SOME ASPECTS OF CONTACT DYNAMICS

J C Marrero (University of La Laguna, Spain)
e-mail: jcmarrer@ull.edu.es

Geometry and Differential Equations Seminar

March 23, 2022

Warsaw, March 23, 2022

- Contact geometry is a classical topic in differential geometry

(Arnold's book, 1989; Libermann-Marle's book 1987)

- A renewed interest in the theory after the contributions by the Warsaw team (Grabowska, Grabowski and collaborators) and other people (Vitagliano and collaborators, ...)

Recently, contact dynamics has been extensively discussed by several authors:

Argentina (Instituto Balseiro)

Italy (Naples)

Mexico (Mexico City)

Spain (Barcelona, La Laguna, Madrid)

UK (Lancaster)

USA (Texas)

⋮

The plan of the talk

1. Contact geometry
2. Contact dynamics
3. Some recent and old topics in contact dynamics
4. Conclusions and outlook

1: Contact geometry (Libermann-Marle, 1987; Arnold, 1989)

1.1 Contact 1-forms

C a smooth manifold ; $\dim C = 2m+1$

A contact 1-form on C is a 1-form η such that $\eta \wedge (d\eta)^m$ is a volume form on C

η a contact 1-form on C
$\Rightarrow$ C is orientable

η a contact 1-form on C $\Rightarrow$ $\exists !$
$\mathcal{R} \in \mathfrak{X}(M)$ /

$$i_{\mathcal{R}} d\eta = 0, \quad i_{\mathcal{R}} \eta = 1$$

$\mathcal{R}$ is the Reeb vector field of M

Some examples:

- $\mathbb{R}^{2n+1}$ (q^i, p_i, z)

$$\eta = dz - p_i dq^i$$

- $T^*Q \times \mathbb{R} = J^*Q \xrightarrow{\pi_Q^0} Q$

$$\eta_Q = dz - \lambda_Q$$

λ_Q the Liouville 1-form on T^*Q

- (Q, g) a Riemannian manifold

$$\underbrace{S_g(T^*Q)(1)}_{\text{The spheric cotangent bundle}} = \{\alpha \in T^*Q \mid \|\alpha\|_g = 1\}$$

$$i^1 : S_g(T^*Q)(1) \hookrightarrow T^*Q$$

$$\lambda_Q^1 = (i^1)^*(\lambda_Q)$$

The symplectification of C

$$S = C \times \underset{(p)}{\mathbb{R}}, \quad \omega = d(e^p \eta)$$

ω is a homogeneous symplectic structure $\quad \mathcal{L}_{\partial/\partial p}\, \omega = \omega$

The inverse process (the contactification of an exact symplectic manifold)

$(S, \omega = d\lambda)$ an exact symplectic manifold

$$\Downarrow$$

$(C = S \times \underset{(z)}{\mathbb{R}},\ \eta = dz - \lambda)$ a contact structure

Examples:

- $(C = J^*Q = T^*Q \times \mathbb{R}, \eta_Q)$

$$C \times \mathbb{R} \xrightarrow{\simeq} T^*Q \times \mathbb{R} \times \mathbb{R}^+ \subseteq T^*(Q \times \mathbb{R})$$

$$((\alpha, z), p) \longrightarrow e^p(\alpha_q - dz|_z)$$

is a symplectic isomorphism between $(C \times \mathbb{R}, d(e^p \eta_Q))$ and $(T^*Q \times \mathbb{R} \times \mathbb{R}^+, \omega_{Q \times \mathbb{R}})$

- $(C = S_g(T^*Q)(1), \lambda_Q^1)$

$$S_g T^*(Q)(1) \times \mathbb{R} \xrightarrow{\simeq} T^*Q - O(Q)$$

$$(\alpha, p) \longrightarrow e^p \alpha$$

is a symplectic isomorphism between $(S_g(T^*Q)(1) \times \mathbb{R}, d(e^p \lambda_Q^1))$ and $(T^*Q - O(Q), \underset{-d\lambda_Q}{\underset{\shortparallel}{\omega_Q}})$

Another interesting example:

η a contact structure on G

$\Downarrow$

$(G \times \mathbb{R}, \omega)$ the symplectification of G

$\Downarrow$ $\longrightarrow$ (Yano, Ishihara, 1973)

$(T(G \times \mathbb{R}), \omega^c)$ $\leadsto$ the complete lift of ω

$= (TG \times \mathbb{R}) \times \mathbb{R}$

$\quad (p) \quad (z)$

$$\mathcal{L}_{(\partial/\partial p)^c} \omega^c = (\mathcal{L}_{\partial/\partial p} \omega)^c = \omega^c$$

$\Downarrow$

$\mathcal{T}G = TG \times \mathbb{R}$ admits a contact structure

$\quad (z)$

The tangent contact structure η^T

$$\boxed{\eta^T = z\eta^v + \eta^c}$$

$\eta^v \equiv$ the vertical lift of η

$\eta^c \equiv$ the complete lift of η

But, we are in Warsaw! So, ...

1.2 Contact structures

> A contact structure $\mathcal{D}$ on C is a distribution of codimension 1 which is maximally non-integrable

$x \in C \Rightarrow \exists\, U \subseteq M$ an open subset / $x \in U$

$$\mathcal{D}|_U = \langle \eta_U \rangle^\circ$$

$\eta_U \wedge (d\eta_U)^m$ is a volume form on U

D a distribution of codimension 1 on G^{2m+1}

D is a contact structure on G

$\Updownarrow$

$S = D^{\circ} - \{0\} \hookrightarrow T^*G$ is a symplectic submanifold of T^*G

$\Updownarrow$

$\mathbb{R}^{\times} = Gl(1, \mathbb{R}) = \mathbb{R} - \{0\}$

$\mathbb{R}^{\times} \curvearrowright S \xrightarrow{p} G$ is a principal $\mathbb{R}^{\times}$-bundle which admits a homogeneous symplectic structure ω

(Arnold's book, 1989; Grabowski, 2013; Bruce, Grabowska, Grabowski, 2017)

Remark (the connection with the symplectification)

η a contact 1-form on $C \Rightarrow$

$$S = C \times \mathbb{R}^\times = (C \times \mathbb{R}^+) \cup (C \times \mathbb{R}^-)$$

$$(C \times \mathbb{R}^+, \omega) \simeq (C \times \mathbb{R}^-, \omega) \simeq (C \times \mathbb{R}, d(e^t \eta))$$

The typical example (the projective cotangent bundle)

$$\begin{tikzcd} \mathbb{R}^\times \arrow[r, bend left, "\Delta_Q"] & T^*Q - O(Q) \arrow[d, "p"] \\ & \boxed{\mathbb{P}T^*Q = \dfrac{T^*Q - O(Q)}{\mathbb{R}^\times}} \end{tikzcd}$$

$\Delta_Q \rightsquigarrow$ The Liouville vector field on T^*Q

Remark

$\mathbb{P}T^*Q$ is not, in general, orientable

$\Downarrow$

$\mathbb{P}T^*Q$ doesn't admit, in general, contact 1-forms $\square$

M a smooth manifold

$\tilde{M}$ the double cover

$\shortparallel$

$\{(x, \mathcal{O}_x) \mid x \in M, \mathcal{O}_x \text{ is an orientation in } T_xM\}$

$\pi : \tilde{M} \longrightarrow M$ the covering map

$\shortparallel$

$\cancel{M_{/\mathbb{Z}_2}}$

$\tilde{M}$ is orientable

$M = C'$ admits a contact structure D

$$\Downarrow$$

$\tilde{C}$ admits a contact 1-form with contact distribution π-projectable over D

(Blair, 2010)

Example (Q,g) a Riemannian manifold

The double cover $\widetilde{\mathbb{P}T^*Q}$ of $\mathbb{P}T^*Q$ is diffeomorphic to ST^*Q

$$\widetilde{\mathbb{P}T^*Q} \xrightarrow{\cong} ST^*Q / \mathbb{Z}_2$$

$$[\alpha] \longmapsto \left[\frac{\alpha}{\|\alpha\|}\right]$$

Darboux theorem

D a contact structure on C^{2m+1}

$$\Downarrow$$

$$\boxed{\begin{array}{l} \forall x \in M \;\; \exists U \subseteq C \text{ an open subset,} \\ x \in \bar{U} \text{ and local coordinates} \\ (q^i, p_i, z) \text{ on } U \text{ such that} \\ D^0_U = \langle dz - p_i\, dq^i \rangle \end{array}}$$

Legendrian submanifolds

D a contact structure on C^{2m+1}

$S \hookrightarrow C$ a submanifold of C

$$\boxed{\begin{array}{l} \bullet\, S \text{ an integral submanifold of } D \\ \Rightarrow \dim S \leq m \\ \bullet\, \text{There exist integral submanifolds} \\ \text{of } D \text{ of maximum dimension } m \end{array}}$$

$$\boxed{\begin{array}{l} \text{A Legendrian submanifold of } (C, D) \\ \text{is an integral submanifold of } D \\ \text{of dimension } m \end{array}}$$

$L \hookrightarrow C$ a submanifold of C

$$\begin{tikzcd} \mathbb{R}^\times \arrow[r, phantom, "\circlearrowright"] & S \arrow[r, "p"] & C \end{tikzcd}$$ the homogeneous symplectic principal $\mathbb{R}^\times$-bundle

$$\boxed{\begin{array}{c} L \text{ is a Legendrian submanifold of } C \\ \Updownarrow \\ p^{-1}(L) \text{ is a Lagrangian submanifold of } S \end{array}}$$

2 Contact dynamics

In what follows, contact structure is a contact 1-form

(Libermann-Marle, 1987; Arnold, 1989)

2.1 Hamiltonian contact dynamics

η a contact 1-form on C^{2m+1}

$\mathcal{R}$ the Reeb vector field on C

$H : C \longrightarrow \mathbb{R} \in C^\infty(C)$

The Hamiltonian vector field of H is characterized by

$$i_{\mathcal{H}_H} d\eta = dH - \mathcal{R}(H)\eta, \quad i_{\mathcal{H}_H}\eta = -H$$

Remark: $\mathcal{H}_{-1} = \mathcal{R}$

Local expression:

(q^i, p_i, z) Darboux coordinates in C

$$\mathcal{H}_H = \frac{\partial H}{\partial p_i}\, \partial/\partial q^i - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right) \partial/\partial p_i + \left(p_i \frac{\partial H}{\partial p_i} - H\right) \partial/\partial z$$

Contact Hamilton equations

$$\boxed{\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} - p_i\frac{\partial H}{\partial z} \\ \frac{dz}{dt} = p_i\frac{\partial H}{\partial p_i} - H}$$

Remarks

- H is not, in general, a first integral of $\mathcal{X}_H$

$$\dot{H} = \frac{dH}{dt} = -H\,\mathcal{R}(H)$$

- $\mathcal{X}_H$ doesn't preserve, in general, the contact structure

$$\mathcal{L}_{\mathcal{X}_H}\eta = -\mathcal{R}(H)\,\eta$$

- $\mathcal{X}_H$ doesn't preserve, in general, the Liouville volume

$$\mathcal{L}_{\mathcal{X}_H}\left(\eta\wedge(d\eta)^m\right) = -(m+1)\,\mathcal{R}(H)\;\eta\wedge(d\eta)^m$$

However,

$$\mathcal{L}_{\mathcal{R}}\eta = 0, \quad \mathcal{L}_{\mathcal{R}}\left(\eta\wedge(d\eta)^m\right) = 0$$

2.2 Contact Lagrangian Mechanics

The Herglotz variational principle

(Herglotz, 1930)

$L: T\mathcal{Q} = TQ \times \mathbb{R} \longrightarrow \mathbb{R}$ a Lagrangian function

$$\Omega(q_0, q_1) = \{c : [0,1] \longrightarrow Q \text{ a smooth curve} \,/\, c(0) = q_0,\ c(1) = q_1\}$$

$$\mathcal{Z} : \Omega(q_0, q_1) \longrightarrow C^\infty([0,1] \longrightarrow \mathbb{R})$$

$$\boxed{\begin{cases} \dfrac{d\,\mathcal{Z}(c)}{dt} = L(c, \dot{c}, \mathcal{Z}(c)) \\ \mathcal{Z}(c)(0) = z_0 \end{cases}}$$

The contact action functional

$$\boxed{\begin{aligned} \mathcal{A} : \Omega(q_0, q_1) &\longrightarrow \mathbb{R} \\ c &\longrightarrow \mathcal{A}(c) = \mathcal{Z}(c)(1) - \mathcal{Z}(c)(0) \\ &= \int_0^1 L(c(t), \dot{c}(t), \mathcal{Z}(c(t)))\,dt \end{aligned}}$$

c is a critical point of $\mathcal{A}$

$$\Updownarrow$$

$$\boxed{\begin{array}{l} \dot{q}^i = \dfrac{dq^i}{dt}, \quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) - \dfrac{\partial L}{\partial q^i} = \dfrac{\partial L}{\partial \dot{q}^i}\dfrac{\partial L}{\partial z} \\ \dot{z} = L(q^i(t), \dot{q}^i(t), z(t)) \end{array}}$$

Contact Euler-Lagrange Eqs

Remark

The Lagrangian energy E_L is not a first integral

$E_L = \Delta(L) - L$, Δ the Liouville vector field on TQ

$$\dot{E}_L = \frac{dE_L}{dt} = -E_L \frac{\partial L}{\partial z}$$

2.3 The Legendre map and the Hamiltonian formalism (de León and collaborators)

$L : J Q = TQ \times \mathbb{R} \longrightarrow \mathbb{R}$ the Lagrangian function

The Legendre map

$$\mathrm{Leg}_L : JQ \longrightarrow J^*Q$$
$$(t, \bar{v}) \longrightarrow (t, \mathrm{Leg}_L(t, v))$$

$$\boxed{\begin{array}{l} \langle \mathrm{Leg}_L(t, \bar{v}), u \rangle = \dfrac{d}{ds}\Big|_{s=0} L(t, v + s u) \\ u, v \in T_q Q \end{array}}$$

L is regular $\Longleftrightarrow \left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$ is a regular matrix

$\Updownarrow$

$\mathrm{Leg}_L : JQ \longrightarrow J^*Q$ is a (local) diffeomorphism

The Hamiltonian function

$$H = E_L \circ Leg_L^{-1} : \mathcal{T}^*Q \longrightarrow \mathbb{R}$$

$$\Downarrow$$

$\mathcal{X}_H$ the contact Hamiltonian vector field of H in $\mathcal{T}^*Q$

The equivalence

$c : I \longrightarrow Q$ is a solution of the contact Euler-Lagrange equations for L

$$\Updownarrow$$

$\sigma = Leg_L \circ j^1 c : I \longrightarrow \mathcal{T}^*Q$ is a solution of the contact Hamilton equations for H

$$\begin{tikzcd}
\mathcal{T}Q = TQ \times \mathbb{R} \arrow[rr, "Leg_L"] \arrow[dr, "\tilde{\tau}_Q"'] & & \mathcal{T}^*Q = T^*Q \times \mathbb{R} \arrow[dl, "\tilde{\pi}_Q"] \\
& Q & \\
& I \arrow[u, "c"] \arrow[uul, bend left, "j^1 c"] \arrow[uur, bend right, "\sigma = Leg_L \circ j^1 c"'] &
\end{tikzcd}$$

3 Some recent and old topics in contact dynamics

3.1 Connection with linear dissipation

$L : TQ \longrightarrow \mathbb{R}$ a Lagrangian function

$E_L = \Delta(L) - L$ the Lagrangian energy

- We have an special dissipation of the Lagrangian energy for the contact Euler-Lagrange equations

$$\boxed{\dot{E}_L = E_L \frac{\partial L}{\partial z}}$$

η a contact structure on C

$H : C \longrightarrow \mathbb{R}$ a Hamiltonian function

X_H the Hamiltonian vector field of H

- We have an special dissipation of the Hamiltonian energy for the contact Hamilton equations

$$\boxed{\dot{H} = -H \, \mathcal{R}(H)}$$

The typical examples

$$(q^i, \dot{q}^i, z) \longrightarrow L(q^i, \dot{q}^i, z) = \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j - V(q) + \gamma z$$

$$(q^i, p_i, z) \longrightarrow H(q^i, p_i, z) = \frac{1}{2} M^{ij} p_i p_j + V(q) + \gamma z$$

$$\boxed{\begin{array}{c} \text{Contact Lagrangian (Hamiltonian) dynamics} \\ \Updownarrow \\ \text{Standard Lagrangian (Hamiltonian) dynamics} \\ \text{with linear dissipation.} \end{array}}$$

This fact has been extensively used by some Spanish groups (Madrid, Barcelona)

3.2 Connection with standard homogeneous symplectic dynamics

η a contact 1-form on C

$H : C \longrightarrow \mathbb{R}$ a Hamiltonian function

$$\begin{tikzcd}
T(C\times\mathbb{R}) \arrow[d, "T\mathrm{pr}_1"] & (C\times\mathbb{R}, \underset{}{\overset{d(e^p\eta)}{\overset{\shortparallel}{\omega}}}) \arrow[l, "X_{e^p H}"'] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, "H_{hom} = e^p H"] & \\
TC & (C, \eta) \arrow[l, "\mathcal{X}_H"'] \arrow[r, "H"'] & \mathbb{R}
\end{tikzcd}$$

$$\boxed{\begin{array}{c} \text{Contact dynamics in } C \\ \Updownarrow \\ \text{Homogeneous symplectic dynamics in } C\times\mathbb{R} \end{array}}$$

Even more (Libermann-Marle, 1987; Arnold, 1989)

$$\begin{tikzcd} & \mathbb{R}^\times \arrow[d, bend left] & \\ TS \arrow[d, "Tp"'] & (S, \omega) \arrow[l, "X_{H_{hom}}"'] \arrow[r, "H_{hom}"] \arrow[d, "p"] & \mathbb{R} \\ TC & (C, D) \arrow[l, "\mathcal{H}_{H_{hom}}"'] & \end{tikzcd}$$

Δ the infinitesimal generator of the $\mathbb{R}^\times$-action on S

$$\mathcal{L}_\Delta \omega = \omega, \quad \mathcal{L}_\Delta H_{hom} = H_{hom}$$

$\boxed{\text{$\mathcal{H}_{H_{hom}}$ is the Hamiltonian vector field on C associated with the homogeneous Hamiltonian function H_{hom} on S}}$

3.3 Contact dynamics and Legendrian submanifolds

Hamiltonian side

η a contact structure on C

$H : C \longrightarrow \mathbb{R}$ a Hamiltonian function

$\Rightarrow (X_H, \mathcal{R}(H)) : C \longrightarrow (\mathcal{T}C = TC \times \mathbb{R}, \eta^{\mathcal{T}})$

is a Legendrian embedding

(Ibáñez, de León, JCM, Martín de Diego, 1997)

$$
\begin{tikzcd}
 & \mathcal{T}T^*Q \arrow[d, "\tau^{\circ}_{T^*Q}"'] & N_H = (X_H, \mathcal{R}(H))(C) \arrow[l, hook'] \\
 & C = T^*Q \arrow[r, "H"] & \mathbb{R} \\
 & I \arrow[u, "\sigma"] \arrow[uu, bend left=60, "{(\dot\sigma, \mathcal{R}(H)\circ\sigma) = \sigma^{\mathcal{T}}}"] &
\end{tikzcd}
$$

σ is a solution of the Hamilton eqs for $H \Leftrightarrow \sigma^{\mathcal{T}}(I) \subseteq N_H$

The Lagrangian side

$$(TT^*Q, \eta_Q^T) \xrightarrow{\alpha^c} (T^*T^*Q, \eta_{T^*Q})$$

a contact isomorphism (a contact analogous of the symplectic Tulczyjew isomorphism)

$L : TQ \longrightarrow \mathbb{R}$ a Lagrangian function

$$\Downarrow$$

$$T^*L : TQ \longrightarrow T^*TQ$$

$$\| $$

$$(dL, L)$$

The Legendrian submanifold

$$\boxed{N_L = \left((\alpha^c)^{-1} \circ T^*L\right)(TQ)}$$

$$\begin{tikzcd}
T^*TQ \arrow[dr, shift right, "\pi^0_{TQ}"'] & & TT^*Q \arrow[ll, "\alpha^c"'] \arrow[dl, "T\pi^0_Q"] \\
& TQ \arrow[ul, shift right, "T^*L"'] \arrow[r, "L"] & \mathbb{R}
\end{tikzcd}$$

The contact Euler-Lagrange dynamics

$$
\begin{tikzcd}
 & & N_L \arrow[d, bend right] & \\
T^*TQ \arrow[dr, "\pi^\circ_{TQ}"'] & & TT^*Q \arrow[ll, "\alpha^c"'] \arrow[dl, "T\pi^\circ_Q"] \arrow[dr, "\tau_{T^*Q}"] & \\
 & TQ \arrow[rr, "Leg_L"] & & T^*Q \\
 & I \arrow[u, "c"'] \arrow[urr, "Leg_L \circ c"'] \arrow[uur, bend right=60, "(Leg_L \circ c)^T"'] & &
\end{tikzcd}
$$

c is a solution of the contact Euler-Lagrange equations for L

$$\Updownarrow$$

$$(Leg_L \circ c)^T(I) \subseteq N_L$$

$$(Leg_L \circ c)^T(t) = \left(\frac{d}{dt}(Leg_L \circ c), \left. \frac{\partial L}{\partial z} \right|_{c(t)} \right)$$

(Esen, Lainz, de León, Marrero, 2021)

The Warsaw team: Extension for the more general case when the involved contact structures don't come from contact 1-forms

3.4 Connection with Reeb dynamics

η a contact structure on C

i) $H = -1 \Rightarrow \mathcal{H}_H = \mathcal{H}_{-1} = \mathcal{R}$ the Reeb vector field

ii) $\boxed{\text{Under certain regularity conditions, contact Hamiltonian dynamics is Reeb-Liouville dynamics}}$

(Bravetti, de León, JCM, Padrón, 2020)

$H : C \longrightarrow \mathbb{R}$ a Hamiltonian function

$U = \{x \in M \mid H(x) \neq 0\}$ an open subset of M

$\boxed{\eta_H = -\frac{\eta}{H}\Big|_U \text{ is a contact 1-form on } U \text{ and the Reeb vector field } \mathcal{R}_H \text{ is } \mathcal{R}_H = (\mathcal{H}_H)|_U}$

$\Downarrow$

$$\boxed{X_{H|U} \text{ preserves the volume form } -\frac{1}{H^{m+1}}\eta\wedge(d\eta)^m}$$

What happens in the subset $S=H^{-1}(0)$?

0 is a regular value of H, R is transverse to the hypersurface $i: S=H^{-1}(0) \hookrightarrow M$ and $\Theta = i^*\eta$

- $\omega = d\Theta$ is an exact symplectic structure on $S=H^{-1}(0)$
- Δ the Liouville vector field of S $(i_\Delta d\Theta = \Theta) \Rightarrow X_{H|S} = -(R(H)\circ i)\,\Delta$
- The contact Hamiltonian dynamics in S is, up to reparametrization, Liouville dynamics.

The previous description is useful to discuss the existence of invariant volume forms for $X_{H|S}$

Under certain regularity and completeness conditions, $X_{H|S}$ preserves a volume form on S

$$\Updownarrow$$

- $\mathcal{C}$ may be identified with the contactification of S and, in turn, S may be identified with the symplectification of a contact submanifold B of $\mathcal{C}$ of codimension 2

Under the previous identification, $X_{H|S}$ admits a global rectification.

$$\overset{\text{contact}}{\mathcal{C}} \longleftrightarrow \overset{\text{symplectic}}{S\times\mathbb{R}} \longleftrightarrow \overset{\text{contact}}{(B\times\mathbb{R})}\times\mathbb{R}$$

"the symplectic sandwich with contact bread"

(Bravetti, de León, Marrero, Padrón, 2020)

Future collaboration between "Warsaw team" and "La Laguna team": Extension of the previous results for the more general case when C only admits a contact structure

iii) The Jacobi-Maupertuis principle

$H : T^*Q \longrightarrow \mathbb{R}$ $\quad H = K_g + V \circ \pi_Q$

$K_g \equiv$ the kinetic energy induced by a Riemannian metric on Q

$V : Q \longrightarrow \mathbb{R} \in C^\infty(Q)$ the potential energy

Assume that V is bounded

$$V(q) < e, \quad \forall q \in Q$$

(for instance, if Q is compact; if not, we can consider the open subset

$$U_e = \{ q \in Q \mid V(q) < e \})$$

$$\Downarrow$$

$g_e = (e - V)g$ a new Riemannian geometry on Q (the Jacobi metric)

$K_{g_e} : T^*Q \longrightarrow \mathbb{R}$ the kinetic energy associated with g_e

(note that $X_{K_{g_e}}$ is just the geodesic flow of g_e in T^*Q)

(Godbillon, 1969)

- The trajectories of the Hamiltonian system (T^*Q, H) with energy e are, up to reparametrization, geodesics of g_e with Riemannian length 1
- The restriction of $X_{K_{g_e}}$ to $S_{g_e}(T^*Q)(1)$ is tangent to $S_{g_e}(T^*Q)(1)$. In fact,

$$X_{K_{g_e}|S_{g_e}(T^*Q)(1)} = (e - V)\, \mathcal{R}^1_Q$$

$\mathcal{R}^1_Q \equiv$ the Reeb vector field of the standard contact structure on $S_{g_e}(T^*Q)(1)$

A very interesting connection!

- Standard Hamiltonian systems
  (Mechanical Hamiltonian trajectories with fixed energy)

$\updownarrow$

- Riemannian geometry
  (Unit Riemannian geodesics)

$\updownarrow$

- Contact Hamiltonian dynamics
  (Orbits of the Reeb vector field)

Important consequences

- Geodesic flows on closed Riemannian manifolds with negative curvature (Anosov, 1969; previously Arnold)

Flows with structural stability, many dense orbits, ergodicity, ...

The typical example:

$$Q = \mathbb{H}^2 / \Gamma \qquad \mathbb{H}^2 \equiv \text{the hyperbolic plane}$$
$$\Gamma \equiv \text{a discrete isometric group}$$

- Periodic orbits in general:

Periodic orbits for $R_Q^1 \longleftrightarrow$ periodic Hamiltonian trajectories of energy e

Famous Weinstein conjecture: The Reeb flow on a compact contact manifold carries at least one periodic orbit

(Viterbo, 1994; Hofer-Viterbo, 1992; Taubes 2007)

- Integrability

First integrals of the mechanical Hamiltonian system (T^*Q, H) $\updownarrow$

First integrals of the geodesic flow on $S_{g_e}(T^*Q)(1)$ (or the Reeb vector field on $S_{g_e}(T^*Q)(1)$)

Linear first integrals $\equiv$ Noether theorem for the cotangent lift of an isometric action

Higher order first integrals $\equiv$ Killing tensors

- Variational integrators

Hybrid variational integrators based on the Jacobi-Maupertuis principle of least action

(Naira, Ober-Blöbaum, Marsden, 2009)

4. Conclusions and outlook

Contact dynamics plays an important role in several places when you discuss geometric mechanics

In my opinion, two important connections:

i) Symplectic homogeneous Hamiltonian dynamics

Potential applications: • Thermodynamics

• Scaling-symmetries + Standard symmetries + reduction (Marsden-Weinstein, ...)

(Work in progress with Bravetti, Grillo and Padrón)

ii) Reeb dynamics and standard Hamiltonian mechanical systems

Jacobi-Maupertuis principle

- Extension of the principle to the more general Poisson setting (work in progress with Iglesias and Padrón)

But, a lot of work remains to be done here on homogeneous Poisson dynamics!

(Relative) structural stability, periodic orbits, ergodicity, ...

THANKS !

DZIĘKUJĘ !