equations of a Finlser metric with $K \equiv 1$.

Paraphrasing: Ignoring global issues, the path geometry of a positive Weyl structure (i.e. whose paths are the geodesics of the associated Weyl connection) is dual to the path geometry of a Finsler metric with $K \equiv 1$.

Dynamical aspects of $K \equiv 1$ Finsler metrics

Theorem (Bryant, Foulon, Ivanov, Matveev, Ziller, 2017). *Let F be a $K \equiv 1$ Finsler metric on S^2. Then there exists a shortest closed geodesic of length $2\pi\ell \in (\pi, 2\pi]$ and the following holds:*

- *If $\ell = 1$, all geodesics are closed and have the same length 2π,*
- *If ℓ is irrational, there exist two closed geodesics with the same image, and all other geodesics are not closed. The length of the second closed geodesic is $2\pi\ell/(2\ell - 1)$. Moreover, the metric admits a Killing vector field.*
- *If $\ell = p/q \in (\frac{1}{2}, 1)$ with $p, q \in \mathbf{N}$ and $\gcd(p, q) = 1$, and in this case all unit-speed geodesics have a common period $2\pi p$. Furthermore, there exists at most two closed geodesics with length less than $2\pi p$. A second one exists only if $2p - q > 1$, and its length is $2\pi p/(2p - q) \in (2\pi, 2p\pi)$.*

In particular, if all geodesics of a Finsler metric on S^2 are closed, then its geodesic flow is periodic with period $2\pi p$ for some integer p.

They also show that the case when F admits a Killing field can be deformed (via a Zermelo deformation) to the case $\ell = 1$.

A duality result

A Weyl structure $[(g, \theta)]$ is called **Besse** if the associated Weyl connection has the property that all of its maximal geodesic are closed.

Theorem (Lange–M., 2019). *There is a one-to-one correspondence between Finsler metrics on S^2 with $K \equiv 1$ and all geodesics closed on the one hand, and positive Besse–Weyl structures on weighted projective spaces $\mathbf{CP}(a_1, a_2)$ with $c := \gcd(a_1, a_2) \in \{1, 2\}$, $a_1 \geqslant a_2$, $2|(a_1 + a_2)$ and $c^3|a_1 a_2$ on the other hand. More precisely,*

1. *such a Finsler metric with shortest closed geodesic of length $2\pi\ell \in (\pi, 2\pi]$, $\ell = p/q \in (\frac{1}{2}, 1]$, $\gcd(p, q) = 1$, gives rise to a positive Besse–Weyl structure on $\mathbf{CP}(a_1, a_2)$ with $a_1 = q$ and $a_2 = 2p - q$, and*
2. *a positive Besse–Weyl structure on such a $\mathbf{CP}(a_1, a_2)$ gives rise to such a Finsler metric on S^2 with shortest closed geodesic of length $2\pi \left(\frac{a_1+a_2}{2a_1}\right) \in (\pi, 2\pi]$,*

and these assignments are inverse to each other. Moreover, two such Finsler metrics are isometric if and only if the corresponding Besse–Weyl structures coincide up to a diffeomorphism.

Weighted projective space

Projective space $\mathbf{CP}^1$ is $\mathbf{C}^2 \setminus \{0\}$ modulo the action

$$\lambda \cdot (z, w) = (\lambda z, \lambda w), \quad \lambda \in \mathbf{C}^*$$

Weighted projective space $\mathbf{CP}(a_1, a_2)$ for weights $(a_1, a_2) \in \mathbf{N}^2$ is $\mathbf{C}^2 \setminus \{0\}$ modulo the action

$$\lambda \cdot (z, w) = (\lambda^{a_1} z, \lambda^{a_2} w), \quad \lambda \in \mathbf{C}^*$$

$\mathbf{CP}(1,1) = \mathbf{CP}^1$, weighted projective space is in general an **orbifold**

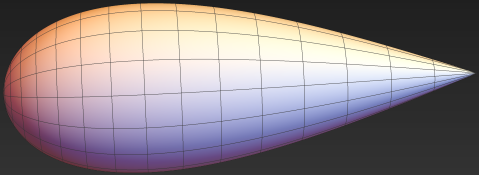
There exists a natural generalisation g_{FS} of the **Fubini–Study metric** to $\mathbf{CP}(a_1, a_2)$

g_{FS} is a Besse orbifold metric of strictly positive Gauss curvature ($K_{g_{FS}} \neq const$).

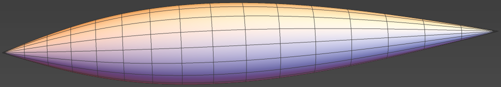
Try to deform g_{FS} among the class of positive Besse-Weyl structures to construct new examples of $K \equiv 1$ Finsler structures.

Isometric embeddings

$$4g_{FS} = \left(\frac{a_1 + a_2}{2} + \frac{a_1 - a_2}{2}\cos(r)\right)^2 \mathrm{d}r^2 + \sin^2(r)d\phi^2, \qquad (r,\phi) \in (0,\pi) \times S^1$$



$\mathbf{CP}(3,1)$



$\mathbf{CP}(5,3)$

Twistor space

Twistor bundle $\mathrm{J}^+ \to M$

$\mathrm{J}_p^+ := \{\text{linear complex structures } J \text{ on } T_pM : (v, Jv) \text{ is pos. oriented } \forall\, v \neq 0\}$

Bundle with fibre

$$\mathrm{GL}^+(2, \mathbf{R})/\mathrm{GL}(1, \mathbf{C}) \simeq \mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$$

Conformal structure $\leftrightarrow$ orientation compatible complex structure

$$J_p : T_pM \to T_pM, \quad J_p = \text{counterclockwise rotation by } \pi/2$$

Conformal structure defines section $[g] : M \to \mathrm{J}_+$.

Proposition (O'Brian & Rawnsley, Dubvois-Violette). *Torsion-free ∇ on TM equips J^+ with an integrable almost complex structure J_p which does only depend on the projective equivalence class of ∇.*

At $j \in \mathrm{J}^+$ lift j horizontally and use complex structure on the fibre vertically.

Holomorphic curves

Proposition (M., 2014). *The Weyl connection* ${}^{(g,\theta)}\nabla$ *belongs to* $\mathfrak{p}$ *iff* $[g] : M \to (\mathrm{J}^+, J_\mathfrak{p})$ *is a holomorphic curve.*

Same statement holds for orbifolds.

Proposition (M., 2014). *For the projective structure on* S^2 *whose geodesics are the great circles, we have* $\mathrm{J}^+ \hookrightarrow \mathbf{CP}^2$

Proposition (Lange–M., 2019). *For the projective structure arising from the Fubini–Study metric* g_{FS} *on* $\mathbf{CP}(a_1, a_2)$*, we have* $\mathrm{J}^+ \hookrightarrow \mathbf{CP}(a_1, (a_1 + a_2)/2, a_2)$*. Furthermore, the holomorphic curve*

$$[g_{FS}] : \mathbf{CP}(a_1, a_2) \to \mathbf{CP}(a_1, (a_1 + a_2)/2, a_2)$$

corresponds to the **Veronese embedding**

$$[z, w] \mapsto [z^2, zw, w^2].$$

Suitable deformations of the Veronese embedding yield positive Besse–Weyl structure on $\mathbf{CP}(a_1, a_2)$ and hence new examples of Finsler 2-spheres with $K \equiv 1$.