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Lipschitzowska objętość symplicjalna

Rozprawa doktorska
Doctoral Dissertation

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Streszczenie

We study the Lipschitz simplicial volume, which is a metric version of the simplicial volume. We introduce the piecewise straightening procedure for singular chains, which allows us to generalize the proportionality principle and the product inequality to the case of complete Riemannian manifolds of finite volume with sectional curvature bounded from above.

Słowa kluczowe

objętość symplicjalna, ograniczone kohomologie, homologie singularne, procedura prostowania

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

53-xx Differential geometry
53C-xx Global differential geometry
53C23 Global geometric and topological methods (à la Gromov); differential geometric analysis on metric spaces

Tytuł pracy w języku angielskim

Lipschitz simplicial volume

Contents

Streszczenie

0.1. Objętość symplicjalna

Celem niniejszej pracy jest pokazanie nowych własności Lipschitzowskiej objętości symplicjalnej. Objętość symplicjalna zamkniętej, orientowalnej rozmaitości M wymiaru n jest zdefiniowana jako

$$\|M\| := \inf\{|c|_1 \ : \ c \in C_*(M;\mathbb{R}) \text{ jest cyklem podstawowym}\},$$

gdzie $C_*(M,\mathbb{R})$ jest singularnym kompleksem łańcuchowym na M o współczynnikach rzeczywistych, zaś $|\cdot|_1$ oznacza normę ℓ^1 na łańcuchach singularnych ze względu na bazę złożoną z sympleksów singularnych. Jest to niezmiennik homotopijny, wykazujący jednak związki ze sztywniejszymi strukturami na rozmaitości (na przykład z objętością Riemannowską).

Objętość symplicjalna została wprowadzona i użyta po raz pierwszy przez Gromowa w jego dowodzie twierdzenia Mostowa o sztywności [26, 33]. Z tego względu jest również czasem nazywana *normą Gromowa*. Gromow udowodnił, iż dla dwóch zwartych rozmaitości hiperbolicznych M_1, M_2 tego samego wymiaru (rozumianych jako ilorazy przestrzeni hiperbolicznej przez grupę działającą na niej w sposób wolny i właściwy przez izometrie) ich objętości symplicjalne są dodatnie i proporcjonalne do objętości Riemannowskiej, czyli

$$\frac{\|M_1\|}{\mathrm{vol}(M_1)} = \frac{\|M_2\|}{\mathrm{vol}(M_2)}.$$

Następnie wykorzystał ten fakt w dowodzie twierdzenia Mostowa by pokazać, że dwie homotopijnie równoważne zwarte rozmaitości hiperboliczne muszą mieć tę samą objętość. Objętość symplicjalna została jednak zbadana przez Gromowa dużo dokładniej w jego pracy [12]. Uogólnił on w niej powyższą zasadę na dowolną parę rozmaitości Riemannowskich o izometrycznych nakryciach uniwersalnych, jak również wykazał inne związki objętości symplicjalnej i Riemannowskiej, między innymi udowodnił istnienie oszacowań z góry i z dołu na objętość symplicjalną za pomocą objętości Riemannowskiej, o ile spełnione są pewne warunki dotyczące krzywizny. Gromow w omawianej pracy opisał również szereg zastosowań objętości symplicjalnej.

Objętość symplicjalną stosuje się do badań minimalnej objętości Riemannowskiej [12], hiperbolicznych chirurgii Dehna [33], rozpoznawania rozmaitości grafowych [2] oraz obliczania liczby wielostycznych trajektorii trawersujących pól wektorowych [1]. Jednak jednym z podstawowych zastosowań objętości symplicjalnej są twierdzenia o stopniach odwzorowań. Jeżeli $f : M \to N$ jest ciągłym odwzorowaniem między rozmaitościami zamkniętymi i orientowalnymi tego samego wymiaru, wtedy oczywiście

$$\|N\| \cdot |\deg(f)| \leqslant \|M\|,$$

skąd wniosek, że jeżeli $\|N\| > 0$, to

$$|\deg(f)| \leqslant \frac{\|M\|}{\|N\|}.$$

Objętość symplicjalna może więc pomóc uzyskiwać stałe ograniczające możliwy stopień odwzorowań między rozmaitościami. O ile jednak nie potrafimy obliczać objętości symplicjalnej, powyższa metoda jest tyleż prosta, co bezużyteczna. Pomocne mogą być już wszelkie oszacowania objętości symplicjalnej, czy choćby przykłady rozmaitości, dla których objętość ta jest niezerowa. Szczęśliwie, jest kilka własności pozwalających dowodzić dodatniości objętości symplicjalnej, mianowicie:

1. *Dodatniość przy ujemnej krzywiźnie* [12, 20, 29]

   Jeżeli M jest zamkniętą rozmaitością ujemnie zakrzywioną, wtedy ma dodatnią objętość symplicjalną. Co więcej, jeżeli $\sec(M) \leqslant -1$, wtedy istnieje dodatnia stała C_n zależna jedynie od wymiaru M, taka, że

   $$\|M\| \geqslant C_n \cdot \mathrm{vol}(M).$$

   Można również pokazać, że jest to również prawdą dla zwartych przestrzeni lokalnie symetrycznych niezwartego typu przy ustalonej metryce odpowiedniej przestrzeni symetrycznej [20, 29].

2. *Zasada proporcjonalności* [12]

   Jeżeli M i N są zamkniętymi rozmaitościami Riemannowskimi o izometrycznych nakryciach uniwersalnych, wtedy

   $$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

3. *Nierówność produktowa* [12]

   Dla dowolnych dwóch zamkniętych rozmaitości M i N zachodzą nierówności

   $$\|M\| \cdot \|N\| \leqslant \|M \times N\| \leqslant \binom{\dim M + \dim N}{\dim M} \|M\| \cdot \|N\|.$$

4. *Addytywność ze względu na sumy spójne* [12]

   Jeżeli M i N są tego samego wymiaru $\geqslant 3$, wtedy

   $$\|M \# N\| = \|M\| + \|N\|.$$

Z drugiej strony jest jednak dużo rozmaitości o zerowej objętości symplicjalnej, jak choćby wszystkie rozmaitości o średniowalnej grupie podstawowej [12].

Naturalnym pytaniem jest, co dzieje się w przypadku rozmaitości niezwartych. Najprostszym uogólnieniem objętości symplicjalnej jest infimum norm ℓ^1 lokalnie skończonych cykli podstawowych. Definicja ta redukuje się do klasycznej w przypadku zwartym, jednakże nie wnosi zbyt wiele dla rozmaitości niezwartych. Z wyżej wymienionych faktów jedynie addytywność ze względu na sumy spójne pozostaje prawdziwa w przypadku niezwartym, nie ma więc w istocie zbyt wielu narzędzi, które pozwalałyby podać przykład rozmaitości niezwartej o dodatniej objętości symplicjalnej. Co więcej, wiele takich 'potencjalnych przykładów' ma w istocie zerową objętość symplicjalną. Warto tu wspomnieć choćby fakt udowodniony przez Gromova [12], że produkt trzech otwartych rozmaitości ma zawsze zerową objętość symplicjalną.

Gromov zasugerował jednak rozwiązanie, które pozwoliłoby uniknąć powyższych niedogodności. Mianowicie, można rozważać objętość symplicjalną liczoną na lokalnie skończonych Lipschitzowskich cyklach podstawowych, tzn. takich, które składają się z sympleksów o jednostajnej stałej Lipschitza. W ten sposób uzyskujemy *Lipschitzowską objętość symplicjalną*:

$$\|M\|_{\mathrm{Lip}} := \inf\{|c|_1 \ : \ c \in C_*^{lf}(M;\mathbb{R}) \text{ is a fundamental cycle, } \mathrm{Lip}(c) < \infty\},$$

gdzie $\mathrm{Lip}(\sum_i a_i\sigma_i) = \sup_i \mathrm{Lip}(\sigma_i)$. Tak zdefiniowana objętość symplicjalna redukuje się do klasycznej dla gładkich rozmaitości zwartych, daje jednak ciekawe rezultaty również w przypadku niezwartym.

W pracy udowodnione są poniższe twierdzenia, które potwierdzają przypuszczenie, jakoby Lipschitzowska objętość symplicjalna była właściwym uogólnieniem objętości symplicjalnej.

Twierdzenie A. *Jeżeli M jest zupełną rozmaitością Riemannowską ujemnie zakrzywioną, wtedy ma dodatnią Lipschitzowską objętość symplicjalną. Co więcej, jeżeli $\sec(M) \leqslant -1$, wtedy istnieje dodatnia stała C_n zależna jedynie od wymiaru M taka, że*

$$\|M\|_{\mathrm{Lip}} \geqslant C_n \cdot \mathrm{vol}(M).$$

Twierdzenie B (Nierówność produktowa)**.** *Dla dowolnych dwóch rozmaitości o ograniczonej z góry krzywiźnie sekcyjnej M i N zachodzą nierówności*

$$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leqslant \|M \times N\|_{\mathrm{Lip}} \leqslant \binom{\dim M + \dim N}{\dim M} \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}.$$

Twierdzenie C (Zasada proporcjonalności)**.** *Jeżeli M i N są zupełnymi rozmaitościami Riemannowskimi o izometrycznych nakryciach uniwersalnych i ograniczonej z góry krzywiźnie sekcyjnej, wtedy*

$$\frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)} = \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)}.$$

Powyższe twierdzenia są przedstawione w pracy jako Twierdzenia 2.2.7, 2.2.10 oraz 2.2.6. Dowód Twierdzenia A jest znany specjalistom, jednak nie jest szczegółowo opisany w literaturze, dlatego jest podany dla kompletności. Dowody Twierdzeń B oraz C są nowe i wykorzystują kawałkową procedurę prostowania, wprowadzoną przez autora w pracy [32]. Warto wspomnieć, że twierdzenia te (B i C) zostały ostatnio udogólnione przez Franceschiniego na przypadek rozmaitości bez ograniczeń na krzywiznę [14].

By pokazać, że Lipschitzowska objętość symplicjalna zachowuje się bardzo podobnie do klasycznej objętości symplicjalnej, dla zupełności udowodnione jest też poniższe twierdzenie, przedstawione w pracy jako Twierdzenie 2.2.12. Jest ono znane, lecz nigdzie dotychczas nieopublikowane.

Twierdzenie D. *Jeżeli M jest rozmaitością taką, że $\|M\|_{\mathrm{Lip}} < \infty$ oraz $\pi_1(M)$ jest średniowalna, wtedy $\|M\|_{\mathrm{Lip}} = 0$.*

W pracy korzystamy z poniższych technik wykorzystywanych do badania objętości symplicjalnej, mających również swoje wersje w przypadku Lipschitzwoskim. Zasada dualności została wprowadzona przez Gromowa [12] i zaadoptowana do przypadku Lipschitzowskiego przez Löh i Sauera [25]. Dyfuzja łańcuchów została również wprowadzona pierwotnie przez Gromowa [12], opisujemy ją jednak w inny od niego sposób, w szczególności bez użycia multikompleksów.

Zasada dualności

Jeżeli M jest n-wymiarową zamkniętą, orientowalną rozmaitością, wtedy jej objętość symplicjalną da się wyrazić za pomocą półnormy ℓ^∞ jej podstawowej klasy kohomologii. Dokładniej

$$\|M\| = \frac{1}{\|[M]^*\|_\infty},$$

gdzie $[M]^*$ jest podstawową klasą kohomologii w $H^n(M;\mathbb{R})$, $\|[M]^*\|_\infty = \inf\{\|\phi\|_\infty : \phi \in C^n(M;\mathbb{R})\,, [\phi] = [M]^*\}$ oraz $\|\phi\|_\infty = \sup_{\sigma\in C(\Delta^n,M)} |\phi(\sigma)|$ dla $\phi \in C^n(M;\mathbb{R})$. Równość ta jest w wielu przypadkach bardzo pomocna przy obliczaniu objętości symplicjalnej. Nie jest ona jednak prawdziwa dla Lipschitzowskiej objętości symplicjalnej, nawet gdy zastosujemy kohomologiczną klasę podstawową o zwartym nośniku. Istnieje jednak pewien (niestety bardziej skomplikowany) wariant zasady dualności, działający dla Lipschitzowskiej objętości symplicjalnej, opisany w pracy jako Twierdzenie 2.3.2.

Twierdzenie. *Niech M będzie zupełną rozmaitością orientowalną, spójną. Wtedy dla dowolnej lokalnie skończonej rodziny $A \subset C(\Delta^n, M)$ złożonej z sympleksów o jednostajnej stałej Lipschitza zachodzi równość*

$$\|M\|^A = \frac{1}{\|[M]^*_{\mathrm{Lip}}\|^A_\infty},$$

gdzie $[M]^*_{\mathrm{Lip}}$ jest kohomologiczną klasą podstawową o Lipschitzowsko-zwartym nośniku (Definicja 1.2.20), $\|\cdot\|^A$ jest półnormą indukowaną przez normę

$$|c|_1^A := \begin{cases} |c|_1 & \textit{jeżeli } \operatorname{supp}(c) \subset A, \\ \infty & \textit{w przeciwnym przypadku} \end{cases}$$

na łańcuchach singularnych, zaś $\|\cdot\|^A_\infty$ jest półnormą indukowaną przez normę $\|\phi\|^A_\infty = \sup_{\sigma\in A} |\phi(\sigma)|$ na kołańcuchach singularnych.

Dyfucja łańcuchów

Niech M będzie rozmaitością, zaś $K \subset M$ jej łukowo spójnym, zwartym podzbiorem. Wtedy jako $\Pi(M,K)$ oznaczamy grupę złożoną z odwzorowań (niekoniecznie ciągłych!) przyporządkowujących punktom z K klasy homotopii ścieżek w M względem ich końców, takich, że dla $g \in \Pi(M,K)$, $g = (\gamma_x)_{x\in K}$:

- $\gamma_x(0) = x$;
- $\gamma_x(1) \in K$;
- g ma skończony nośnik, tzn. jedynie dla skończenie wielu $x \in K$ ścieżki γ_x są nietrywialne;
- odwzorowanie $x \mapsto \gamma_x(1)$ wyznacza bijekcję na K.

Dyfuzja łańcuchów jest techniką polegającą na modyfikowaniu danego łańcucha singularnego za pomocą działania powyższej grupy. W szczególności, jeżeli powyższa grupa jest średniowalna, zaś łańcuch składa się z sympleksów o różnych wierzchołkach, jesteśmy w stanie 'uśrednić' łańcuch tak, by usunąć pewne sympleksy z obliczeń (Lipschitzwoskiej) objętości symplicjalnej. Prowadzi to do następującego stwierdzenia, pojawiającego się w pracy jako Stwierdzenie 2.4.1, które jest głównym technicznym rezultatem uzyskiwanym za pomocą dyfuzji łańcuchów.

Stwierdzenie. *Niech M będzie rozmaitością, zaś $K \subset M$ łukowo spójnym zwartym podzbiorem takim, że obraz przekształcenia $\pi_1(K) \to \pi_1(M)$ jest średniowalny. Wówczas dla dowolnego (lokalnie skończonego, Lipschitzwoskiego) łańcucha singularnego $c = \sum_i a_i\sigma_i$ takiego, że każdy sympleks σ_i ma różne wierzchołki, zachodzi następująca nierówność.*

$$\|[c]\|_1 \leqslant \sum_{\{i\,:\,\sigma_i \text{ nie ma krawędzi w } K\}} |a_i|.$$

Nietrudnym wnioskiem z techniki dyfuzji jest Twierdzenie D. By je udowodnić, wystarczy najpierw podzielić barycentrycznie dowolny łańcuch i zmodyfikować go tak, by wszystkie sympleksy miały różne wierzchołki, a następnie zastosować dyfuzję łańcuchów na odpowiednio dużym zwartym podzbiorze badanej rozmaitości.

0.2. Kawałkowa procedura prostowania

Kawałkowa procedura prostowania, która stanowi główną techniczną część pracy, jest nowa i opisana przez autora również w jego pracy [32]. Klasyczna procedura prostowania, możliwa do zastosowania jedynie dla rozmaitości niedodatnio zakrzywionych, działała następująco. Mając dany sympleks singularny σ na niedodatnio zakrzywionej jednospójnej rozmaitości Riemannowskiej M, istnieje dokładnie jeden sympleks geodezyjny (zdefiniowany indukcyjnie jako geodezyjny stożek nad podstawą, będącą sympleksem geodezyjnym) o takich samych (i tak samo uporządkowanych) wierzchołkach. Przekształcenie przyporządkowujące sympleksowi ten właśnie sympleks geodezyjny, zwany *wyprostowaniem* σ i oznaczanym przez $\mathrm{str}(\sigma)$, rozszerza się do przekształcenia łańcuchowego o normie ℓ^1 mniejszej lub równej 1 i łańcuchowo homotopijnego z identycznością na $C_*(M;\mathbb{R})$. Podobną procedurę można zastosować na dowolnej niedodatnio zakrzywionej rozmaitości M przez podniesienie danego sympleksu σ do nakrycia uniwersalnego, wyprostowania go tam i opuszczenia rezultatu. Główną zaletą procedury jest możliwość zastąpienia dowolnego łańcucha singularnego łańcuchem dużo bardziej regularnym bez zwiększenia normy, co pozwala czasem znacząco uprościć obliczenia objętości symplicjalnej.

Kawałkowa procedura prostowania jest uogólnieniem klasycznej procedury prostowania na przypadek rozmaitości o krzywiźnie ograniczonej z góry i Lipschitzowskiej objętości symplicjalnej. Dla danego Lipschitzwoskiego sympleksu singularnego σ i rozmaitości Riemannowskiej M się ona z następujących etapów.

- σ należy rozdrobnić barycentrycznie m razy, gdzie minimalne m, dla którego procedura zadziała, zależy od stałej Lipschitza σ, wymiaru M i górnego ograniczenia na krzywiznę M

- Każdy sympleks σ' z m-krotnie rozdrobnionego sympleksu σ podnosimy do pewnego *otoczenia eksponencjalnego*. Dla danego punktu $x \in M$ jego otoczenie eksponencjalne V_x jest zdefiniowane jako kula o odpowiednim promieniu (ograniczonym z dołu przez stałą zależną od krzywizny M) w przestrzeni stycznej T_xM, z metryką Riemannowską indukowaną z przekształcenia $\exp_x : T_xM \to M$.

- Eksponencjalne otoczenie każdego punktu ma tę własność, że punkty w pewnym (jednostajnym) otoczeniu zera w V_x dla $x \in M$ mają jednostajnie dodatni promień włożoności. Możemy więc klasycznie wyprostować podniesienie każdego sympleksu σ', o ile jest on odpowiednio mały, a to gwarantuje nam wybór m.

- Opuszczamy wyprostowane w powyższy sposób sympleksy. Ponieważ otoczenia eksponencjalne dopuszczają między sobą lokalnie izometryczne odwzorowania przejścia, powyższa procedura zachowuje brzegi sympleksów i możemy je wszystkie skleić ponownie do jednego sympleksu $\mathrm{str}_m(\sigma)$.

W szczególności uzyskujemy następujące stwierdzenie (Wniosek 3.1.20).

Stwierdzenie. *Jeżeli M jest zupełną rozmaitością Riemannowską o ograniczonej z góry krzywiźnie, dla każdego cyklu $c \in C_*^{lf,\mathrm{Lip}}(M;\mathbb{R})$ istnieje $m \in \mathbb{N}$ takie, że cykl $\mathrm{str}_m(c)C_*^{lf,\mathrm{Lip}}(M;\mathbb{R})$ jest dobrze zdefiniowany, m-ty podział barycentryczny $\mathrm{str}_m(c)$ składa się z sympleksow geodezyjnych, $[c] = [\mathrm{str}_m(c)] \in H_*^{lf,\mathrm{Lip}}(M;\mathbb{R})$ oraz $|\,\mathrm{str}_m(c)|_1 \leqslant |c|_1$. W szczególności Lipschitzowską objętość symplicjalną można obliczać jedynie na kawałkami wyprostowanych łańcuchach.*

Homologie kawałkami gładkie

Homologie kawałkami gładkie to homologie podkompleksu singularnego kompleksu łańcuchowego składającego się jedynie z łańcuchów złożonych z sympleksów kawałkami gładkich (Definicja 3.2.2). Definicję tę można uogólnić na przypadek lokalnie skończony i Lipschitzowski. Wprowadzamy również kawałkami gładki wariant homologii Milnora-Thurstona, tzn. homologii, gdzie łańcuchy są odpowiednimi miarami Borelowskimi na zbiorze sympleksów singularnych. Są one wyposażone w pół-normę indukowaną przez normę absolutnej wariacji na miarach. Stosując kawałkową procedurę prostowania do tychże miar łatwo dowodzimy następującego stwierdzenia (Stwierdzenie 3.2.10).

Stwierdzenie. *Niech M będzie zupełną rozmaitością Riemannowsą o ograniczonej z góry krzywiźnie. Wówczas kawałkami gładkie, lokalnie skończone Lipschitzowskie ℓ^1-homologie oraz kawałkami gładkie Lipschitzowskie homologie Milnora-Thurstona są izometrycznie izomorficzne.*

0.3. Zastosowania procedury prostowania

Pod koniec pracy, korzystając z kawałkowej procedury prostowania, dowodzone są własności Lipschitzowskiej objętości symplicjalnej, mianowicie Twierdzenia A, B oraz C.

Ograniczenie dolne przez objętość Riemannowską dla ujemnie zakrzywionych rozmaitości (Twierdzenie A)

Dowód faktu, iż dla zupełnej rozmaitości Riemannowskiej M o krzywiźnie ograniczonej z góry przez -1 zachodzi nierówność

$$\|M\|_{\mathrm{Lip}} \geqslant C_n \cdot \mathrm{vol}(M)$$

dla pewnej stałej $C_n > 0$ zależnej jedynie od $n = \dim M$, wynika z tego, iż procedura prostowania (klasyczna) działa dla Lipschitzowskich lokalnie skończonych łańcuchów oraz z ograniczenia górnego na objętość geodezyjnych sympleksów w przestrzeni ujemnie zakrzywionej.

Nierówność produktowa (Twierdzenie B)

Nierówność

$$\|M \times N\|_{\mathrm{Lip}} \leqslant \binom{\dim M + \dim N}{\dim M} \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}$$

dość łatwo wynika z własności symplicjalnej aproksymacji produktu krzyżowego oraz tego, że produkt krzyżowy (lokalnie skończonych) klas podstawowych jest klasą podstawową.

Dowód nierówności $\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leqslant \|M \times N\|_{\mathrm{Lip}}$ dla rozmaitości o ograniczonej z góry krzywiźnie jest nieco trudniejszy. Mówimy, że lokalnie skończona rodzina $(k+l)$-wymiarowych jednostajnie Lipschitzowskich sympleksów singularnych A na $M \times N$ jest (k,l)-*rzadka*, jeżeli rodziny sympleksów A_M i A_N na M i N złożone z rzutowania pewnych k i l wymiarowych ścian sympleksów na M i N odpowiednio są lokalnie skończone (Definicja 4.2.1). Ponieważ produkt krzyżowy jest dobrze określony dla kołańcuchów o Lipschitzowsko zwartych nośnikach, mamy nierówność

$$\|\phi \times \psi\|_\infty^A \leqslant \|\phi\|_\infty^{A_M} \cdot \|\psi\|_\infty^{A_N}$$

dla dowolnych dwóch kołańcuchów $\phi \in C^*_{cs,\mathrm{Lip}}(M,\mathbb{R})$, $\psi \in C^*_{cs,\mathrm{Lip}}(N,\mathbb{R})$. Korzystając z zasady dualności dla Lipschitzowskiej objętości symplicjalnej i powyższej nierówności otrzymujemy nierówność

$$\|M \times N\|^A \geqslant \|M\|^{A_M} \cdot \|N\|^{A_N}.$$

Jedyne, czego brakuje by udowodnić nierówność produktową to stwierdzenie, iż Lipschitzowską objętość symplicjalną da się obliczać na cyklach o rzadkich nośnikach. To zaś wynika z zastosowania kawałkowej procedury prostowania dla odpowiednio dobranego zbioru wierzchołków prostowanych sympleksów.

Zasada proporcjonalności (Twierdzenie C)

Niech M i N będą dwiema rozmaitościami Riemannowskimi o izometrycznych nakryciach uniwersalnych. Głównym narzędziem w dowodzie zasady proporcjonalności jest istnienie wprowadzonego przez Thurstona [33] odwzorowania rozsmarowującego smear_* między kawałkami gładkim kompleksem singularnym na M i kawałkami gładkim kompleksem Milnora-Thurstona na N. Odwzorowanie to jest łańcuchowe i nie zwiększa normy, ponadto

$$\langle \mathrm{dvol}_M, \mathrm{smear}_*([M])\rangle := \int_{\mathscr{P}C^1(\Delta^n,M)} \int_{\Delta^n} \sigma^* \,\mathrm{dvol}_M \; d\,\mathrm{smear}_*([M])(\sigma) = \mathrm{vol}(M),$$

gdzie $[M]$ jest klasą podstawową M, zaś $\mathscr{P}C^1(\Delta^n, M)$ jest zbiorem sympleksów kawałkami gładkich.

Korzystając z kawałkowej procedury prostowania, dla danego cyklu podstawowego c na M jesteśmy w stanie skonstruować Lipschitzowski, lokalnie skończony cykl singularny c' taki, że $[c'] = [\mathrm{smear}_*(c)]$ oraz $|c'| \leqslant \|\,\mathrm{smear}_*(c)\|$ (gdzie na cyklach Milnora-Thurstona rozważamy normę absolutnej wariacji miary). Stąd wniosek, iż

$$\frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \|N\|_{\mathrm{Lip}} \leqslant \|M\|_{\mathrm{Lip}}.$$

Podobną nierówność w drugą stronę możemy łatwo uzyskać zamieniając obie rozmaitości miejscami w powyższym rozumowaniu.

Introduction

The simplicial volume is a homotopy invariant of manifolds defined for a closed manifold M as

$$\|M\| := \inf\{|c|_1 \ : \ c \text{ is a fundamental cycle in with real coefficients}\},$$

where $|\cdot|_1$ is the ℓ^1 norm on $C_*(M;\mathbb{R})$ with respect to the basis consisting of singular simplices. Despite relatively simple definition, it has many applications. Most of them use the property that although the simplicial volume is a homotopy invariant, it has many connections with more rigid structures on manifolds such as the Riemannian volume.

The simplicial volume was first introduced by Gromov in his proof of Mostov's rigidity theorem [26, 33], therefore it is often referred to as the *Gromov norm.* He proved that the simplicial volume of a closed hyperbolic manifold is non-zero and proportional to its volume, that is for any two hyperbolic manifolds of the same dimension one has

$$\frac{\|M_1\|}{\mathrm{vol}(M_1)} = \frac{\|M_2\|}{\mathrm{vol}(M_2)}.$$

He used this fact to show that two homotopy equivalent hyperbolic manifolds need to have the same volume. In particular, the simplicial volume was the ingredient of the proof that linked the topological structure of a hyperbolic manifold with its geometry. In his remarkable work [12], he generalised this property, called the *proportionality principle*, to every pair of manifolds with isometric universal covers.

Moreover, Gromov showed that the simplicial volume can be applied to much wider variety of problems, e.g. degree theorems. If $f : M \to N$ is a continuous map between compact n-dimensional manifolds, then

$$\|N\| \cdot |\deg(f)| \leqslant \|M\|.$$

Therefore, if $\|N\| \neq 0$ one has the estimate

$$|\deg(f)| \leqslant \frac{\|M\|}{\|N\|}.$$

In particular, this implies that manifolds with positive simplicial volume are rigid in the sense all self maps $M \to M$ must be of degree 0 or ± 1.

The above estimate on the degree is straightforward, but useless, unless one can compute or estimate the simplicial volume, or at least decide if it is zero or not. The exact computation of the simplicial volume is usually a very difficult problem unless it is 0, e.g. in the case of manifolds with an amenable fundamental group. There are only a few classes of examples of manifolds for which the simplicial volume is known and positive, including hyperbolic spaces [12], closed manifolds locally isometric to products of surfaces of genus $\geqslant 2$ [4] and connected sums of these. Fortunately, Gormov introduced some estimates of the simplicial volume in the case of Riemannian manifolds. Namely, if the sectional curvature of M satisfies $\sec(M) \leqslant -1$ then

$$\|M\| \geqslant C_n \operatorname{vol}(M)$$

for some constant C_n, depending only on the dimension of M. On the other hand, if $\operatorname{Ricci}(M) \geqslant -(n-1)$ then there exists a constant D_n depending on $\dim M$, such that

$$\|M\| \leqslant D_n \operatorname{vol}(M).$$

Therefore if $f : M \to N$ is a map between n-dimensional closed Riemannian manifolds such that $\sec(N) \leqslant -1$ and $\operatorname{Ricci}(M) \geqslant -(n-1)$ then

$$|\deg(f)| \leqslant \frac{D_n}{C_n} \cdot \frac{\operatorname{vol}(M)}{\operatorname{vol}(N)}.$$

Note that the above technique of obtaining non-trivial and useful degree theorems depends on the examples of manifolds with positive simplicial volume. The only examples described by Gromov in [12] were manifolds obtained by taking the products and connected sums of negatively curved manifolds. He used the facts that

$$\|M\| \cdot \|N\| \leqslant \|M \times N\| \leqslant \binom{\dim M + \dim N}{\dim M} \|M\| \cdot \|N\|,$$

and if $\dim M = \dim N \geqslant 3$ then also

$$\|M \# N\| = \|M\| + \|N\|.$$

However, Gromov stated the following conjecture.

Conjecture. *Let M be a closed Riemannian manifold with $\sec(M) \leqslant 0$ and $\operatorname{Ricci}(M) < 0$. Then $\|M\| > 0$.*

The problem is still open. However, there is some progress. Namely, Lafont and Schmidt showed that the simplicial volume is positive for locally symmetric spaces of non-compact type [20, 29]. Using the estimates of the simplicial volume by the Riemannian volume and the proportionality principle, they showed that if N is a locally symmetric space of non-compact type and $\operatorname{Ricci}(M) \geqslant -(n-1)$, then there exists a constant E_n depending only on the dimension of N and M such that

$$|\deg(f)| \leqslant E_n \frac{\operatorname{vol}(M)}{\operatorname{vol}(N)}.$$

There are also other applications of the simplicial volume. Gromov in [12] used it to estimate the minimal volume, while Thurston in [33] used the simplicial volume to show that the hyperbolic volume decreases under the hyperbolic Dehn surgery. In the theory of 3-manifolds, the simplicial volume was also used to recognise graph manifolds [2]. Nowadays, the simplicial volume is applied to compute the number of multi-tangent trajectories of traversing vector fields [1].

A natural question to ask is if there is a generalisation of this invariant to non-compact case. The answer is: yes, and it can be defined for an orientable manifold by

$$\|M\| = \{|c|_1 \; : \; c \in C_n^{lf}(M, \mathbb{R}) \text{ is a locally finite fundamental cycle}\}.$$

The above invariant obviously restricts to the classical simplicial volume for compact manifolds. The problems arise when one checks if various properties of the simplicial volume are valid also for non-compact manifolds. Almost all the techniques that in the compact case can be used to prove the positivity of the simplicial volume do not work without the assumption on compactness.

- There is no lower bound on the simplicial volume (for negatively curved manifolds) of the form
$$\|M\| \geqslant C_n \operatorname{vol}(M).$$
A counterexample is given by the hyperbolic space. It has infinite volume and constant negative curvature, but there exist proper maps $f : \mathbb{H}^n \to \mathbb{H}^n$ of arbitrary large degree, hence $\|\mathbb{H}^n\| = 0$.

- The estimate from below on the simplicial volume of the product of manifolds
$$\|M\| \cdot \|N\| \leqslant \|M \times N\|$$
does not hold if both manifolds M and N are open. In particular, there is a theorem of Gromov [12] that if M is the product of three open manifolds, then $\|M\| = 0$. There is also a similar result that $\|M\| = 0$ if M is a locally symmetric space of $\mathbb{Q}$-rank$\geqslant 3$ [25].

  On the other hand, non-compact locally symmetric spaces of non-compact type of $\mathbb{Q}$-rank one have positive simplicial volume [25]. However, very little is known for non-compact locally symmetric spaces of $\mathbb{Q}$-rank two and products of two open manifolds.

  Question. *Does the simplicial volume of a product of two open manifolds is either 0 or infinite?*

  Does the simplicial volume of a non-compact locally symmetric space of $\mathbb{Q}$-rank 2 is either 0 or infinite?

- The proportionality principle holds in some special cases, e.g. for some families of locally symmetric spaces of $\mathbb{Q}$-rank 1 [7], but not in general. The easiest example can be given by the hyperbolic space $\mathbb{H}^n$ and any closed hyperbolic n-manifold M. If the proportionality principle was true, then the fact that $\|M\| > 0$ would imply $\|\mathbb{H}^n\| = \infty$, but we know it is 0. The proportionality principle fails even if we restrict our attention to manifolds of finite volume. Indeed, let M' be a non-compact hyperbolic n-manifold of finite volume. Then although M^3 and $(M')^3$ are both covered by $(\mathbb{H}^n)^3$, we have $\|M\|^3 > 0$ and $\|M'^3\| = 0$.

As we may observe, the properties that do not generalize to the non-compact case are exactly these which can be used to show the positivity of the simplicial volume of certain manifolds. In fact, many manifolds that we would expect to have positive simplicial volume actually have vanishing one. Therefore applications of the simplicial volume are very limited in the non-compact case.

To avoid these inconveniences, Gromov proposed in [12] also other variants of the simplicial volume for non-compact manifolds. Namely, he considered the simplicial volume computed on chains consisting of simplices of bounded size.
$$\|M\|_{new} := \inf\{|c|_1 \ : \ c \in C_*^{lf}(M;\mathbb{R}) \text{ is a fundamental cycle, } \text{"}size\text{"}(c) < \infty\}.$$
By varying the definition of 'size', we can obtain various simplicial volumes. One of the most promising ways of defining "size" is requiring the uniform Lipschitz constant for every simplex in a given chain. It is, however, not the only definition that leads to interesting results. Using another definition that a chain has finite "size" if all simplices of this chain are of bounded diameter, Gromov showed the existence of extremal manifolds in every dimension, i.e. complete Riemannian manifolds with the sectional curvature bounded by ± 1 that cannot be deformed in a way decreasing the volume. But that is another story...

The Lipschitz definition of "size" leads to the Lipschitz simplicial volume, defined for a complete Riemannian manifold M as

$$\|M\|_{\mathrm{Lip}} := \inf\{|c|_1 \ : \ c = \sum_i a_i\sigma_i \in C_*^{lf}(M;\mathbb{R}) \text{ is a fundamental cycle, } \exists_{L<\infty}\forall_i \operatorname{Lip}(\sigma_i) \leqslant L\}.$$

This version of the simplicial volume is obviously designed for manifolds with finite volume. Every fundamental cycle with the Lipschitz constant L has norm at least $\frac{\mathrm{vol}(M)}{L^n \mathrm{vol}(\Delta^n)}$, hence if the volume of M is infinite, so is the Lipschitz simplicial volume. However, because on compact manifolds every cycle can be approximated by a smooth one without increasing the ℓ^1 norm, and finite smooth chains are Lipschitz, the Lipschitz simplicial volume equals the classical one for compact manifolds.

In this work we prove the following theorems, presented later as Theorems 2.2.7, 2.2.6 and 2.2.10 respectively. They justify the hope that the Lipschitz simplicial volume is a proper generalization of the simplicial volume.

Theorem A. *If M is a complete Riemannian manifold with $\sec(M) \leqslant -1$ then there exists a constant C_n, depending only on $n = \dim M$, such that*

$$\|M\|_{\mathrm{Lip}} \geqslant C_n \operatorname{vol}(M).$$

Theorem B (Product inequality)**.** *For any two complete Riemannian manifolds M and N such that $\sec(M), \sec(N) < K < \infty$ there are inequalities*

$$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leqslant \|M \times N\|_{\mathrm{Lip}} \leqslant \binom{\dim M + \dim N}{\dim M} \|N\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}.$$

Theorem C (Proportionality principle)**.** *If M and N are two complete Riemannian manifold with $\sec(M), \sec(N) < K < \infty$ which have isometric fundamental covers then*

$$\frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)} = \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)}.$$

The proofs of the product inequality and the proportionality principle are original and made by the author, who published them also in his present work [32]. However, both these theorems hold without curvature assumptions by the recent work of Franceschini [14]. The proof of Theorem A is known for specialists and described here for the sake of completeness, because it was not described in details before. For the similar reason we prove also the following theorem, which is denoted in the work as Theorem 2.2.12.

Theorem D. *Let M be a complete manifold such that $\|M\|_{\mathrm{Lip}} < \infty$ and $\pi_1(M)$ is amenable. Then $\|M\|_{\mathrm{Lip}} = 0$.*

The above theorems indicate in particular that the Lipschitz simplicial volume is positive in many situations when one would expect it. In contrast to the 'natural' generalisation of the simplicial volume to the non-compact case, this one has more chances of having interesting applications.

We present one of such applications, following Löh and Sauer [25]. If N is a Riemannian locally symmetric space of non-compact type of finite volume, then by [3] there exists a compact locally symmetric space N' with isometric universal cover. The fact that $\|N'\| \geqslant C_n \cdot \mathrm{vol}(N')$ for some constant $C_n > 0$ (depending only on $n = \dim N'$) [20] combined with the proportionality principle for the Lipschitz simplicial volume yields

$$\|N\|_{\mathrm{Lip}} \leqslant C_n \operatorname{vol}(N').$$

Therefore if M is a Riemannian manifold with $\sec(M) \leqslant 1$ and $\mathrm{Ricci}(M) \geqslant -(n-1)$ then for any proper Lipschitz map $f : M \to N$, we have

$$|\deg(f)| \leqslant \frac{D_n}{C_n} \cdot \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)}.$$

Let us comment on the proofs of the product inequality and the proportionality principle. All known proofs of these properties for compact manifolds use bounded cohomology at some point. It is defined as cohomology of the chain complex $C_b^*(M, \mathbb{R})$ of bounded singular chains, i.e.

$$C_b^k(M; \mathbb{R}) := \{\phi \in C^k(M; \mathbb{R}) \ : \ \sup_{\sigma \in C(\Delta^k, M)} |\phi(\sigma)| < \infty\}.$$

Note that bounded cochains are endowed with the canonical ℓ^∞ norm, which induce the ℓ^∞ semi-norm on cohomology classes. For compact manifolds, bounded cohomology is linked with the simplicial volume by the *duality principle*

$$\|M\| = \frac{1}{\|[M]^*\|_\infty},$$

where $[M]^* \in H^n(M, \mathbb{R})$ is a fundamental cohomology class. Although computing the ℓ^∞ norm of a given class is usually a serious problem, in many cases it simplifies significantly the computation of the simplicial volume. However, the applications of this tool to non-compact generalisations of the simplicial volume are often limited. For the 'classical' generalization of the simplicial volume, the duality principle holds, but in a much weaker setting, namely

$$\|M\|^A = \frac{1}{\|[M]^*_{cs}\|^A_\infty},$$

where $[M]^*_{cs}$ is a compactly supported fundamental cohomology class, A is any locally finite family of simplices and the norms $\|\cdot\|^A$ and $\|\cdot\|^A_\infty$ are the ℓ^1 semi-norm on (homology classes of) chains with supports in A and ℓ^∞ semi-norm on cohomology computed on the simplices in A respectively. This principle is not only more complicated than the original one, but involves also bounded cohomology with compact supports, where the cup product is not well defined. This flaw is one of the direct reasons why the proof of the product inequality does not generalize to the non-compact case.

For the Lipschitz simplicial volume, one has the duality principle of the form

$$\|M\|^A_{\mathrm{Lip}} = \frac{1}{\|[M]^*_{cs,\mathrm{Lip}}\|^A_\infty},$$

where $[M]^*_{cs,\mathrm{Lip}}$ is fundamental cohomology class with Lipschitz compact support (see Definition 1.2.20), A is any locally finite family of Lipschitz simplices and the norms $\|\cdot\|^A$ and $\|\cdot\|^A_\infty$ are the ℓ^1 semi-norm on (homology classes of) Lipschitz chains with supports in A and ℓ^∞ semi-norm on cohomology computed on the simplices in A respectively. At the first glance, this version does not seem any better than in the non-Lipschitz case. However, it is, though not much. The only (but significant) advantage of the cohomology with Lipschitz compact supports over the cohomology with just compact supports is that the cocycles are closed under taking the cross product. This allows us to make some progress in proving the product inequality, but it is not enough. Moreover, it does not help much in establishing other properties. What distinguishes significantly the Lipschitz and non-Lipschitz simplicial volumes of non-compact manifolds, is the existence of the straightening procedure.

The straightening procedure, in the first form was introduced by Dupont in [11], but it was Thurston [33], who first used it in the context involving the simplicial volume. The main goal of the procedure is to simplify singular chains in order to make them more suitable for various computations. The procedure in its standard form was originally defined for hyperbolic manifolds, but almost without changes can be applied to all non-positively curved manifolds. It works as follows. Given a manifold M with $\sec(M) \leqslant 0$ and a singular simplex, lift it (in any way) to the universal covering $\widetilde{M}$, which is $CAT(0)$, hence the geodesics there are unique. Therefore there exists a simplex with the same set of vertices which is *geodesic*. A geodesic simplex with a given set of vertices is defined inductively as the geodesic cone over one of its faces, which are lower dimensional geodesic simplices. In particular, a geodesic simplex is determined uniquely by its set of vertices. Now, define the straightening of the original simplex as the image of this unique geodesic simplex under the covering map. One can check that it defines a chain operator on $C_*(M;\mathbb{R})$ homotopic to the identity and not increasing the ℓ^1 norm. The first application of the procedure follows form the observation that if M is negatively curved, then the volume of any geodesic simplex is bounded by a universal constant depending on a curvature. Hence the simplicial volume must be positive, because for any fundamental cycle $c = \sum_i a_i\sigma_i$ one has

$$\sum_i |a_i| \operatorname{vol}(\operatorname{im}(\sigma_i)) \geqslant \operatorname{vol}(M).$$

However, there are many other applications. But before we present them, let us make a comment on the straightening procedure for non-compact manifolds. For the classical simplicial volume it does not work, because the straightening of a locally finite chain does not need to be locally finite. The easiest example of such chain is the family of arcs $k \cdot e^{it}$ for $t \in [0, \pi]$, $k \in \mathbb{N}_+$, on the complex plane $\mathbb{C}$. It forms a locally finite chain, but its straightening (with respect to the flat Riemannian metric on $\mathbb{R}^2 \cong \mathbb{C}$) is the family $[-k, k] \times \{0\}$ for $k \in \mathbb{N}_+$, which is not locally finite. The obstruction lies in the fact that the original chain consists of simplices of arbitrary large diameter. In the case of the Lipschitz simplicial volume, however, it is impossible because we require a universal Lipschitz constant for any simplex in a chain. Therefore the straightening can be generalized without modifications to the Lipschitz case.

Another standard application of the straightening procedure is the possible completion of Thurston's proof of the proportionality principle. The main step of the proof for two manifolds M and N is the use of their common (up to isometry) universal cover to establish the smearing map from smooth singular homology of M to Milnor-Thurston homology of N. Milnor-Thurston homology theory is a homology theory in which cycles are compactly supported finite measures on the singular simplices, endowed with the absolute variation semi-norm. In particular, an ordinary singular chain is interpreted as a discrete measure. The smearing map has all the properties that would suffice to prove the proportionality principle, but for the Milnor-Thurston homology. To prove it for the singular theory, one needs to approximate a Milnor-Thurston chain by a singular one without increasing the ℓ^1 norm. Thurston did not finish his proof, it was completed later by Löh in [31, 23], but using bounded cohomology to establish an abstract isometric isomorphism between both mentioned homology theories. In her later work with Sauer [25], they completed the proof for the Lipschitz simplicial volume using a geometric approximation constructed by the straightening procedure. The straightening, together with a careful study of cohomology with Lipschitz compact supports, allowed them also to prove the product inequality. However, both proofs are valid only for non-positively curved manifolds.

The main technical part of this dissertation is a generalization of the straightening procedure to the case of manifolds with sectional curvature bounded from above. General idea is as follows. Geodesics on positively curved manifolds are not unique in general, but they are locally unique if they join sufficiently close points. Hence although there is a little (if any) hope that we could define geodesic simplices with arbitrary sets of vertices, we can still define them if their vertices are sufficiently close. Therefore given a simplex, we can subdivide it into sufficiently small pieces, straighten each piece and then glue them all back. The presented procedure is called the *piecewise straightening*. There are, however, some technical gaps in this reasoning that need to be filled. The biggest of them (if there is any way to measure the size of a gap of a mathematical reasoning) is that in order to define the above procedure for a locally finite Lipschitz chain on the one hand one needs to subdivide every simplex some number of times, depending on the injectivity radii of points in its image, but on the other hand we need to subdivide all the simplices the same number of times. This is possible only if the injectivity radius of every point of a given manifold M is uniformly bounded from below. However, we solve this problem by introducing an *exponential neighbourhood* for every point of M. For a point $x \in M$, it is just a neighbourhood of 0 in the tangent space T_xM on which the exponential map is locally diffeomorphic, with modified metric in such a way that the exponential map becomes a local isometry. These neighbourhoods were first introduced by Gromov in [12], however, we present much more detailed study of such spaces. They have many useful properties, one of them is that their injectivity radii around the origins are uniformly bounded from below because of the curvature bound. Now, instead of straightening the simplices directly on M, we lift them to the appropriate exponential neighbourhoods and straighten them there.

To end this introduction, let us also make a comment on the additivity of the simplicial volume and possible generalizations to non-compact manifolds. The original result for compact manifolds, proved in [12] and later in [6, 19], relies heavily on the use of bounded cohomology. Gromov indicated that the result holds also for the classical simplicial volume in the case of non-compact manifolds, however, he sketched only the part of the proof, namely the proof that

$$\|M\#N\| \leqslant \|M', \partial M'\| + \|N', \partial N'\|,$$

where M' and N' are manifolds obtained from M and N respectively by removing an embedded disc. Using the results of Löh from [24, Proposition 5.19] and Kuessner from [19] one can possibly fill in the details of the proof, but it would still rely on bounded cohomology. On the other hand, taking the connected sum of two manifolds modifies the components only locally, hence there is a big chance that the additivity holds also for the Lipschitz simplicial volume. The proof, however, would require establishing either a new, geometric proof of the additivity in the compact case and generalizing it to the non-compact case, or a clever method that would allow to restrict the non-compact case to the compact one.

Organization of this work

In Chapter 1 we recall basic notions concerning Riemannian geometry, singular homology and the amenability of groups. Most of this material is standard and can be skipped by the reader with basic knowledge on these topics. The only non-standard material is contained in Section 1.2.3, where Lipschitz homology theories are presented, following [25], and in Section 1.3, where we describe the properties of amenable actions, following [12].

In Chapter 2 we introduce the definition of the simplicial volume and the Lipschitz simplicial volume. We describe also the basic properties of both invariants and some standard techniques used to study the Lipschitz simplicial volume. These involve the duality principle and the diffusion of chains. The latter one was introduced by Gromov in [12], however, we introduce it in an alternative way, in particular we do not use multicomplexes. As a corollary we obtain the proof of Theorem D. The proofs of Theorems A, B and C are given in Chapter 4, because more machinery is needed.

Chapter 3 is devoted to the study of the piecewise straightening procedure. It is the main technical part of this work, the results were obtained by the author and described in [32]. Section 3.2, devoted to the study of the piecewise C^1 homology theories is also original. It does not have any direct connection with the Lipschitz simplicial volume, but we describe there some technical results which turn out to be useful when combined with the piecewise straightening procedure.

In Chapter 4 we use the piecewise straightening procedure to prove Theorems A, B and C. The proof of Theorem A is known for specialists, but has not been described in details in the literature, therefore we give it for the sake of completeness. The proofs of Theorems B and C are based on the proofs given in [25] for non-positively curved manifolds. However, we use the piecewise straightening procedure to generalize them to the case of the manifolds with curvature bounded form above.

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Chapter 1

Preliminaries

The purpose of this chapter is to remind the reader the notions and facts which will be used throughout the rest of this work. It involves material from differential and Riemannian geometry (Section 1.1), homology theory (Section 1.2) and geometric group theory, more precisely the material concernig the amenability of groups (Section 1.3). Most of the following definitions and facts are described in the standard textbooks, such as [22, 21] in the case of Riemannian geometry, [8, 16, 21, 34] in the case of homology and [27, 28] in the case of amenable groups. However, there is some amount of material concerning Lipschitz homology theories (Subsection 1.2.3) that, although elementary, is dedicated to the study of the Lipschitz simplicial volume and is taken from [25].

1.1. Riemannian geometry

In the following section we recall basic facts and terminology concerning Riemannian geometry. The material is mostly standard. In Section 1.1.1 we introduce basic concepts of Riemannian geometry, while in Section 1.1.2 we make a brief introduction to geodesics and the exponential map. Finally, in Section 1.1.3 we recall how to integrate differential forms and functions on Riemannian manifolds.

For the rest of this section, we assume that M is a complete, oriented, smooth, n-dimensional manifold. We will also use the fact that we can view vector fields on M as $C^\infty(M)$-linear maps $C^\infty(M) \to C^\infty(M)$ satisfying Leibniz rule and use the adequate notation.

1.1.1. Riemannian metric and curvature

Definition 1.1.1. We say that M is *Riemannian* if it is equipped with the *Riemannian structure* (or the *Riemannian metric*), i.e. a family of inner products

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

for $x \in M$, which are smooth, in the sense that for every two smooth vector fields $X, Y \in C^\infty(M, TM)$ the mapping

$$x \mapsto g_x(X(x), Y(x))$$

is smooth.

We will often denote the inner product of two vectors $u, v \in T_xM$ simply by $\langle u, v\rangle$.

In fact every smooth manifold can be equipped with some Riemannian structure [22, Exercise 3.1], but we should mention one of the most standard examples.

Example 1.1.2. $\mathbb{R}^{n+1}$, viewed as an affine space with a standard inner product is a Riemannian manifold. A little bit more interesting example, though similarly basic, is the singular simplex Δ^n. We view it as a subset $\Delta^n \subset \mathbb{R}^{n+1}$, where it can be defined as the convex hull of the standard basis, that is,

$$\Delta^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \ : \ \sum_{i=0}^{n} x_i = 1 \, , \, x_i \geqslant 0 \text{ for } i = 0, ..., n\}.$$

We equip the simplex with the metric and Riemannian structure induced from $\mathbb{R}^n$. In particular, it is a metric space of diameter $\sqrt{2}$.

If not otherwise stated, we will assume that if M is Riemannian, its metric is compatible with the Remannian structure. To state this properly, let us recall that the length of a smooth path $\gamma : [0,1] \to M$ on a Riemannian manifold is

$$l(\gamma) := \int_0^1 \|\frac{d}{dt}\gamma(t)\| dt = \int_0^1 \sqrt{\langle \frac{d}{dt}\gamma(t), \frac{d}{dt}\gamma(t)\rangle} dt.$$

Knowing this, we say that the metric on a complete manifold is *induced from its Riemannian structure* if for every two points their distance equals the length of the shortest (smooth) path joining them, i.e. for every $x, y \in M$,

$$d(x,y) = \inf\{l(\gamma) \ : \ \gamma : [0,1] \to M \, , \, \gamma(0) = x \, , \, \gamma(1) = y\}.$$

Next we would like to define the curvature and some related notions.

Definition 1.1.3. An *affine connection* on a smooth manifold is a bilinear map

$$\begin{array}{rcl} C^\infty(M, TM) \times C^\infty(M, TM) & \to & C^\infty(M, TM) \\ (X, Y) & \mapsto & \nabla_X Y \end{array}$$

such that for any smooth function $f \in C^\infty(M)$ and vector fields $X, Y \in C^\infty(M, TM)$, we have

1. $\nabla_{fX} Y = f \nabla_X Y$;
2. $\nabla_X fY = X(f) \cdot Y + f \nabla_X Y$.

The above definition could be in fact stated in the case of arbitrary smooth vector bundles by substituting the second variable (and the output) by smooth sections of a given vector bundle. However, we will not need such generality.

As in the case of the Riemannian metric, every manifold admits an affine connection. However, we will be interested in some special connections, namely these satisfying two additional conditions.

1. *Riemannian metric compatibility*: for any three vector fields $X, Y, Z \in C^\infty(M, TM)$,

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

2. *Torsion-freeness*: for any two vector fields $X, Y \in C^\infty(M, TM)$,

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y]$ is the Lie bracket (i.e. commutator) of vector fields.

We have the following fact.

Theorem 1.1.4 ([22, Theorem 5.4])**.** *For each Riemannian manifold there exists a unique torsion-free connection which preserves the Riemannian metric. This connection is called* Levi-Civita connection*.*

If not otherwise stated, we will assume that every connection we are using on a Riemannian manifold is the Levi-Civita connection.

Now we are ready to define the (Riemannian) curvature.

Definition 1.1.5. For a Riemannian manifold M the *Riemannian curvature tensor* is a bilinear function

$$\begin{aligned} R : C^\infty(M, TM) \times C^\infty(M, TM) &\to \mathrm{Hom}(C^\infty(M, TM), C^\infty(M, TM)) \\ R(X, Y)Z &:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned}$$

where ∇ is the Levi-Civita connection.

Remark 1.1.6. Although the domain of the Riemannian curvature tensor is formally the product of vector fields, in fact the local behaviour of the curvature depends only on certain tangent vectors. Therefore by the notation

$$R(u, v)w$$

for vectors $u, v, w \in T_xM$, $x \in M$, we will understand the value in x of $R(U, V,)W$ for every vector fields $U, V, W \in C^\infty(M, TM)$ such that $U(x) = u$, $V(x) = v$ and $W(x) = w$. Whenever we use this notation in different contexts we will understand automatically that the result does not depend on the choice of corresponding vector fields.

The Riemannian curvature carries a lot of information about a given manifold, but in many cases less information is required or given. Therefore there are several other definitions of the curvature based on the Riemannian one. We list only two of them.

Definition 1.1.7. The *sectional curvature* of a plane $\pi \subset T_xM$, for $x \in M$, is

$$K(\pi) = \frac{\langle R(u, v)v, u\rangle}{\|u\|^2\|v\|^2 - \langle u, v\rangle^2}.$$

where $u, v \in T_xM$ are any two vectors such that $\pi = \mathrm{span}\{u, v\}$.

If $K(\pi) \leqslant K$ for all planes $\pi \subset TM$ then we denote this fact by $\sec(M) \leqslant K$ and we say that the (sectional) curvature is bounded by K. Moreover we say that a manifold is

- *negatively (non-positively) curved* if $\sec(M) < 0$ ($\sec(M) \leqslant 0$);
- *positively (non-negatively) curved* if $\sec(M) > 0$ ($\sec(M) \geqslant 0$);

We list below the most typical examples of simply connected manifolds with a given curvature.

- Spheres S^n, for $n \geqslant 2$, are positively curved.
- Euclidean spaces $\mathbb{R}^n$, for $n \geqslant 1$, are flat, i.e. $\sec(\mathbb{R}^n) = 0$.
- Hyperbolic spaces $\mathbb{H}^n$, for $n \geqslant 2$, are negatively curved.

The examples presented above are in some sense special-they are simply connected manifolds of constant sectional curvatures. Observe that having (non) positive/negative curvature is a local property, hence it is inherited by all locally isometric images, in particular by covering maps. This implies e.g. that all tori are flat, and all hyperbolic manifolds are negatively curved.

Definition 1.1.8. The *Ricci curvature tensor* is a bilinear map

$$\mathrm{Ricci} : C^\infty(M, TM) \times C^\infty(M, TM) \to C^\infty(M)$$
$$\mathrm{Ricci}(X, Y)(x) := \mathrm{tr}(v \mapsto R(v, X)Y(x)),$$

where $x \in M$, $v \in T_xM$ and $\mathrm{tr} : \mathrm{Hom}(T_xM, T_xM) \to \mathbb{R}$ is the trace operator.

We say that $\mathrm{Ricci}(M) \geqslant K$ for some $K \in \mathbb{R}$ if $\mathrm{Ricci}(v, v) \geqslant K$ for every vector $v \in T_xM$, $x \in M$, such that $\|v\| = 1$.

Note that if $\sec(M) \geqslant K$, then if $x \in M$ and $v_1, ..., v_n$ is an orthonormal basis of T_xM, then

$$\mathrm{Ricci}(v_i, v_i) = \sum_{j=1}^{n} \langle R(v_j, v_i)v_i(x), v_j(x)\rangle \geqslant (n-1)K.$$

1.1.2. Geodesics and exponential map

Using the definition of an affine connection it is easy to give a concise definition of a geodesic.

Definition 1.1.9. A smooth path $\gamma : [0, C] \to M$ on a Riemannian manifold M is called *a geodesic* if it is constant (the degenerated case) or it is locally injective and

$$\nabla_{\frac{d}{dt}\gamma} \frac{d}{dt}\gamma = 0.$$

Remark 1.1.10. In the above definition we made some simplifications. First of all, $\frac{d}{dt}\gamma$ is a vector field defined only on the image of γ, while to define properly the value of a connection we need it to be defined in some open neighbourhood of $\mathrm{im}\,\gamma$. However, we can choose some extension of $\frac{d}{dt}\gamma$ to any open neighbourhood of $\mathrm{im}\,\gamma$ and the above definition does not depend on this choice (as in Remark 1.1.6). The other problem with this definition is that it might not be well stated in the case that γ is not injection. However, if it is locally injective we can view the condition in the definition as a local one, and if it is not, it must be constant.

This definition of a geodesic coincides partially with more general (and intuitive) definition of a geodesic as a shortest path joining its endpoints. Namely, geodesics in the Riemannian sense are the paths which are locally geodesic in the metric sense.

Theorem 1.1.11 ([22, Theorem 6.12])**.** *Let $\gamma : [0, C] \to M$ be a geodesic. Then it is locally minimizing, i.e. for every $t \in [0, C]$ there exists $\varepsilon > 0$ such that for every $s \in (t - \varepsilon, t + \varepsilon) \cap [0, C]$, we have*

$$d(\gamma(t), \gamma(s)) = l(\gamma|_{[t,s]}).$$

Moreover, by the completeness of M, every two points x, y are joined by some (not necessarily unique) geodesic which minimize the distance between these points [22, Corollary 6.15]. We call such geodesic a *shortest geodesic* joining $x, y \in M$. If this geodesic is unique, we denote it by $[x, y]$.

Note that every geodesic γ is locally determined uniquely by a point $x \in \mathrm{im}\,\gamma$ and a tangent vector $v \in T_xM$ in this point [22, Theorem 4.10]. This allows us to define the exponential map.

Definition 1.1.12. For $x \in M$ let $\exp : T_xM \to M$ be the map defined as

$$\exp_x(v) = \gamma_v^x(1),$$

where $\gamma_v^x : [0,1] \to M$ is the unique geodesic such that $\gamma(0) = x$ and $\frac{d}{dt}\gamma|_{t=0} = v$. We call this map the *exponential map* at x.

The exponential map can be defined for any Riemannian manifold in some neighbourhood of the zero section $M \subset TM$. However, because M is complete we can do it for the whole tangent space TM by the Hopf-Rinov theorem [22, Theorem 6.13]. One of the crucial properties of this map is the following.

Proposition 1.1.13 ([22, Lemma 5.10])**.** *For every $x \in M$ there exists an open neighbourhood $U_x \subset T_xM$ containing 0 such that the map*

$$\exp_x|_{U_x} : U_x \to M$$

is a diffeomorphism. Moreover, the map

$$\begin{array}{ccc} TM & \to & M \times M \\ (x,v) & \mapsto & (x, \exp_x(v)) \end{array}$$

is a diffeomorphism in some open neighbourhood of the zero section $M \subset TM$.

We will be particularly interested in the existence of the lower bound on the radii of balls $B_x \subset T_xM$ for $x \in M$, for which the maps $\exp_x|_{B_x}$ are local diffeomorphisms. Fortunately, such bounds do exists in the presence of bounded curvature.

Proposition 1.1.14 ([22, Proposition 10.11, Corollary 11.3])**.** *If $\sec(M) < K$, where $0 < K < \infty$, then the map*

$$\begin{array}{ccc} \exp : TM & \to & M \times M \\ (x,v) & \mapsto & (x, \exp_x(v)) \end{array}$$

is a local diffeomorphism on the set

$$T_{\frac{\pi}{\sqrt{K}}}M = \{(x,v) \in TM \ : \ \|v\| < \frac{\pi}{\sqrt{K}}\}.$$

If $\sec(K) \leqslant 0$, then a much stronger statement is possible.

Theorem 1.1.15 (Cartan-Hadamard Theorem [22, Theorem 11.5])**.** *If M is a complete, connected manifold such that $\sec(M) \leqslant 0$, then for any point $x \in M$ the map $\exp_x$ is a covering map. In particular, the universal covering of M is diffeomorphic to $T_xM \cong \mathbb{R}^n$.*

We shall also recall the following definition.

Definition 1.1.16. Let $x \in M$. Then the *injectivity radius* of x is a positive number

$$\mathrm{injrad}_x(M) := \sup\{r \in \mathbb{R} \ : \ \exp_x|_{B_{T_xM}(0,r)} \text{ is a diffeomorphism onto its image}\}.$$

Moreover, we can define the injectivity radius of M as

$$\mathrm{injrad}(M) := \inf_{x \in M} \mathrm{injrad}_x(M).$$

Because the function $x \mapsto \mathrm{injrad}_x(M)$ is continuous, if M is compact, then $\mathrm{injrad}(M) > 0$. This does not have to be true without the compactness assumption. Moreover, note that if $\sec(M) < K$, where $K > 0$, it gives us no information about the injectivity radius of M, even if M is simply connected. However, if $\sec(M) \leqslant 0$ then by Theorem 1.1.15, for every $x \in M$ the map $\exp_x$ is a covering map, hence simply-connectedness implies $\mathrm{injrad}(M) = \infty$.

1.1.3. Integration and volume

Having a differential n-form $\omega \in \Omega^n(M) = \bigwedge^n T^*M$ on a Riemannian manifold, we can integrate it as follows.

- If $M = \mathbb{R}^n$ then ω is of the form
$$\omega(x) = f(x)dx_1 \wedge ... \wedge dx_n$$
for some $f \in C^1(\mathbb{R}^n)$. Then
$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x_1, ..., x_n)dx_1...dx_n = \int_{\mathbb{R}^n} f(x_1, ..., x_n)d\mu(x),$$
where μ is the Lebesgue measure on $\mathbb{R}^n$.
- If ω is supported on an open subset $A \subset M$ contained in the image of some smooth chart $\phi : \mathbb{R}^n \to M$ then
$$\int_M \omega = \int_{\mathbb{R}^n} \phi^*\omega.$$
- If ω is arbitrary, let $(\psi_j)_{j\in\mathbb{N}}$ be a locally finite partition of unity on M such that each ψ_j, $j \in \mathbb{N}$, is supported in the image of some smooth chart $\phi_j : \mathbb{R}^n \to M$, $j \in \mathbb{N}$. Then
$$\int_M \omega = \sum_{j\in\mathbb{N}} \int_M \psi_j\omega = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}^n} \phi_j^*(\psi_j\omega).$$

To define the above integral we do not need the Riemannian structure. However, the presence of such provides a natural definition of the volume form.

Proposition 1.1.17 ([22, Lemma 3.2]). *On any oriented Riemannian manifold M there exists a unique n-form $\mathrm{dvol}_M \in \Omega^n(M)$ such that*
$$\mathrm{dvol}_M(v_1, ..., v_n) = 1$$
for any oriented orthonormal basis $v_1, ..., v_n \in T_xM$, where $x \in M$. We call this form the volume form *on M.*

Having this form and a smooth function $f \in C^1(M)$, we can define
$$\mathrm{vol}(M) := \int_M \mathrm{dvol}_M .$$

We recall a few classical results concerning the curvature and volume of surfaces. Note that on a Riemannian surface M the sectional curvature can be considered as a function $\sec : M \to \mathbb{R}$.

Theorem 1.1.18 (Gauss-Bonnet formula, [22, Theorem 9.3]). *Let Δ be a triangle with geodesic edges on a Riemannian surface M and let ε_i, $i = 1, 2, 3$, be the exterior angles of Δ. Then*
$$\int_\Delta \sec(x)\,\mathrm{dvol}_M + \sum_{i=1}^{3} \varepsilon_i = 2\pi.$$

Because every closed surface can be obtained by gluing such triangles, one can obtain the following.

Theorem 1.1.19 (Gauss-Bonnet theorem, [22, 9.7])**.** *If M is a closed oriented Riemannian surface, then*

$$\int_M \sec(x)\, \mathrm{dvol}_M = 2\pi\chi(M).$$

In particular, if M has the constant curvature $K = \pm 1$ then

$$\mathrm{vol}(M) = 2\pi|\chi(M)|.$$

1.2. Homology and cohomology

The notion of (co)homology is one of the central concepts in the algebraic topology. In the following section we recall some properties and notions concerning it. The topic is quite broad, therefore we concentrate on certain specific facts and examples that will be used in further chapters. In Section 1.2.1, we recall the most basic definitions and examples concerning homology. In Section 1.2.2, we concentrate on the functorial behaviour of (co)homology and recall some examples of chain maps. In Section 1.2.3, we introduce Lipschitz homology and cohomology theories and we show that they are isomorphic to the corresponding non-Lipschitz ones. In particular, this section contains material which (in spite of its elementariness) is not classical and is dedicated specifically to the study of the Lipschitz simplicial volume, therefore we give detailed proofs of mentioned facts. Finally, in Section 1.2.4 we recall the notion of the fundamental class and in Section 1.2.5 we discuss product structures in various (co)homology theories, including the Lipschitz ones.

1.2.1. Various (co)homology theories

For the rest of this section we will assume that R is a commutative ring. However, in further chapters we will use almost only the case $R = \mathbb{R}$. In this particular case, and more generally if R is a field, many aspects of homology theories simplify a lot. Sometimes, though, we will be interested in a more general setting.

Definition 1.2.1. A *chain complex* C_* (over R) is a sequence of R-modules $(C_k)_{k\in\mathbb{N}}$ equipped with boundary maps $\partial : C_k \to C_{k-1}$ for $k \in \mathbb{N}$, such that $\partial \circ \partial = 0$.

We call the R-modules

$$Z_k(C_*) := \ker(\partial : C_k \to C_{k+1}) \subset C_k$$

and

$$B_k(C_*) := \mathrm{im}(\partial : C_{k+1} \to C_k) \subset C_k$$

the *cycles* and the *boundaries* of C_* respectively. The *homology* of the chain complex C_* is a sequence of R-modules

$$H_k(C_*) := Z_k(C_*)/B_k(C_*)$$

Finally, we call two cycles $c, c' \in C_k$ *homologuous* if $[c] = [c'] \in H_k(C_*)$, i.e. $c = c' + \partial d$ for some $d \in C_{k+1}$.

Similarly, a *cochain complex* is a sequence of vector spaces $(C^k)_{k\in\mathbb{N}}$, equipped with coboundary maps $\delta : C^k \to C^{k+1}$ for $k \in \mathbb{N}$, such that $\delta \circ \delta = 0$. We also call the R-modules

$$\begin{aligned} Z^k(C^*) &:= \ker(\delta : C^k \to C^{k+1}) \subset C^k, \\ B^k(C^*) &:= \mathrm{im}(\delta : C^{k-1} \to C^k) \subset C^k, \\ H^k(C^*) &:= Z^k(C^*)/B^k(C^*) \end{aligned}$$

the *cocycles*, *coboundaries* and *cohomology* of C^* respectively. We call also two cocycles $\phi, \phi' \in C^k$ *cohomologuous* if $[\phi] = [\phi'] \in H^k(C^*)$, i.e. $\phi = \phi' + \delta\psi$ for some $\psi \in C^{k-1}$.

The most interesting example of chain complexes and homology from our view are singular ones for a topological space. Given the simplex $\Delta^n \subset \mathbb{R}^{n+1}$ defined as in Example 1.1.2, there is the sequence of (isometric) embeddings $\iota_i : \Delta^{k-1} \to \Delta^k$ for $i = 0, ..., k$ defined as

$$\iota_i(x_0, ..., x_{k-1}) = (x_0, ..., x_{i-1}, 0, x_i, ..., x_{k-1}).$$

Example 1.2.2. Let X be a topological space. A *singular (k-)simplex* is a continuous map $\sigma : \Delta^k \to X$. We have maps $\partial_i : C(\Delta^k, X) \to C(\Delta^{k-1}, X)$ for $i = 0, ..., k$, defined as

$$\partial_i(\sigma) = \sigma \circ \iota_i.$$

Let $C_k(X; R)$ be the free R-module with the basis consisting of singular k-simplices. That is, elements of $C_k(X; R)$ are finite formal sums of singular simplices with coefficients in R. We extend operators ∂_i linearly to $\partial_i : C_k(X; R) \to C_{k-1}(X; R)$ and define

$$\partial = \sum_{i=0}^{k} (-1)^i \partial_i : C_k(X; R) \to C_{k-1}(X; R).$$

Then $C_*(X; R)$, together with ∂, forms a chain complex called the *singular chain complex of* X. We call its homology the *(singular) homology of* X with coefficients in R.

Moreover, let $C^k(X; R) = \mathrm{Hom}(C_k(X, \mathbb{Z}); R)$ and let $\delta : C^k(X; R) \to C^{k+1}(X; R)$, where $k \in \mathbb{N}$, be defined as

$$\delta\phi(\sigma) = \phi(\partial\sigma),$$

where $\sigma \in C(\Delta^k, X)$. Then $C^*(X; R)$, together with δ, forms a cochain complex called the *singular cochain complex* and we call its cohomology the *(singular) cohomology of* X with coefficients in R.

Remark 1.2.3. Although by varying the coefficients of a (co)homology theory one might get sometimes very interesting results, as mentioned before, we will use further almost always real (co)homology, i.e. with coefficients in $\mathbb{R}$. In that case we will write for short $H_*(X)$ ($H^*(X)$) instead of $H_*(X; \mathbb{R})$ ($H^*(X; \mathbb{R})$).

In the following chapters we will often investigate the geometric properties of singular chains, therefore we need a little bit more terminology. For a singular chain $c = \sum_i a_i \sigma_i \in C_k(X; R)$ we will denote by $\mathrm{supp}(c)$ the set $\bigcup_i \{\sigma_i\} \subset C(\Delta^k, X)$. More generally, by the j-skeleton of c for $j = 0, ..., k$, we will understand the set

$$\bigcup_i \{\sigma_i|_F \ : \ F \text{ is a } j\text{-dimensional face of } \Delta^k\} \subset C(\Delta^j, X).$$

We recall also that the 0-skeleton of c is called the set of *vertices* of c and the 1-skeleton is called the set of *edges* of c.

A very useful fact about singular homology and cohomology is that we can evaluate the cohomology classes on the homology classes. Namely, the mapping $H^*(X; R) \to \mathrm{Hom}(H_*(M; R), R)$ defined as

$$\langle [\phi], [c] \rangle = \phi(c),$$

where $\phi \in Z^*(X; R)$ and $c \in Z_*(X; R)$ for some $k \in \mathbb{N}$, does not depend on the choice of representatives. Indeed, if $c' = c + \partial d$ and $\phi' = \phi + \delta\psi$ for some $d \in C_{k+1}(X; R)$ and $\psi \in C^{k-1}(X; R)$ then

$$\phi'(c') = (\phi + \delta\psi)(c + \partial d) = \phi(c) + \phi(\partial c) + \delta\psi(c) + \delta\psi(\partial d) = \phi(c) + \delta\phi(c) + \psi(\partial c) + \psi(\partial\partial d) = \phi(c).$$

There are also other classical variants of (co)homology that will be used throughout the next chapters. For a topological space X let $K(X)$ be the family of compact sets $K \subset X$.

Definition 1.2.4. Let X be a locally compact space and let $C_*^{lf}(X;R)$ be the chain complex of (possibly infinite) sums of singular simplices such that for every compact set $K \subset X$ only finitely many simplices intersect K. That is,

$$C_*^{lf}(X;R) := \{\sum_{i\in\mathbb{N}} a_i\sigma_i \ : \ a_i \in R,\ \sigma_i \in C(\Delta^*, X)\ ;\ \forall_{K(X)}\#\{\sigma \ : \ \operatorname{im}\sigma \cap K \neq \emptyset\} < \infty\}.$$

We call it the *locally finite singular chain complex* and the corresponding homology theory $H_*^{lf}(X;R)$ *locally finite homology.*

The corresponding 'dual' cohomology theory (in the sense that one can evaluate cocycles on locally finite cycles) is cohomology with compact supports.

Definition 1.2.5. For a locally compact space X let $C_{cs}^*(X;R) \subset C^*(X;R)$ be the cochain complex consisting of cochains with compact supports, i.e.

$$C_{cs}^*(X;R) := \{\phi \in C^*(X;R) \ : \ \exists_{K\in K(X)}(\operatorname{im}(\sigma) \cap K = \emptyset) \Rightarrow \phi(\sigma) = 0\}.$$

We call the corresponing cohomology theory $H_{cs}^*(X;R)$ *cohomology with compact supports.*

The next homology theory is an interesting generalisation of the classical real singular homology.

Definition 1.2.6. Let $\mathcal{C}_k(X)$ be the set of compactly supported Borel measures on $C(\Delta^k, X)$ endowed with the compact-open topology and let $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$, for $k \in \mathbb{N}$, be defined as

$$(\partial\mu)(\sigma) := \mu(\partial\sigma),$$

where $\sigma \in \mathcal{C}_k(X)$. Then $(\mathcal{C}_k(X), \partial)$ forms a chain complex called the *Milnor-Thurston chain complex* and we call the corresponding homology theory $\mathcal{H}_*(X)$ *Milnor-Thurston homology.*

Note that we can evaluate ordinary real singular cochains on Milnor-Thurston chains by the formula

$$\phi(\mu) = \int_{C(\Delta^k,X)} \phi(\sigma)d\mu(\sigma),$$

where $\phi \in C^k(X)$ and $\mu \in \mathcal{C}_k(X)$. The value of $\phi(\mu)$, as in the case of singular homology, does not depend on the choice of respective representatives.

There is also one more classical cohomology theory that will be used, namely de Rham cohomology. It is not based on any singular chain complex as the previous examples and is defined only for smooth manifolds.

Definition 1.2.7. Let M be a smooth manifold and let $\Omega^k(M)$, for $k \in \mathbb{N}$, be the space of smooth sections of $\bigwedge^k T^*M$, i.e. of the k-th exterior power of the cotangent bundle of M. Then for $k \in \mathbb{N}$ we have the operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined for $\omega \in \Omega^k(M)$ and vector fields $V_0, ..., V_k$ on M as

$$\begin{aligned} d\omega(V_0,...,V_k) &= \sum_{i=0}^{k}(-1)^i V_i(\omega(V_0,...,\hat{V_i},...,V_k)) \\ &+ \sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega([V_i,V_j],V_0,...,\hat{V_i},...,\hat{V_j},...,V_k), \end{aligned}$$

where $\hat{V_i}$ denotes that we omit a corresponding term in the sequence, i.e.

$$(V_0, ..., \hat{V_i}, ..., V_k) = (V_0, ..., V_{i-1}, V_{i+1}, ..., V_k).$$

One can check that with this operator Ω_* forms a cochain complex, and the corresponding cohomology theory $H^*_{\mathrm{dR}}(M)$ is called *de Rham cohomology.*

1.2.2. Chain maps and homotopies

This section, in which some functorial properties of (co)homology are described, is primarily a reminder of certain examples of chain maps, which will be used throughout the rest of this dissertation.

Definition 1.2.8. Let (C_*, ∂_C) and (D_*, ∂_D) be two chain complexes. A map $f : C_* \to D_*$ is called a *chain map* if $f(C_k) \subset D_k$ and it commutes with the boundaries:

$$\partial_D \circ f = f \circ \partial_C.$$

We say that two chain maps $f, g : C_* \to D_*$ are *(chain) homotopic* if there exists an operator $H : C_* \to D_{*+1}$ such that

$$H\partial_C + \partial_D H = f - g$$

called a *(chain) homotopy* between f and g.

Proposition 1.2.9 ([34, Lemma 1.4.5])**.** *A chain map $f : C_* \to D_*$ induces a homomorphism on homology spaces $f_* : H_*(C_*) \to H_*(D_*)$. Moreover, if two chain maps $f, g : C_* \to D_*$ are homotopic then $f_* = g_*$.*

Example 1.2.10 ([16, Proposition 2.9, Theorem 2.10])**.** If $f : X \to Y$ is a continuous map between topological spaces, f induces maps on the corresponding chain and cochain complexes

$$\begin{array}{ll} f_* : C_*(X) \to C_*(Y) & f_*(\sigma) = f \circ \sigma; \\ f^* : C^*(Y) \to C^*(X) & f^*(\phi)(\sigma) = \phi(f \circ \sigma). \end{array}$$

which are chain maps, thus inducing homomorphisms

$$f_* : H_*(X) \to H_*(Y)$$

and

$$f^* : H^*(Y) \to H^*(X).$$

Moreover, if $f, g : X \to Y$ are two maps homotopic to each other, there exist a homotopy $H(f, g)_* : C_*(X) \to C_{*+1}(Y)$ ($H(f, g)_* : C^*(Y) \to C^{*-1}(X)$) between f_* and g_* (f^* and g^*).

The same statements are true for locally finite homology and cohomology with compact supports, but we need to assume additionally the properness of maps and homotopies.

The proof that homology is invariant under homotopy equivalences is classical and (after certain modifications) is applied in the proofs of many other 'classical' facts concerning homology of spaces. However, we will sometimes need some of the technical aspects of this proof. Therefore we introduce the following lemma, which follows from the mentioned methods.

Lemma 1.2.11. *Let $C_*^{(1)} \subset C_*^{(2)} \subset C_*^{lf}(M;R)$ be two chain subcomplexes and let*

$$\Sigma_k^j := \{\sigma \in C(\Delta^k, M) \ : \ \sigma \in C_*^{(j)}\}$$

for $j = 1,2$. Let also $N \in \mathbb{N} \cup \{\infty\}$ and let $H_\sigma : \Delta^k \times I \to X$ be a system of maps for $\sigma \in \Sigma_k^1$, $k \leqslant N$, such that for each $k \leqslant N$, $\sigma \in \Sigma_k^1$ and $i = 0, ...k$,

- *$H_\sigma|_{\Delta^k \times \{0\}} = \sigma$;*
- *$H_\sigma|_{\partial_i \Delta^k \times I}$ depends only on $H_{\partial_i \sigma}$ and $H_\sigma|_{\partial_i \Delta^k \times \{1\}} = H_{\partial_i \sigma}|_{\Delta^{k-1} \times \{1\}}$;*
- *for any chain $c = \sum_i a_i \sigma_i \in C_k^{(2)}$, the chain $\sum_i a_i P(H_{\sigma_i}) \in C_{k+1}^{(2)}$, where $P : C(\Delta^k \times I, X) \to C_{k+1}(X;R)$ is the prism operator defined e.g. in [16, Proof of Theorem 2.10] (see Section 1.2.5 for details). In particular, $H_\sigma|_{\Delta^k \times \{1\}} \in \Sigma_k^2$.*

Then the inclusion of chains $\iota_* : C_{*\leqslant N}^{(1)} \to C_*^{(2)}$ and the map $\eta_* : C_{*\leqslant N}^{(1)} \to C_*^{(2)}$ defined for $k \leqslant N$ as

$$\eta_k(\sum_i a_i \sigma_i) = \sum_i a_i H_{\sigma_i}|_{\Delta^k \times \{1\}},$$

are chain homotopic.

Proof (sketch). Note that η_* is a chain map, because for every simplex $\sigma \in \Sigma_k^1$ we have

$$\eta_k(\partial \sigma) = \sum_{i=0}^{k} (-1)^k H_{\partial_i \sigma}|_{\Delta^{k-1} \times \{1\}} = \sum_{i=0}^{k} (-1)^k H_\sigma|_{\partial_i \Delta^k \times \{1\}} = \partial \eta_k(\sigma).$$

One can check ([16, Proof of Theorem 2.10]) that a homotopy $h : C_k^{(1)}(X;R) \to C_{k+1}^{(2)}(X;R)$ between η_* and ι_* for $k \leqslant N$ is given by

$$h(\sum_i a_i \sigma_i) = \sum_i a_i P(H_{\sigma_i}).$$

□

Remark 1.2.12. The above lemma can be stated also in the case of subcomplexes of the Milnor-Thurston chain complex or its variants without any changes in the proof. We need only to assume that H_σ depend in the Borel way on σ and for any compact set $A \subset C(\Delta^k, M)$ the set

$$\bigcup_{\sigma \in A} \operatorname{supp} P(H_\sigma) \subset C(\Delta^{k+1}, M)$$

has compact closure. In particular, for $\mu \in \mathcal{C}_k(X)$, by $P(H_\mu)$ we understand the measure on $C(\Delta^{k+1}, X)$ which is a sum of push forwards of μ under the maps $\sigma \mapsto H_\sigma(\Delta') \in C(\Delta^{k+1}, X)$ for $\Delta' \in \operatorname{supp} P(\Delta^k \times I)$ (see section 1.2.5). More precisely, for a Borel set $A \subset C(\Delta^{k+1}, X)$ we have

$$P(H_\mu)(A) = \sum_{\Delta' \in \operatorname{supp} P(\Delta^k \times I)} \mu(\{\sigma \ : \ H_\sigma(\Delta') \in A\}).$$

Now we give a few more concrete examples of chain maps.

Example 1.2.13. Let M be a smooth manifold and let $C_*^{sm}(M;R)$ be the chain complex of smooth chains, i.e. consisting of smooth singular simplices. Then the inclusion of chains $\iota_* : C_*^{sm}(M;R) \to C_*(M;R)$ induces an isomorphism on homology. To see this, note that by [22, Theorem 6.19] every singular simplex σ is homotopic to some smooth simplex, and if $\partial\sigma$ is smooth, then we can choose this homotopy to be the identity on the boundary. Therefore for $C_*^{(1)} = C_*^{(2)} = C_*(M;R)$, we can construct operators H_σ satisfying all the conditions of Lemma 1.2.11 by induction on k such that $H_\sigma|_{\Delta^*\times\{1\}}$ are smooth, and if σ is smooth then $P(H_\sigma)$ is a smooth chain. Then the chain map $\eta_* : C_*(M;R) \to C_*^{sm}(M;R)$ (where use the notation from Lemma 1.2.11) induces an inverse to the map $H_*^{sm}(M;R) \to H_*(M;R)$, because both $\iota_* \circ \eta_*$ and $\eta_* \circ \iota_*$ are chain homotopic to identities by Lemma 1.2.11. The same applies to the smooth and ordinary singular cohomology theories.

Example 1.2.14. Because we can treat every real singular cycle as a discrete measure on the set of singular simplices, there is an inclusion of chain complexes $C_*(X) \to \mathcal{C}_*(X)$ which is a chain map. The induced map on the homology is an isomorphism for CW-complexes [35, Theorem 5.0] (and by [23] isometric with respect to the ℓ^1 semi-norm on $C_*(X)$ and absolute variation semi-norm on $\mathcal{C}_*(X)$, see Sections 2.1 and 3.2.2 for more details), but we will not use this fact.

Example 1.2.15. Another example of a chain map, which is quite important to us, is the *barycentric subdivision operator* $S : C_*(X) \to C_*(X)$. It has the following properties.

1. S is homotopic to $Id_{C_*(X)}$.
2. $S(\Delta^n) = \sum_i a_i\sigma_i$, where every $\sigma_i \in C(\Delta^n, \Delta^n)$ is an affinely embedded simplex such that
$$\operatorname{diam}(\operatorname{im}\sigma_i) \leqslant \frac{n-1}{n}\operatorname{diam}(\Delta^n).$$
3. There exists a chain homotopy $T : C_*(X) \to C_{*+1}(X)$ between S and $Id_{C_*(X)}$ such that for any singular simplex $\sigma \in C(\Delta^n, X)$,
$$T(\sigma) = \sum_i a_i\sigma_i = \sum_i a_i(\sigma' \circ \sigma_i'),$$
where $\sigma_i' \in C(\Delta^{n+1}, \Delta^n \times I)$ are affine embeddings and $\sigma' \in C(\Delta^n \times I, X)$ is defined by $\sigma'(x,t) = \sigma(x)$.

See [16, Proposition 2.21] for more details. The second property implies in particular that the diameters (and Lipschitz constants, if a given simplex is Lipschitz) of simplices obtained by the iterated barycentric subdivision tend uniformly to 0. Moreover, if σ is Lipschitz with the Lipschitz constant less than L, then all the simplices in $S(\sigma)$ and $T(\sigma)$ are Lipschitz with the Lipschitz constants less than L' for some $L' < \infty$ depending only on L.

Example 1.2.16. The symmetric group S_{n+1} acts on the set of singular simplices $C(\Delta^n, M)$ by permuting the order of the vertices of Δ^n. That is, for $s \in S_n$, we define the map $s_\Delta : \Delta^n \to \Delta^n$ as the unique affine map such that
$$s_\Delta(v_i) = v_{s(i)}$$
for $i = 0, ..., n$, where v_i is the i-th vertex of Δ^n, and $s(\sigma) = \sigma \circ s_\Delta$. Then the map Alt : $C_*(X) \to C_*(X)$ defined for a singular simplex $\sigma \in C(\Delta^k, X)$ as
$$\operatorname{Alt}(\sigma) := \frac{1}{(k+1)!}\sum_{s\in S_{k+1}} \operatorname{sgn}(s)\cdot s(\sigma),$$

where $\operatorname{sgn}(s)$ is the sign of s, is a chain map. Moreover, it is chain homotopic to the identity on $C_*(X)$ [15, Appendix B].

Example 1.2.17. Another very interesting example of a chain map is the de Rham map

$$\mathrm{dR}_* : \Omega^*(M) \to C^*_{sm}(M).$$

It is defined for a differential k-form $\omega \in \Omega^k(M)$ as

$$\mathrm{dR}_*(\omega)(\sigma) = \int_{\Delta^k} \sigma^*\omega,$$

where $\sigma : \Delta^k \to M$ is a smooth singular simplex. One can check that this is a chain map by the Stokes theorem [21, Theorem 11.31]. Moreover, by the de Rham theorem [21, 11.34] it induces an isomorphism on cohomology groups

$$H^*_{\mathrm{dR}}(M) \cong H^*_{sm}(M) \cong H^*(M).$$

1.2.3. Lipschitz homology and cohomology

There is a variant of homology that we will be particularly interested in. It is Lipschitz homology, which is obtained simply by considering only Lipschitz singular simplices. Let M be a connected Riemannian manifold and let $c = \sum_i a_i\sigma_i \in C_*(X)$ (or $C^{lf}_*(M)$) be a (locally finite) singular chain. The *Lipschitz constant of* c is the number

$$\mathrm{Lip}(c) := \sup_i \mathrm{Lip}(\sigma_i),$$

where $\mathrm{Lip}(\sigma_i)$ is the (optimal) Lipschitz constant of σ_i, i.e. if it is Lipschitz and $+\infty$ otherwise. If $\mathrm{Lip}(c) < \infty$, we call this chain *Lipschitz*.

Definition 1.2.18. Let $C^{\mathrm{Lip}}_*(M) \subset C_*(M)$ be the chain complex consisting of Lipschitz chains:

$$C^{\mathrm{Lip}}_*(M) := \{c \in C_*(M) \,:\, \mathrm{Lip}(c) < \infty\}.$$

We call this chain complex the *Lipschitz chain complex* and the corresponding homology theory $H^{\mathrm{Lip}}_*(M)$ *Lipschitz homology.*

Similarly, we call the corresponding chain complex $C^{lf,\mathrm{Lip}}_*(M) \subset C^{lf}_*(M)$ *Lipschitz locally finite chain complex* and the homology theory $H^{lf,\mathrm{Lip}}_*(M)$ *Lipschitz locally finite homology.*

Because in the case of $C^{\mathrm{Lip}}_*(M)$ any chain is a finite linear combination of simplices, Lipschitz chains are exactly these which contain only Lipschitz simplices. However, in the case of $C^{lf,\mathrm{Lip}}_*(M)$ the assumption that for a given chain all the simplices have the same bound on the Lipschitz constant is significant. Note that these homology groups are functorial with respect to Lipschitz (and proper, in the locally finite case) homotopy classes of maps.

For technical reasons, we will also use the following variant of Lipschitz homology. For $L < \infty$ let

$$C^{<L}_*(M) := \{c \in C_*(M) \,:\, \mathrm{Lip}(c) < L\}.$$

be the chain complex of L-Lipschitz chains. We will denote the homology of the above chain complex by $H^{<L}_*(M)$. There is also the corresponding locally finite variant $H^{lf,<L}_*(M)$ defined in a similar way.

In fact these theories are isomorphic to the non-Lipschitz ones.

Theorem 1.2.19 ([25, Theorem 3.3, Lemma 3.5])**.** *Let M be a connected Riemannian manifold. Then the natural inclusions of chains $C_*^{\mathrm{Lip}}(M) \to C_*(M)$ and $C_*^{lf,\mathrm{Lip}}(M) \to C_*^{lf}(M)$ induce isomorphisms $H_*^{\mathrm{Lip}}(M) \to H_*(M)$ and $H_*^{lf,\mathrm{Lip}}(M) \to H_*^{lf}(M)$.*

There is also a 'dual' cohomology theory defined as follows.

Definition 1.2.20. Let $\phi \in \mathrm{Hom}(C_*^{\mathrm{Lip}}(M), \mathbb{R})$. We say that ϕ has *Lipschitz compact support* if for every $L < \infty$ it has compact support when restricted to L-Lipschitz simplices, i.e. there exists a compact set $K_L \subset M$ depending on L, such that

$$\forall_{\sigma \in C(\Delta^*, M)\,,\,\mathrm{Lip}(\sigma) < L} (\mathrm{im}(\sigma) \subset M \setminus K_L) \Rightarrow \phi(\sigma) = 0.$$

We denote the cochain complex of Lipschitz compactly supported cocycles $C^*_{cs,\mathrm{Lip}}(M) \subset \mathrm{Hom}(C_*^{\mathrm{Lip}}, \mathbb{R})$ and call the corresponding cohomology theory $H^*_{cs,\mathrm{Lip}}(M)$ *cohomology with Lipschitz compact supports.*

Also in this case the above theory turns out to be isomorphic to the corresponding non-Lipschitz one.

Theorem 1.2.21 ([25, Theorem 3.8])**.** *Let M be a connected Riemannian manifold. Then the natural inclusion of cochains $C^*_{cs}(M) \to C^*_{cs,\mathrm{Lip}}(M)$ induces an isomorphism $H^*_{cs}(M) \cong H^*_{cs,\mathrm{Lip}}(M)$.*

1.2.4. Fundamental class

It is well known fact that the homology of the disc relative to its boundary is one dimensional, i.e. $H_n(D^n, S^n; \mathbb{Z}) \cong \mathbb{Z}$ [16, Example 2.17]. This implies that if M is an n-dimensional manifold, then for every $x \in M$ one has $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$. Moreover, we have the following fact

Theorem 1.2.22 ([16, Theorem 3.26(a)])**.** *If M is a closed, connected, orientable manifold, the restriction map $H_n(M, \mathbb{Z}) \to H_n(M, M \setminus \{x\})$ is an isomorphism for every $x \in M$.*

Let $[M] \in H_n(M, \mathbb{Z})$ be an element such that its image in every group $H_n(M, M \setminus \{x\})$, $x \in M$, is a generator (which can be chosen in a canonical way if M is oriented). We call this element the *fundamental class* of M and every cycle that represents it a *fundamental cycle*. Because most of the time we will use only homology with real coefficients, by $[M] \in H_*(M)$ we will also denote the image of the fundamental class under the change of coefficients homomorphism $H_*(M, \mathbb{Z}) \to H_*(M, \mathbb{R})$ and also call it the fundamental class. It would be clear from the context if we are dealing with real or integral fundamental class/cycle.

In the case of cohomology, there is an important result binding e.g. singular homology and cohomology.

Theorem 1.2.23 ([8, Theorem V.7.1])**.** *Let C_* be a chain complex of free R-modules, where R is a principal ideal domain, and let $C^*(A) := \mathrm{Hom}(C_*; A)$ for an R-module A be an associated cochain complex. Then for every $k \in \mathbb{N}$, there is an exact sequence*

$$0 \to \mathrm{Ext}_R(H_{k-1}(C_*), A) \to H^k(C^*(A)) \to \mathrm{Hom}(H_k(C_*), A) \to 0.$$

In particular, if R is a field then $\mathrm{Ext}_R(H_{k-1}(C_*), R) = 0$ and $H^k(C^*(R)) \cong (H_k(C_*))^*$.

By the above theorem, we know that $H^*(M) \cong \mathrm{Hom}(H_*(M), \mathbb{R})$. Let $[\phi]$ be the unique class such that $\phi([M]) = 1$. We call this class the *fundamental cohomology class.*

Fundamental classes are also present in other theories. We assume that M is a connected, oriented manifold.

- For the relative homology there is the fundamental class $[M, M \setminus K] \in H^*(M, M \setminus K)$ for any connected compact subset $K \subset M$ [16, Lemma 3.27(a)].
- For non-compact manifolds there is the fundamental class $[M]_{lf} \in H_*^{lf}(M)$ [24, Theorem 5.4]. Dually, we have the compactly supported fundamental cohomology class $[M]^*_{cs} \in H^*_{cs}(M)$ [24, Corollary 5.7].
- By Theorem 1.2.19, there is the locally finite Lipschitz fundamental class $[M]_{\mathrm{Lip}} \in H_*^{lf,\mathrm{Lip}}(M)$, and by Theorem 1.2.21, there is the Lipschitz fundamental cohomology class $[M]^*_{\mathrm{Lip}} \in H^*_{cs,\mathrm{Lip}}(M)$.
- Similarily, if M is smooth, closed manifold, then by Example 1.2.13 there exist the smooth fundamental class in $H_*^{sm}(M)$ and smooth fundamental cohomology class in $H^*_{sm}(M)$.
- In the case of de Rham cohomology, we know that by the de Rham theorem (Example 1.2.7) there exists the fundamental cohomology class. On the other hand, there is a canonical cocycle in $\Omega^n(M)$, namely the volume form. One could ask how this cocycle is related to the fundamental class. It turns out that

  $$\langle \mathrm{dR}_*(\mathrm{dvol}_M), c\rangle = \int_c \mathrm{dvol}_M = \int_M \mathrm{dvol}_M = \mathrm{vol}(M)$$

  for every smooth fundamental cycle $c \in C_n^{sm}(M)$, hence $[\mathrm{dR}_*(\mathrm{dvol})] = \mathrm{vol}(M) \cdot [M]^*$. This fact can be used to recognise fundamental cycles.

  More generally, if $\omega \in \Omega^n(M)$ is a form with the support in a compact set K, then $\mathrm{dR}_*(\omega)$ lies in $C^*_{sm}(M, M \setminus K)$ and

  $$\langle \mathrm{dR}_*(\omega), [M, M \setminus K]\rangle = \int_K \omega = \int_M \omega.$$

In fact, the volume form can be used to detect fundamental cycles also in the locally finite, Lipschitz case.

Proposition 1.2.24 ([25, Proposition 4.4])**.** *Let M be a Riemannian n-manifold, and let $c = \sum_{k\in\mathbb{N}} a_k \sigma_k \in C_n^{lf,\mathrm{Lip}}(M)$ be a cycle with $|c|_1 < \infty$. Then c is a fundamental cycle if and only if*

$$\sum_{k\in\mathbb{N}} a_k \cdot \langle \mathrm{dvol}_M, \sigma_k\rangle = \mathrm{vol}(M)$$

1.2.5. Products

In this section we would like to present various products in singular (co)homology. Before we do that, let us present how to subdivide the product of simplices into simplices, following [16, Chapter 3.B]. To construct such a subdivision, it is more convenient to present the simplex in an alternative way. We will present the k-simplex as a subset of $\mathbb{R}^k$ as

$$\dot{\Delta}^k := \{(x_1, ..., x_k) \in \mathbb{R}^k \ : \ 0 \leqslant x_1 \leqslant ... \leqslant x_k \leqslant 1\}.$$

Note that there is the diffeomorphism $g : \Delta^k \to \dot{\Delta}^k$ defined as

$$g(x_0, ..., x_k) = (x_0, x_0 + x_1, ..., \sum_{i=0}^{k-1} x_i)$$

with an inverse

$$g^{-1}(y_1,...,y_k) = (y_1, y_2 - y_1, ..., y_k - y_{k-1}, 1 - y_k).$$

Therefore without loss of generality we can identify Δ^k with $\dot{\Delta}^k$.

The product $\dot{\Delta}^k \times \dot{\Delta}^l$ is the set

$$\{(x_1,...,x_k,y_1,...,y_l) \in \mathbb{R}^{k+l} \ : \ 0 \leqslant x_1 \leqslant ... \leqslant x_k \leqslant 1 \ ; \ 0 \leqslant y_1 \leqslant ... \leqslant y_l \leqslant 1\}.$$

Let $Z_{k,l}$ be the set of such permutations of the sequence $(1,...,k+l)$ that preserve the orders of $(1,...,k)$ and $(k+1,...,k+l)$. That is, for $z \in Z_{k,l}$, if $z(i_1) = j_1$ and $z(i_2) = j_2$ for $i_1 \leqslant i_2 \leqslant k$, then $j_1 \leqslant j_2$, and similarly if $k < i_1 \leqslant i_2$. Note that for any permutation $z \in Z_{k,l}$, the set

$$\Delta_z := \{(x_1,...,x_{k+l}) \in \mathbb{R}^{k+l} \ : \ 0 \leqslant x_{z(1)} \leqslant ... \leqslant x_{z(k+l)} \leqslant 1\}$$

is a subset of $\dot{\Delta}^k \times \dot{\Delta}^l$ diffeomorphic to $\dot{\Delta}^{k+l}$. Moreover,

$$\bigcup_{z \in Z_{k,l}} \Delta_z = \dot{\Delta}^k \times \dot{\Delta}^l,$$

hence to the product of simplices $\dot{\Delta}^k \times \dot{\Delta}^l$ we can assign the singular $k+l$-chain $\sum_{z \in Z_{k,l}} \Delta_z \in C_{k+l}(\dot{\Delta}^k \times \dot{\Delta}^l)$. For topological spaces X, Y, this allows us to define the *Prism operator*

$$\begin{aligned} P : C(\Delta^k, X) \times C(\Delta^k, Y) \ &\rightarrow \ C_{k+l}(X \times Y) \\ \sigma \times \tau \ &\mapsto \ (\sigma \times \tau)_*(\sum_{z \in Z_{k,l}} \Delta_z) \end{aligned}$$

for $k, l \in \mathbb{N}$.

Note also that if X, Y are metric spaces, then if the simplices $\sigma : \Delta^k \rightarrow X$ and $\tau : \Delta^l \rightarrow Y$ are Lipschitz, then $P(\sigma \times \tau)$ is a Lipschitz chain with the Lipschitz constant depending only on k, l, $\mathrm{Lip}(\sigma)$ and $\mathrm{Lip}(\tau)$.

The above construction allows us to define the *cross product* in various homology theories:

$$\begin{aligned} \times : H_k(X) \otimes H_l(Y) \ &\rightarrow \ H_{k+l}(X \times Y), \\ \times : H_k^{lf}(X) \otimes H_l^{lf}(Y) \ &\rightarrow \ H_{k+l}^{lf}(X \times Y), \\ \times : H_k^{lf,\mathrm{Lip}}(X) \otimes H_l^{lf,\mathrm{Lip}}(Y) \ &\rightarrow \ H_{k+l}^{lf,\mathrm{Lip}}(X \times Y), \end{aligned}$$

for $k, l \in \mathbb{N}$. It is defined on the chain level by the Eilenberg-Zilber map

$$\sum_i a_i \sigma_i \otimes \sum_j b_j \tau_j \mapsto \sum_{i,j} a_i b_j P(\sigma_i \times \tau_j).$$

It is a classical fact that for two discs D^m, D^n the cross product of fundamental cycles in $C_m(D^m, \partial D^m)$ and $C_n(D^n, \partial D^n)$ yields a fundamental cycle in $C_{m+n}(D^{m+n}, \partial D^{m+n})$. From the local definition of the fundamental cycle, the following proposition follows.

Proposition 1.2.25. *Let $c_1 \in C_*(M)$ and $c_2 \in C_*(N)$ be fundamental cycles of manifolds M and N respectively. Then $c_1 \times c_2$ is a fundamental cycle of $M \times N$. The same applies to locally finite and Lipschitz locally finite fundamental cycles.*

Now we would like to examine the cross product in cohomology. It is constructed with the use of the Alexander-Whitney map

$$\begin{aligned} AW : C_{n+m}(X \times Y) \ &\rightarrow \ \sum_{k+l=n+m} C_k(X) \otimes C_l(Y) \\ \sigma \ &\mapsto \ \sum_{k+l=n+m} \pi_X \circ \sigma\rfloor_k \otimes \pi_Y \circ {}_l\lfloor\sigma, \end{aligned}$$

where $\sigma \in C(\Delta^{n+m}, X \times Y)$, $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the canonical projections, $\sigma\rfloor_k$ is the k-dimensional face of σ spanned by the first $k+1$ vertices and ${}_l\lfloor\sigma$ is the l-dimensional face of σ spanned by the last $l+1$ vertices. We define the cohomological cross product on the cochain level as

$$(\phi \times \psi)(\sigma) = (\phi \otimes \psi)(AW(\sigma)) = \phi(\pi_X \circ \sigma\rfloor_n) \cdot \psi(\pi_Y \circ {}_m\lfloor\sigma).$$

for $\phi \in C^n(X)$, $\psi \in C^m(Y)$ and $\sigma \in C(\Delta^{n+m}, X \times Y)$. This induces the cross product on the cohomology level

$$\times : H^*(X) \otimes H^*(Y) \to H^*(X \times Y).$$

There is also a cross product in cohomology with compact supports, which is induced by the relative cross product

$$\times : H^*(X, X \setminus K_X) \otimes H^*(Y, Y \setminus K_X) \to H^*(X \times Y, (X \times Y) \setminus (K_X \times K_Y)),$$

where $K_X \subset X$ and $K_Y \subset Y$ are compact. However, this cross product is not defined directly on the cochain level. One may observe that if $\phi \in C^*(X, X \setminus K_X)$ and $\psi \in C^*(Y, Y \setminus K_Y)$, then $\phi \times \psi \in C^*(X \times Y)/(C^*((X \setminus K_X) \times Y) + C^*(X \times (Y \setminus K_Y)))$, i.e. it is a cochain which vanishes on the chains supported on $C_*(X \setminus K_X) \times Y$ and $C_*(X \times (Y \setminus K_Y))$. Using the barycentric subdivision operator and the fife lemma [8, Lemma IV.5.10] one can show (see [16, Chapter 3.2] for more details) that

$$H^*(X \times Y, ((X \setminus K_X) \times Y) + (X \times (Y \setminus K_Y))) \cong H^*(X \times Y, (X \times Y) \setminus (K_X \times K_Y)).$$

As a result the cross product is defined only on the level of cohomology. However, in the case of cohomology with Lipschitz compact supports the cross product can be defined on the cochain level.

Proposition 1.2.26 ([25, Lemma 3.15])**.** *Let X and Y be two complete metric spaces. Then the cross product*

$$\begin{array}{ccc} C^n(X) \otimes C^m(Y) & \to & C^{n+m}(X \times Y) \\ \phi \otimes \psi & \mapsto & \phi \times \psi \end{array}$$

restricts to the cross product $C^n_{cs,\mathrm{Lip}}(X) \otimes C^m_{cs,\mathrm{Lip}}(Y) \to C^{n+m}_{cs,\mathrm{Lip}}(X \times Y)$.

1.3. Amenability

We will need a few facts concerning the amenability of groups. The amenability of groups can be defined both analytically and geometrically, and its intuitive meaning is that the group allows certain averaging constructions. We present here only a few facts and definitions that are sufficient for our applications, for more information and examples see [27, 28].

Definition 1.3.1. A group G is *amenable* if it admits a G-invariant mean, i.e. an element $\mu \in (\ell^\infty(G))^*$, such that

1. $\mu(\mathbf{1}_G) = 1$;
2. $\mu(g \cdot f) = \mu(f)$ for every $g \in G$, $f \in \ell^\infty(G)$.

In most cases only finitely generated amenable groups are considered. Then one has nice alternative characterisations. However, some of these characterisations can be stated also in the general case. Recall that we say that an element $\mu \in \ell^1(G)$ is a *probability measure*, if it is non-negative and $\sum_{g\in G} \mu(g) = 1$. The following theorem, although not stated as above in [27], is in fact proved there.

Proposition 1.3.2 ([27, Theorem 3.3.2]). *If G is amenable, then for any $\varepsilon > 0$ and any finite set $\Gamma \subset G$ there exists a finitely supported probability measure $\mu \in \ell^1(G)$ such that*

$$\sup_{g \in \Gamma} \|\mu - g \cdot \mu\|_1 \leqslant \varepsilon.$$

Remark 1.3.3. Note that if $\mu \in \ell^\infty(G)$ is a mean invariant with respect to the left action of G on ℓ^∞, the mean defined as

$$\mu_r(f) = \mu(\mathrm{inv}(f)),$$

where $\mathrm{inv}(f)(g) = f(g^{-1})$ for $g \in G$, defines an invariant mean with respect to the right action of G on $\ell^\infty(G)$. The same applies to the above proposition, namely it is also valid for the right action on $\ell^1(G)$.

The simplest class of examples of amenable groups are finite groups. It can be checked that an invariant mean for a finite group G is given by

$$\mu(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

However, one can obtain much more examples using the following facts.

Proposition 1.3.4 ([28, Proposition 3.55]). *Let*

$$1 \to A \to B \to C \to 1$$

be an exact sequence of groups. Then

1. *if B is amenable, then A and C are;*
2. *if A and C are amenable, then B is.*

Proposition 1.3.5 ([10, Theorem 3.4]). *The direct limit (over any, non necessarily countable, linearly ordered system) of amenable groups is an amenable group. In particular, if $(G_j)_{j \in J}$ is a family of amenable groups for any indexing set J, then $\bigoplus_{j \in J} G_j$ is amenable.*

The following proposition is a characterisation of amenability used by Gromov in [12]. Note that if G acts on a set X, then $\ell^1(G)$ acts on $\ell^1(X)$ by

$$(\mu * f)(x) = \sum_{g \in G} \mu(g) f(g^{-1}x),$$

where $\mu \in \ell^1(G)$ and $f \in \ell^1(X)$. Certainly, $\|\mu * f\|_1 \leqslant \|\mu\|_1 \cdot \|f\|_1$.

Proposition 1.3.6 ([12, Section 4.2]). *Let $f : X \to \mathbb{R}$ be a finitely supported function and let G be an amenable group acting transitively on the set X. Then for every $\varepsilon > 0$ there exists a finitely supported probability measure μ on G such that*

$$\|\mu * f\|_1 \leqslant \varepsilon + |\sum_{x \in X} f(x)|.$$

Corollary 1.3.7. *Let $f : X \to \mathbb{R}$ be a finitely supported function and let G be an amenable group acting on the set X with a finite set of orbits $X_0, ..., X_N \subset X$. Then for every $\varepsilon > 0$ there exists a finitely supported probability measure μ on G such that*

$$\|\mu * f\|_1 \leqslant \varepsilon + \sum_{i=1}^{N} |\sum_{x \in X_i} f(x)|$$

Proof. Consider $f_i = f|_{X_i} \in \ell^1(X_i)$ for $i = 1, ..., N$. In particular, $\|f\| = \sum_{i=1}^N \|f_i\|_1$. By induction, choose finitely supported probability measures $\mu_k \in \ell^1(G)$ for $k = 1, ..., N$, such that

$$\|\mu_k * (... * \mu_1 * f_k)\|_1 \leqslant \frac{\varepsilon}{N} + |\sum_{x \in X_k} \mu_{k-1} * ...\mu_1 * f_k(x)| = \varepsilon + |\sum_{x \in X_i} f_k(x)|.$$

Then for $\mu = \mu_N * ... * \mu_1$, we have

$$\begin{aligned} \|\mu * f\|_1 &= \|\mu_N * ... * \mu_1 * f\|_1 = \sum_{i=1}^N \|\mu_N * ...\mu_1 * f_i\|_1 \\ &\leqslant \sum_{i=1}^N \|\mu_N\|_1 \cdot ...\|\mu_{i+1}\|_1 \cdot \|\mu_i * ...\mu_1 * f_i\|_1 \\ &\leqslant \sum_{i=1}^N (\frac{\varepsilon}{N} + |\sum_{x \in X_i} f_i(x)|) = \varepsilon + \sum_{i=1}^N |\sum_{x \in X_i} f(x)|. \end{aligned}$$

□

Chapter 2

Simplicial volume

In this chapter we present the simplicial volume, its variants and some methods used to study them. In Section 2.1 we introduce the notion of the simplicial volume and describe some of its most crucial properties. In Section 2.2 we study generalizations of the simplicial volume to the non-compact case. In particular, we define the Lipschitz simplicial volume and state the results concerning this invariant. Section 2.3 is devoted to the presentation of the duality principle and its Lipschitz variant [12, 25], which are very important tools in the study of the standard and Lipschitz simplicial volumes. Finally, in Section 2.4 we introduce the diffusion technique which is also a useful tool in various computations concerning the simplicial volume. In particular, we prove Theorem D form the introduction. The diffusion of chains was first described by Gromov [12], but we define it in an alternative way.

2.1. Simplicial volume of compact manifolds

Let X be a topological space. Then we can introduce the ℓ^1 norm on the space $C_*(X)$ by

$$|\sum_i a_i\sigma_i|_1 = \sum_i |a_i|.$$

This norm allows us to define the semi-norm on $H_*(X)$ by

$$\|[c]\|_1 = \inf\{|c'|_1 \ :\ c' \in C_*(X)\ ;\ [c] = [c'] \in H_*(X)\}.$$

This short introduction leads us to one of the most important definitions of this work.

Definition 2.1.1. Let M be a closed, oriented manifold. Then the *simplicial volume* of M is

$$\|M\| := \|[M]\|_1.$$

However, the simplicial volume can be defined also for manifolds with boundary as $\|[M,\partial M]\|_1$, and for non-orientable manifolds by

$$\|M\| = \frac{1}{2}\|\widetilde{M}^2\|,$$

where $\|\widetilde{M}^2\|$ denotes a double oriented covering of M.

To get a better feeling how the simplicial volume behaves, let us consider two basic examples, following [12].

Example 2.1.2. We will show that $\|S^1\| = 0$. Let $\gamma_n : I \to S^1$ be defined as $\gamma_n(t) = e^{2n\pi it}$. Then for every $n \in \mathbb{N}_+$ we have the fundamental cycle

$$c_n = \frac{1}{n}\gamma_n,$$

which satisfies $|c_n|_1 = \frac{1}{n}$. Therefore $\|S^1\| = 0$.

Example 2.1.3. We will show that for a closed surface Σ_g of genus $g \geqslant 2$ one has

$$\|\Sigma_g\| = 4(g-1) = 2|\chi(\Sigma_g)|.$$

Because the universal cover of Σ_g is isometric to $\mathbb{H}^2$, we can equip Σ_g with a Riemannian metric of constant curvature -1. For a singular simplex $\sigma : \Delta^2 \to \Sigma_g$ let $\operatorname{str}(\sigma) : \Delta^2 \to \Sigma_g$ be the singular simplex which is the projection of a geodesic simplex in $\widetilde{\Sigma}_g = \mathbb{H}^2$ with the same vertices as some lift of σ to $\widetilde{\Sigma}_g$. In particular, by Theorem 1.1.18,

$$\operatorname{vol}(\operatorname{im}\operatorname{str}(\sigma)) \leqslant \operatorname{vol}(\operatorname{im}(\widetilde{\operatorname{str}(\sigma)})) < \pi.$$

If $c = \sum_i a_i\sigma_i$ is a fundamental cycle, then the cycle $c' = \sum_i a_i \operatorname{str}(\sigma_i)$ is also fundamental, with $|c'|_1 \leqslant |c|_1$. Moreover, by Gauss-Bonnet theorem (Theorem 1.1.19),

$$\pi|c'|_1 > \sum_i |a_i| \operatorname{vol}(\operatorname{im}\operatorname{str}(\sigma_i)) \geqslant \operatorname{vol}(\Sigma_g) = 2\pi|\chi(\Sigma_g)|.$$

Hence $\|\Sigma_g\| \geqslant 2|\chi(\Sigma_g)| = 4(g-1)$.

On the other hand, present Σ_g as an $4g$-gon with identified edges. Then it can be subdivided into $4g-2$ triangles, hence $\|\Sigma_g\| \leqslant 4g-2$. Moreover, for every $n \geqslant 1$ there exists a finite n-folded covering $\Sigma_{n(g-1)+1} \to \Sigma_g$, hence using the similar triangulation of $4(n(g-1)+1)$-gon into $4n(g-1)+2$ triangles, we obtain

$$\|\Sigma_g\| \leqslant \inf_{n\in\mathbb{N}} \frac{4n(g-1)+2}{n} = 4(g-1).$$

We list now some crucial properties of the simplicial volume.

1. *Functoriality*

   Let $f : M \to N$ be a continuous map of n-dimensional closed manifolds. Then because $f_*([M]) = \deg(f) \cdot [N]$, we have

   $$\|N\| \cdot |\deg(f)| \leqslant \|M\|.$$

   Corollary 2.1.4. *If $\|N\| > 0$ then for any map $f : M \to N$,*

   $$|\deg(f)| \leqslant \frac{\|M\|}{\|N\|}.$$

   As one can see, the above corollary is quite useless, unless one could decide if the simplicial volume of a given manifold is 0 or not. The exact value or some fine estimate would be even better, but even in the case where we know only that $\|N\| \neq 0$ we can deduce that there exists an upper bound on the degree of any map from any given closed manifold M. In particular, if $\|N\| \neq 0$ then N is *inflexible*, i.e. the only self-maps $f : N \to N$ are of degree 0 or ± 1. However, the converse is false, because there exist inflexible manifolds with vanishing simplicial volume [9]. Therefore the vanishing of the simplicial volume does not give us any new information and we are particularly interested in the examples of manifolds with non-zero simplicial volume.

2. *Product inequality*

   Although there is no known formula for the simplicial volume of the product of manifolds, there exist some estimates comparing this simplicial volume with the simplicial volumes of factors.

   Theorem 2.1.5 ([12, Section 1.1]). *If M and N are closed manifolds, then*

   $$\|M\| \cdot \|N\| \leqslant \|M \times N\| \leqslant \binom{\dim M + \dim N}{\dim M} \|M\| \cdot \|N\|.$$

   However, it is an open question if the above estimates are optimal. The only exact computation of the simplicial volume of a product of manifolds M_1 and M_2 such that $\|M_1 \times M_2\| > 0$ was made by Bucher in [4], stating that if M_1 and M_2 are two closed surfaces then

   $$\|M \times N\| = \frac{3}{2}\|M\| \cdot \|N\|.$$

   There are also some generalisations of these inequalities to the case of surface bundles, see [13, 5] for details.

3. *Relations with volume*

   In the presence of certain curvature conditions, the simplicial volume, in spite of being a homotopy invariant, has strong connections with the Riemannian volume.

   Theorem 2.1.6 ([12, Section 1.2]). *Let M be a closed Riemannian manifold such that* $\sec(M) \leqslant -1$. *Then there exists positive constant C_n, depending only on* $\dim M$, *such that*

   $$\|M\| \geqslant C_n \operatorname{vol}(M).$$

   In fact, this is true also if M is a closed locally symmetric space of non-compact type [20, 29].

   On the other hand, we have the following.

   Theorem 2.1.7 ([12, Section 2.4 and 2.5]). *If M is a closed Riemannian manifold with* $\operatorname{Ricci}(M) \geqslant -(n-1)$, *then*

   $$\|M\| \leqslant n! \cdot \operatorname{vol}(M).$$

   These two theorems, together with functorial properties of the simplicial volume, are sufficient to yield a non-trivial degree theorem.

   Corollary 2.1.8. *Let $f : M \to N$ be a continuous map between two closed Riemannian n-manifolds such that* $\operatorname{Ricci}(M) \geqslant -(n-1)$ *and* $\sec(N) \leqslant -1$ *(or N is a closed locally symmetric space of non-compact type). Then there exists a constant D_n, depending only on n, such that*

   $$|\deg(f)| \leqslant D_n \frac{\operatorname{vol}(M)}{\operatorname{vol}(N)}.$$

4. *Proportionality principle*

   The theorems in the previous point might indicate that the simplicial volume is in some sense proportional to the Riemannian volume. The following theorem is some kind of confirmation to such suspicions.

Theorem 2.1.9 ([12, Section 2.3])**.** *Let M and N be closed Riemannian manifolds with isometric universal covers. Then*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

5. *Connected sums*

   Another source of manifolds with positive simplicial volume are the connected sums of such manifolds.

   Theorem 2.1.10 ([12, Section 3.5])**.** *Let M and N be closed n-dimensional manifolds, where $n \geqslant 3$. Then*

   $$\|M \# N\| = \|M\| + \|N\|.$$

6. *Vanishing on amenable manifolds*

   Unfortunately, there are many manifolds with trivial simplicial volume. The theorem below yields a large class of such examples.

   Theorem 2.1.11 ([12, Sections 3.1-3.3])**.** *If M is a closed manifold and $\pi_1(M)$ is amenable, then*

   $$\|M\| = 0.$$

   In fact, a stronger statement is true. Namely if there exists an amenable open cover of M with multiplicity $\leqslant \dim M$, then $\|M\| = 0$ [12, Section 4.2].

2.2. Simplicial volume of non-compact manifolds

Let us now examine the non-compact case. Let M be a complete, connected, oriented n-manifold. Then we can define the simplicial volume of $\|M\|$ just as

$$\|M\| := \|[M]_{lf}\|_1,$$

where $[M]_{lf} \in H_n^{lf}(M)$ is the locally finite fundamental class. Note that because we are using locally finite homology, it is invariant with respect to proper homotopy equivalences. Although some of the properties of the simplicial volume for compact manifolds generalise to the non-compact case, some of them do not. We list some of these.

- If M and N are complete manifolds then

  $$\|M \times N\| \leqslant \binom{\dim M + \dim N}{\dim M} \|M\| \cdot \|N\|.$$

  However, the inequality

  $$\|M\| \cdot \|N\| \leqslant \|M \times N\|$$

  does not hold in general. In fact, the simplicial volume of a products is in many cases zero by the following theorem.

  Theorem 2.2.1 ([12, Section 4.2])**.** *Let M_1, M_2, M_3 be open manifolds. Then*

  $$\|M_1 \times M_2 \times M_3\| = 0.$$

One could ask what happens if we take the product of two open manifolds. However, very little is known. In particular, the question whether the simplicial volume of a product of two open manifolds $M_1 \times M_2$ can be finite and positive is still open.

- As in the compact case, there exists a constant $C_n > 0$ depending only on dimension of M, such that if M is a Riemannian manifold with $\mathrm{Ricci}(M) \geqslant -(n-1)$ then

$$\|M\| \leqslant C_n \operatorname{vol}(M).$$

  On the other hand, there is no lower bound on the simplicial volume in terms of the Riemannian volume in general. For example, for the hyperbolic space $\mathbb{H}^n$ for $n \geqslant 2$ one has $\sec(\mathbb{H}^n) \leqslant -1$ and $\operatorname{vol}(\mathbb{H}^n) = \infty$, while $\|\mathbb{H}^n\| = 0$ because there exist proper maps $\mathbb{H}^n \to \mathbb{H}^n$ of arbitrary large degree.

- The proportionality principle does not hold in the non-compact case, even for Riemannian manifolds of finite volume. To see this, let M be a complete, non-compact hyperbolic manifold of finite volume. Let also M' be a closed hyperbolic manifold of the same dimension. Then M and M' have isometric universal covers, but by Theorems 2.1.5 and 2.2.1,

$$\frac{\|(M')^3\|}{\operatorname{vol}((M')^3)} \neq 0 = \frac{\|M^3\|}{\operatorname{vol}(M^3)}.$$

As we can see, although some properties are inherited by the simplicial volume in the non-compact case, some of those which are particularly interesting to us (because they yield some examples of manifolds with non-zero simplicial volume) in most cases do not hold any more. Therefore Gromov in [12] introduced a metric version of the simplicial volume.

Definition 2.2.2. Let M be a complete oriented Riemannian manifold with $\operatorname{vol}(M) < \infty$. Then the *Lipschitz simplicial volume* of M is

$$\|M\|_{\mathrm{Lip}} := \|[M]_{\mathrm{Lip}}\|_1.$$

Note that this definition makes sense only for manifolds with $\operatorname{vol}(M) < \infty$, because in the other case one must have $\|M\|_{\mathrm{Lip}} = \infty$. Indeed, we have the following lemma.

Lemma 2.2.3 ([25, Proposition 4.4(1)])**.** *For every L-Lipschitz singular simplex on M, we have*

$$\langle \mathrm{dvol}_M, \sigma_k \rangle \leqslant L^n \operatorname{vol}(\Delta^n).$$

Now, let $c = \sum_i a_i \sigma_i \in C_n^{lf,\mathrm{Lip}}(M)$ be a fundamental cycle. Then on the one hand one has $\operatorname{vol}(M) = \infty$, but on the other hand, by the above lemma and Proposition 1.2.24,

$$\operatorname{vol}(M) = \langle \mathrm{dvol}_M, c \rangle \leqslant \sum_i |a_i| \operatorname{vol}(\sigma_i) \leqslant \sum_i |a_i| \operatorname{Lip}(c)^n \operatorname{vol}(\Delta^n).$$

It follows that $\sum_i |a_i| = \infty$.

To see that the above definition truly generalises the definition of the simplicial volume, we prove the following.

Proposition 2.2.4. *Let M be a smooth compact manifold. Then $\|M\|_{\mathrm{Lip}} = \|M\|$.*

Proof. By Example 1.2.13, smooth homology is isomorphic to ordinary singular homology. Moreover, by the proof of Lemma 1.2.11, the simplicial volume can be computed only on smooth chains. Because every finite smooth chain is Lipschitz, the proposition follows. □

However, in contrast to the 'naive' generalization of the simplicial volume, the Lipschitz simplicial volume seems to satisfy a lot of properties of the classical one also in the non-compact case.

1. *Functoriality*

   The functorial behaviour of the Lipschitz simplicial volume is very similar to that of the simplicial volume. The only difference is that it is functorial with respect to Lipschitz proper maps. In particular, we have the following proposition.

   Proposition 2.2.5. *If $\|N\|_{\mathrm{Lip}} > 0$ then for any Lipschitz proper map $f : M \to N$,*

   $$|\deg(f)| \leqslant \frac{\|M\|_{\mathrm{Lip}}}{\|N\|_{\mathrm{Lip}}}.$$

   Again, the above proposition is not very useful unless we can estimate somehow $\|\cdot\|_{\mathrm{Lip}}$, or at least give examples of manifolds with positive Lipschitz simplicial volume.

2. *Product inequality*

   In contrast to the standard generalisation of the simplicial volume, the product inequality for the Lipschitz simplicial volume holds.

   Theorem 2.2.6 ([32, Theorem 1.3])**.** *Let M and N are closed manifolds with $\sec(M), \sec(N) < K < \infty$. Then*

   $$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leqslant \|M \times N\|_{\mathrm{Lip}} \leqslant \binom{\dim M + \dim N}{\dim M} \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}.$$

   The proof is postponed to Section 4.2. It has been proved recently by Franceschini [14] that it is true also without the curvature assumption .

3. *Relations with volume*

   Theorem 2.2.7. *Let M be a complete Riemannian manifold such that $sec(M) \leqslant -1$. Then there exists positive constant C_n, depending only on $\dim M$, such that*

   $$\|M\|_{\mathrm{Lip}} \geqslant C_n \cdot \mathrm{vol}(M).$$

   The proof of the above theorem is postponed to Section 4.1.

   Using the proportionality principle for the Lipschitz simplicial volume, one can prove the above estimate also if M is a complete locally symmetric space [25, Corollary 1.6].

   There is also an estimate from above for the Lipschitz simlicial volume by the Riemannian volume. However, we need to add an additional curvature assumption.

   Theorem 2.2.8 ([25, Theorem 1.8])**.** *Let M be a complete Riemannian manifold such that $\mathrm{Ricci}(M) \geqslant -(n-1)$ and $\sec(M) \leqslant 1$. Then there exists a constant D_n, depending on $n = \dim M$, such that*

   $$\|M\| \leqslant D_n \,\mathrm{vol}(M)$$

   Using the above two theorems, we can generalize the corresponding degree theorem to the non-compact case.

Corollary 2.2.9 ([25, Theorem 1.10]). *Let $f : M \to N$ be a proper Lipschitz map between two complete Riemannian n-manifolds of finite volume such that* $\mathrm{Ricci}(M) \geqslant -(n-1)$ *and* $\sec(N) \leqslant -1$ *(or N is a closed locally symmetric space of non-compact type). Then there exists a constant D_n, depending only on n, such that*

$$|\deg(f)| \leqslant D_n \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)}.$$

4. *Proportionality principle*

   For the Lipschitz simplicial volume, the proportionality principle generalizes to the non-compact case.

   Theorem 2.2.10 ([32, Theorem 1.4]). *Let M and N be complete Riemannian manifolds of finite volume with isometric universal covers such that* $\sec(M), \sec(N) < K < \infty$. *Then*

$$\frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)} = \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)}.$$

   The proof is postponed to Section 4.3. This theorem has been also recently generalized in [14] to the case of manifolds without curvature bounds.

5. *Connected sums*

   This case is still open. However, because taking the connected sum of two manifolds is an operation that modifies both manifolds locally, there is a chance that the Lipschitz simplicial volume behaves similarly as the classical simplicial volume of a compact manifold. Moreover, in some special cases the additivity with respect to connected sums can be deduced from existing theorems.

   Proposition 2.2.11. *Let M and N be two manifolds of the same dimension $n \geqslant 3$ which are homeomorphic to the interiors of compact manifolds with boundaries consisting of components with amenable fundamental groups. Assume moreover that $\|M\|_{\mathrm{Lip}}$ and $\|N\|_{\mathrm{Lip}}$ are finite. Then*

$$\|M\# N\|_{\mathrm{Lip}} = \|M\|_{\mathrm{Lip}} + \|N\|_{\mathrm{Lip}}.$$

   Proof (sketch). By [18, Corollary 1.4], we know that if $\|M\|_{\mathrm{Lip}}, \|N\|_{\mathrm{Lip}} < \infty$, $M \cong V$ and $N \cong W$, where V and W are compact manifolds with amenable boundary components, then $\|M\|_{\mathrm{Lip}} = \|V, \partial V\|$ and $\|N\|_{\mathrm{Lip}} = \|W, \partial W\|$. Moreover, by [19, Theorem 1],

$$\|V\# W, \partial V \cup \partial W\| = \|V, \partial V\| + \|W, \partial W\|.$$

   Observe that using again [18, Corollary 1.4], the left-hand side of the above equality equals $\|M\# N\|_{\mathrm{Lip}}$. $\square$

6. *Vanishing on amenable manifolds*

   Vanishing results for the Lipschitz simplicial volume are quite similar to those in the compact case. One has the following theorem.

   Theorem 2.2.12. *Let M be a complete manifold such that $\|M\|_{\mathrm{Lip}} < \infty$. If $\pi_1(M)$ is amenable then*

$$\|M\| = 0.$$

The proof is postponed to Section 2.4.

Note, however, that there are amenable manifolds such that $\|M\|_{\text{Lip}} = \infty$ in every dimension. Consider for example $M_n = S^{n-1} \times \mathbb{R}$ for $n \in \mathbb{N}_+$. Then $\pi(M_n)$ is amenable (in fact trivial for $n \neq 1$), but $\text{vol}(M_n) = \infty$, hence $\|M\|_{\text{Lip}} = \infty$.

2.3. Duality

One of the most important tools in the study of the simplicial volume is bounded cohomology, which is linked to the simplicial volume by the *duality principle*. Bounded cohomology is cohomology of the complex of bounded cochains, which is a subcomplex of the standard singular chain complex. It turns out that the ℓ^1 semi-norm of homology classes can be expressed by a semi-norm of corresponding bounded cohomology classes. Almost all the techniques used for the simplicial volume can be translated to the language of bounded cohomology. Moreover, in many cases bounded cohomology is easier to study. In particular, if an amenable group acts on a set of bounded cochains, we can average this action. This opportunity is exploited in many problems where the amenability of certain subspaces is present [6, 12, 19, 18]. Another application of the duality between the simplicial volume and bounded cohomology in which we are interested in, is the product inequality.

We will not need bounded cohomology itself here. To our applications, it suffices to define the ℓ^∞ semi-norm on singular cochains. Let $\phi \in C^k(M)$ for $k \in \mathbb{N}$. Then we define the (possibly infinite) ℓ^∞ norm of ϕ as

$$\|\phi\|_\infty = \sup_{\sigma \in C(\Delta^k, M)} |\phi(\sigma)|.$$

If this norm is finite, we say that ϕ is *bounded*. This norm induce the ℓ^∞ semi-norm on $H^*(M)$ as

$$\|[\phi]\|_\infty = \inf\{\|\phi'\|_\infty \ : \ [\phi'] = [\phi] \in H^*(M)\}.$$

Obviously, the above semi-norm can be infinite.

We mentioned that the simplicial volume and bounded cohomology are connected by the duality principle. We present it below.

Theorem 2.3.1 (Duality principle, [12, Section 1.1])**.** *Let M be a closed oriented manifold. Then*

$$\|M\| = \frac{1}{\|[M]^*\|_\infty},$$

where $[M]^* \in H^n(M)$ is the fundamental cohomology class.

There is also a version of the duality principle for the Lipschitz simplicial volume. Unfortunately, it is slightly more complicated than the classical one. First of all, the space 'dual' to the space of locally finite chains is the space of bounded cochains with compact supports, which is usually more complicated to study than the space of just bounded cochains. The reason is that it is difficult to express having a compact support in the analytical setting.

We will denote by $S_k^{lf,\text{Lip}}(M)$ the family of subsets of $C(\Delta^k, M)$ such that $A \in S_k^{lf,\text{Lip}}$ if and only if it is locally finite, in the sense that for any given compact subset $K \subset M$ we have $\#\{\sigma \in A \ : \ \sigma \cap K \neq \emptyset\} < \infty$, and consists of L-Lipschitz simplices for some $L < \infty$, depending on A. If $A \in S_k^{lf,\text{Lip}}$, we define the ℓ^1 norm relative to A of a singular locally finite Lipschitz chain $c = \sum_i a_i \sigma_i \in C_k^{lf,\text{Lip}}(M)$ as

$$|c|_1^A := \begin{cases} |c|_1 & \text{if } \operatorname{supp}(c) \subset A, \\ \infty & \text{otherwise.} \end{cases},$$

where $\operatorname{supp}(c) = \bigcup_i \{\sigma_i\} \subset C(\Delta^k, M)$. It induces a semi-norm on $H_k^{lf,\mathrm{Lip}}(M)$ by

$$\|[c]\|_1^A := \inf\{|c'|_1^A \,:\, [c'] = [c] \in H_k^{lf,\mathrm{Lip}}(M)\}.$$

In particular, we will denote by $\|M\|^A$ the simplicial volume relative to A, i.e.

$$\|M\|^A := \|[M]\|_1^A.$$

It is a straightforward observation that

$$\|M\|_{\mathrm{Lip}} = \inf_{A \in S_n^{lf,\mathrm{Lip}}} \|M\|^A.$$

We have also dual norms for cocycles with Lipschitz compact supports. They are defined as

$$\|\phi\|_\infty^A := \sup_{\sigma \in A} |\phi(\sigma)|$$

for $A \in S_k^{lf,\mathrm{Lip}}$. We will denote by $\|\cdot\|_\infty^A$ also the corresponding semi-norm on cohomology.

Theorem 2.3.2 ([25, Proposition 3.12])**.** *Let M be an oriented, connected Riemannian manifold and let $A \in S_n^{lf,\mathrm{Lip}}(M)$. Then*

$$\|M\|^A = \frac{1}{\|[M]^*_{\mathrm{Lip}}\|_\infty^A},$$

where $[M]^*_{\mathrm{Lip}}$ is the Lipschitz compactly supported fundamental class.

2.4. Diffusion of chains

Another technique, invented by Gromov in [12], is the diffusion of chains. The philosophy of this method is to use certain paths to mix vertices of a given chain and observe how does this procedure affect the coefficients of the simplices in this chain. Under some assumptions (concerning amenability), some coefficients might (almost) vanish. This method can be used to exclude certain simplices from the calculation of the simplicial volume. Because it works purely on the geometric level, it can be easily adapted to the Lipschitz case when used locally and yield a proof of Theorem 2.2.12.

The diffusion technique has several variants, we will use the following one.

Proposition 2.4.1. *Let M be a Riemannian manifold and let $K \subset M$ be a path-connected compact set such that the image of the map $\pi_1(K) \to \pi_1(M)$ is amenable. Then for any cycle $c = \sum_i a_i\sigma_i \in C_k^{lf,Lip}(M)$ such that every simplex $\sigma \in \operatorname{supp}(c)$ satisfying $\sigma \cap K \neq \emptyset$ has distinct vertices, we have*

$$\|[c]\|_1 \leqslant \sum_{\{i \,:\, \sigma_i \text{ has no edge in } K\}} |a_i|.$$

Remark 2.4.2. In fact the inequality in the above proposition is true without the assumption on the distinctiveness of vertices. However, the proof is much longer and more technical, and the above version is sufficient for our applications.

Corollary 2.4.3. *Let M be a Riemannian manifold and let $K \subset M$ be a path-connected compact uncountable set such that the image of the map $\pi_1(K) \to \pi_1(M)$ is amenable. Then for any cycle $c = \sum_i a_i\sigma_i \in C_k^{lf,Lip}(M)$, we have*

$$\|[c]\|_1 \leqslant k! \sum_{\{i \,:\, \sigma_i \not\subset K\}} |a_i|.$$

Proof. Note that by homotoping finitely many simplices while keeping their vertices fixed, we can assume that for every simplex $\sigma \in \operatorname{supp}(c)$ such that $\sigma \cap K \neq \emptyset$, its barycentric subdivision $S(\sigma)$ consists of simplices with distinct vertices (here we use the assumption that K is uncountable). Let $S(c) = \sum_j a'_j \sigma'_j$. We have

$$\begin{aligned} \|[c]\|_1 = \|[S(c)]\|_1 &\leqslant \sum_{\{j\,:\,\sigma'_j \text{ has no edge in } K\}} |a'_j| \leqslant \sum_{\{j\,:\,\sigma'_j \not\subset K\}} |a'_j| \\ &\leqslant \sum_{\{i\,:\,\sigma_i \not\subset K\}} \sum_{\{\sigma'_j \in \operatorname{supp}(S(\sigma_i))\}} |a_i| \leqslant \sum_{\{i\,:\,\sigma_i \not\subset K\}} k! \cdot |a_i|. \end{aligned}$$

□

To prove Proposition 2.4.1 we will need a few ingredients. Some of them, which have been already introduced, are the alternation of chains and diffusion of a function in the sense of Corollary 1.3.7. The last one is the *pre-straightening* of chains. Let $\Sigma_K = \bigcup_{k\in\mathbb{N}} \Sigma_K^k$, where $\Sigma_K^k \subset \operatorname{Lip}(\Delta^k, M)$ for $k \in \mathbb{N}$, be a family of Lipschitz simplices satisfying:

1. $\Sigma_K^0 = M$;
2. if $\sigma \in \Sigma_K^k$ and $\sigma' = \partial_i \sigma$ for some $i = 0, ..., k$ then $\sigma' \in \Sigma_K^{k-1}$;
3. Σ_K^k is closed under the action of S_{k+1} for $k \in \mathbb{N}$;
4. if a Lipschitz singular k-simplex σ satisfies $\partial_i \sigma \in \Sigma_K^{k-1}$, for $i = 0, ..., k$, then there exists a unique simplex $\operatorname{pstr}(\sigma) \in \Sigma_K^k$ with the same boundary which is Lipschitz homotopic to σ relative to $\partial\sigma = \partial \operatorname{pstr}(\sigma)$. Moreover, if $\operatorname{im}(\sigma) \subset K$, then $\operatorname{im} \operatorname{pstr}(\sigma) \subset K$.

Such a family always exists and can be easily constructed by induction on $k \in \mathbb{N}$. We will call any such family the *family of pre-straight simplices* (with respect to K).

Definition 2.4.4. We say that a chain c is Σ_K*-admissible* if every simplex $\sigma_i \in \operatorname{supp}(c)$ such that $\sigma_i \cap K \neq \emptyset$ is contained in Σ_K.

Lemma 2.4.5. *Every locally finite Lipschitz chain c is homologuous in $C_*^{lf,\mathrm{Lip}}(M)$ to a Σ_K-admissible chain* $\operatorname{pstr}_*(c)$.

Proof. First of all we construct a system of Lipschitz homotopies $H_\sigma : \Delta^k \times I \to M$ for $\sigma \in \bigcup_{k\in\mathbb{N}} \operatorname{Lip}(\Delta^k, M)$ such that

1. $H_\sigma|_{\Delta^k \times \{0\}} = \sigma$;
2. $H_\sigma|_{\Delta^k \times \{1\}} \in \Sigma_K^k$;
3. for every $k \in \mathbb{N}$ and $i = 0, ..., k$, one has
$$H_\sigma|_{\partial_i \Delta^k \times I}(x, t) = H_{\partial_i \sigma}(x, \min\{2t, 1\}).$$

We define H_σ by induction on $k \in \mathbb{N}$. We can define $H_\sigma(x,t) = x$ for 0-simplices σ. Let H_σ be defined for all simplices $\sigma \in \bigcup_{j<k} \operatorname{Lip}(\Delta^k, M)$ and let $\sigma' \in \operatorname{Lip}(\Delta^k, M)$. Then we define $H'_{\sigma'} : \Delta^k \times \{0\} \cup \partial\Delta^k \times I$ by
$$H'_{\sigma'}|_{\Delta^k \times \{0\}} = \sigma'$$
and
$$H'_{\sigma'}|_{\partial_i \Delta^k \times I} = H_{\partial_i \sigma'}$$

for $i = 0, ..., k$. Because there exists a Lipschitz retraction $r_k : \Delta^k \times I \to \Delta^k \times \{0\} \cup \partial\Delta^k \times I$, we define $H''_{\sigma'} : \Delta^k \times I \to M$ as

$$H''_{\sigma'}(x,t) = H'_{\sigma'}(r_k(x,t)).$$

Let $\sigma'' = H''_{\sigma'}|_{\Delta^k \times \{1\}}$. Then $\partial\sigma''$ is a sum of simplices in Σ_K^{k-1}, hence there exists a unique simplex $\mathrm{pstr}(\sigma'') \in \Sigma_K^k$ which is Lipschitz homotopic to σ'' relative to $\partial\Delta^k$. Let $H'''_{\sigma''} : \Delta^k \times I$ be such homotopy. We define $H_{\sigma'}$ as

$$H_\sigma(x,t) = \begin{cases} H''_{\sigma'}(x,2t) & t \in [0,\frac{1}{2}] \\ H'''_{\sigma''}(x,2t-1) & t \in [\frac{1}{2},1] \end{cases}.$$

Having the homotopies H_σ, a naive way of defining the pre-straightening of $c = \sum_i a_i\sigma_i \in C_*^{lf,\mathrm{Lip}}(M)$ is

$$\mathrm{pstr}_*(c) = \sum_i a_i H_{\sigma_i}|_{\Delta^k \times \{1\}}.$$

However, the resulting chain could be non-Lipschitz because we have no control over the Lipschitz constants of $H_{\sigma_i}|_{\Delta^k \times \{1\}}$. On the other hand, we are interested in modifying only the simplices which have non-empty intersection with K. Let

$$A := \bigcup_{\{i\,:\,\sigma_i \cap K \neq \emptyset\}} \mathrm{im}(\sigma_i),$$

and let $f_c \in \mathrm{Lip}(M,\mathbb{R})$ be a compactly supported Lipschitz function such that $f_c|_A = 1$. Define

$$\mathrm{pstr}_*(c) := \sum_i a_i f_\# H_{\sigma_i}|_{\Delta^k \times \{1\}},$$

where $f_\# H_{\sigma_i}(x,t) = H_{\sigma_i}(x, f(\sigma(x)) \cdot t)$ for $x \in \Delta^k$ and $t \in [0,1]$. Note that the pre-straight chain defined above is a Lipschitz chain homotopic to c by the chain homotopy $\sum_i a_i P(f_\# H_{\sigma_i})$, where P is the prism operator from Section 1.2.5. Moreover, the simplices in $\mathrm{pstr}_*(c)$ which intersect K are contained in Σ_K^k. $\square$

Now let $\Pi(M,K)$ be a set of (not necessarily continuous) maps $K \ni x \mapsto [\gamma_x]$, where $[\gamma]$ is a homotopy class (in M) of a path $\gamma : I \to K$ relative to its endpoints, such that

1. $\gamma_x(0) = x$ for every $x \in K$;
2. the map $x \mapsto \gamma_x(1)$ is a bijection of K.
3. $([\gamma_x])_{x\in K} \in \Pi(M,K)$ has finite support, in the sense that for all but finitely many $x \in K$ the path γ_x is constant.

$\Pi(M,K)$ forms a group, where the multiplication is induced by the concatenation of paths. More precisely, if $([\gamma_x])_{x\in K}, ([\gamma'_x])_{x\in K} \in \Pi(M,K)$, then

$$([\gamma_x])_{x\in K} \cdot ([\gamma'_x])_{x\in K} = ([\gamma'_x * \gamma_{\gamma'_x(1)}])_{x\in K},$$

where $*$ is the concatenation of paths. Note that $\Pi(M,K)$ acts on Σ_K^k. For $g = (\gamma_x)_{x\in K} \in \Pi(M,K)$ and $\sigma \in \Sigma_K^k$ we define this action as follows. Let $v_0, ..., v_k$ be the vertices of σ and let σ_0 be a simplex obtained by homotopically extending each edge $[v_i, v_j]$ to an edge $\gamma_{v_i}^{-1} * [v_i, v_j] * \gamma_{v_j}$ (where we use the convention $\gamma_x(t) = x$ for $x \notin K$, $t \in [0,1]$), then extending this homotopy (defined on the 1-skeleton of Δ^k) to the whole Δ^k. Next we homotopy σ_0

successively to simplices $\sigma_1, ..., \sigma_k$ such that for $i = 1, ..., k$ the homotopy between σ_{i-1} and σ_i fixes the $i-1$-skeleton of Δ^k and the image of i-skeleton of Δ^k under σ_i is contained in Σ^i_K.

Note also that if we repeat the above construction on a chain instead of a single simplex, we will define the action of $\Pi(M,K)$ on Σ_K-admissible chains, which will satisfy $[g \cdot c] = [c] \in H^{lf,\mathrm{Lip}}_*(M)$ for a cycle $c \in C^{lf,\mathrm{Lip}}_*(M)$ and $g \in \Pi(M,K)$. In particular, for a cycle $c \in C^{lf,\mathrm{Lip}}_*(M)$ and a finitely supported probability measure $\mu \in \ell^1(\Pi(M,K))$, we have

$$[\mu * c] = [\sum_{g \in \Pi(M,K)} \mu(g) g \cdot c] = [c].$$

Proof of Proposition 2.4.1. Let

$$T := \{\sigma \in \mathrm{Lip}(\Delta^k, M) \,:\, \sigma \text{ has an edge in } K\}.$$

Let also $c' = \sum_j a'_j \sigma'_j = \mathrm{Alt}(\mathrm{pstr}_*(c))$. It is homologuous to c and by the constructions of both operators one has $|c'|_1 \leqslant |c|_1$. Moreover, by the definition of the pre-straigtening we obtain

$$\sum_{\{j \,:\, \sigma'_j \notin T\}} |a'_j| \leqslant \sum_{\{i \,:\, \sigma_i \notin T\}} |a_i|.$$

Therefore it is enough to prove the statement for c' instead of c.

There is a short exact sequence of groups

$$1 \to \bigoplus_{x \in K} \mathrm{im}(\pi_1(K,x) \to \pi_1(M,x)) \to \Pi(M,K) \to S_{fin}(K) \to 1,$$

where $S_{fin}(K)$ is a group of finitely supported permutations of K_0. Both $\bigoplus_{x \in K} \mathrm{im}(\pi_1(K,x) \to \pi_1(M,x))$ and $S_{fin}(K)$ are the direct limits of amenable groups, hence they are amenable by Proposition 1.3.5. It implies the amenability of $\Pi(M,K)$ by Proposition 1.3.4. Consider the action of $\Pi(M,K)$ on the set $\bigcup_{g \in \Pi(M,K)} \bigcup_{\{\sigma \in \mathrm{supp}(c) \cap T\}} g \cdot \sigma$. Then it obviously has a finite set of orbits $X_1, ..., X_N$. Therefore by Corollary 1.3.7, for every $\varepsilon > 0$ there exists a finitely supported probability measure $\mu \in \Pi(M,K)$ such that if $\mu * c' = \sum_p a''_p \sigma''_p$ then

$$\sum_{\{p \,:\, \sigma''_p \in T\}} |a''_p| \leqslant \varepsilon + \sum_{l=1}^{N} |\sum_{\{j \,:\, \sigma'_j \in X_l\}} a'_j|.$$

Because ε is arbitrary and the action of $\Pi(M,K)$ does not affect simplices not in T, we have

$$\|[c]\|_1 \leqslant \sum_{\{i \,:\, \sigma_i \notin T\}} |a_i| + \sum_{l=1}^{N} |\sum_{\{j \,:\, \sigma'_j \in X_l\}} a'_j|.$$

Therefore it suffices to show that for every $l = 1, ..., N$, we have $\sum_{\{j \,:\, \sigma'_j \in X_l\}} a'_j = 0$.

Let $\sigma'_j \in T$, then it has an edge e contained in K. Note that because the vertices of σ'_j are distinct, if $s \in S_{k+1}$ is the transposition that interchanges the endpoints of e then the simplex $s \cdot \sigma'_j$ is in the same orbit of the action of $\Pi(M,K)$ as σ'_j. To see this, define $g = (\gamma_x)_{x \in K}$ as

$$\gamma_x(t) = \begin{cases} e(t) & x = e(0); \\ e(1-t) & x = e(1); \\ x & x \neq e(0), e(1). \end{cases}$$

Then $g \cdot \sigma = s \cdot \sigma$. However, $c' = \mathrm{Alt}(c')$, therefore the coefficients of σ'_j and $s \cdot \sigma'_j$ cancel out. $\square$

As a corollary, we get the desired vanishing result for the Lipschitz simplicial volume.

Proof of Theorem 2.2.12. Let $c = \sum_{i=1}^{\infty} a_i\sigma_i$ be any Lipschitz fundamental cycle for M such that $|c|_1 < \infty$. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} |a_i| < \varepsilon.$$

Let $K = \bigcup_{i=1}^{N} \operatorname{im}(\sigma_i)$. It is a compact subset of M as a finite sum of compact sets. By Corollary 2.4.3, one obtains

$$\|M\|_{\mathrm{Lip}} = \|[c]\|_1 \leqslant n! \cdot \varepsilon.$$

Because ε was arbitrary, one has $\|M\|_{\mathrm{Lip}} = 0$. □

Corollary 2.4.6. *Let M be a complete Riemannian manifold with* $\operatorname{vol}(M) < \infty$*,* $\sec(M) \leqslant 1$ *and* $\operatorname{Ricci}(M) \geqslant -(n-1)$*. Then if $\pi_1(M)$ is amenable, then $\|M\|_{\mathrm{Lip}} = 0$.*

Remark 2.4.7. In fact, the diffusion technique was originally designed by Gromov to study the classical simplicial volume of non-compact manifolds. He used *locally finite diffusion* [12] and obtained much more general statement.

Theorem ([12, Vanishing-finiteness Theorem 4.2])**.** *Let M be an n-dimensional manifold with an amenable precompact open cover $\mathcal{U} = U_i$, $i \in \mathbb{N}$ such that*

1. *$U_i \to \infty$ in the sense for every compact set K one has*

$$\#\{U_i \in \mathcal{U} \,:\, U_i \cap K \neq \emptyset\} < \infty;$$

2. *the multiplicity of the cover $\mathcal{U}$ is at most n;*

3. *M is amenable at infinity, i.e. for every compact set $K \subset M$ there exists a compact set $K' \supset K$ such that* $\operatorname{im}(\pi_1(M \setminus K') \to \pi_1(M \setminus K))$ *is amenable.*

Then $\|M\| = 0$.

The outline of the proof is as follows. Consider a triangulation of M such that the star of every simplex is contained in one of the sets U_i, $i \in \mathbb{N}$. Then one can subdivide M into disjoint Borel subsets $V_j \in M$, $j \in \mathbb{N}$, such that each of them is contained in some U_i, $i \in \mathbb{N}$. Moreover, because the multiplicity of $\mathcal{U}$ is at most n, one can choose $(V_j)_{j\in\mathbb{N}}$ such that every simplex has at least one edge in some of these sets. Then one can modify the cycle defined by the triangulation such that it has arbitrary small norm, using locally finite diffusion, i.e. locally finite generalization of Proposition 2.4.1 for the family $(V_j)_{j\in\mathbb{N}}$.

In the Lipschitz case the above approach is not possible because when applying locally finite diffusion one cannot control the Lipschitz constant globally. However, the proof that the simplicial volume of a manifold with an amenable fundamental group is 0 generalises to the Lipschitz case, provided the Lipschitz simplicial volume is finite.

Chapter 3

Piecewise straightening procedure

In this chapter we define and investigate the piecewise straightening procedure, which is the main technical result in this work. In Section 3.1 we describe the piecewise straightening itself, while in Section 3.2 we describe piecewise C^1 homology and piecewise C^1 Milnor-Thurston homology.

3.1. Piecewise straightening

The straightening procedure on non-positively curved manifolds is well described and applied successfully to many problems. Roughly speaking, given a complete, simply connected Riemannian manifold M with non-positive curvature and a singular simplex $\sigma : \Delta^k \to M$ with vertices $x_0, ..., x_k$, the straightening of this simplex is the geodesic simplex $[x_0, ..., x_k]$, which is defined inductively to be the geodesic join of x_k with the geodesic simplex $[x_0, ..., x_{k-1}]$. Because geodesics on M joining points are unique, there exists a (unique) geodesic homotopy between σ and $[x_0, ..., x_k]$ which is defined as the geodesic join $[\sigma, [x_0, ..., x_k]]$. We can apply the same procedure to a singular simplex σ on non necessarily simply-connected Riemannian manifold M with non-positive curvature by taking a lift of σ to the universal cover $\widetilde{\sigma} : \Delta^k \to \widetilde{M}$, applying straightening there and pushing down the result. It can be shown that this procedure does not depend on the choice of the lift, hence it commutes with boundaries and defines a chain operator inducing an isomorphism on homology. The same applies to locally finite Lipschitz chains and homology. The straightening procedure has also the advantage that it does not increase the l^1 norm of chains, therefore the isomorphism on homology turns out to be isometric. As a result, the simplicial volume can be computed by considering only straight chains. This fact, together with a careful control of the set of vertices of a given chain, is the key to prove e.g. the proportionality principle and the product inequality for the Lipschitz simplicial volume, assuming all the involved manifolds have non-positive curvature.

However, if we consider simply connected Riemannian manifolds with $\sec(M) < K < \infty$, where $K > 0$, the geodesics do not have to be unique any more. They are unique locally, but unfortunately not uniformly, even if we pass to the universal cover. Therefore the crucial problem in defining the piecewise straightening procedure on M is the choice of a suitable space in which we have such uniform local uniqueness of geodesics. If such a space is provided, one can define piecewise straightening by barycentrically subdividing given singular chain, straighten every small simplex and glue the straightened simplices back together.

In Section 3.1.1, for every point of M we construct an 'exponential neighbourhood' of it which is a space admitting a local isometry to M for which there exists a uniform lower bound (depending on K) of the injectivity radius of points in some (uniform) neighbourhood of the origin. This system of spaces and local isometries on M admits also transition maps (at least locally) which allow one to apply some local constructions independently of the choice of a point for which we consider its exponential neighbourhood. The construction is sketched in [12, 4.3(B)], however, we provide a more detailed approach. In Section 3.1.2, we recall basic notions concerning geodesic simplices and joins and prove that under some curvature and diameter conditions the geodesic join of Lipschitz maps is also Lipschitz. Finally, in Section 3.1.3, we define the piecewise straightening procedure for locally finite Lipschitz chains.

To clarify the notation, we will denote by $B_M(x, r)$ the open ball in a space M centred at x with radius r, and more generally by $B_M(X, r)$ the open r-neighbourhood of a set $X \subset M$. We consider also all Riemannian manifolds as metric spaces with metric induced by the Riemannian structure.

3.1.1. Exponential neighbourhoods

Let M be a connected, complete n-dimensional Riemannian manifold with $\sec(M) < K$, where $K > 0$.

Definition 3.1.1. Let $x \in M$ and let $r \leqslant \frac{\pi}{\sqrt{K}}$. Consider the open ball $B_{T_xM}(0, r)$ in the tangent space T_xM. Then the exponential map $\exp_x : B_{T_xM}(0, r) \to M$ is an immersion by Proposition 1.1.14. We endow $B_{T_xM}(0, r)$ with the Riemannian metric induced from M by $\exp_x$ and obtain a space $V_x(r)$ which we call the $r-$*exponential neighbourhood* of x with a distinguished point $\bar{x} \in V_x(r)$, which corresponds to 0 in $B_{T_xM}(0, r)$ and the canonical local isometry $p_x : V_x(r) \to M$ such that $p_x(\bar{x}) = x$.

If $r = \frac{\pi}{\sqrt{K}}$ we will denote this space for short as V_x.

The spaces V_x are not complete. However, the closures of the open balls $B_{V_x}(\bar{x}, r)$ for any $r < \frac{\pi}{\sqrt{K}}$ are complete as metric spaces and for $y \in V_x(r)$ the map $\exp_y : T_yV_x \to V_x$ is defined for vectors of length less than $\frac{\pi}{\sqrt{K}} - r$. As we will see next, these spaces have all the desired properties described in the introduction of this section. First of all we check that there exists a uniform lower bound on the injectivity radii of points around the origins of V_x.

Proposition 3.1.2. *Let $x \in M$ and let $y \in V_x(\frac{\pi}{4\sqrt{K}})$. Then the injectivity radius of y in V_x is at least $\frac{\pi}{4\sqrt{K}}$.*

Proof. If $y \in B_{V_x}(\bar{x}, \frac{\pi}{4\sqrt{K}})$ then the exponential map $\exp_y : T_yV_x \to V_x$ is defined for vectors of length less than $\frac{3\pi}{4\sqrt{K}}$. Indeed, if $z \in B_{T_yV_x}(0, \frac{3\pi}{4\sqrt{K}})$, then

$$d_{V_x}(\bar{x}, \exp_y(z)) \leqslant d_{V_x}(\bar{x}, y) + d_{V_x}(y, \exp_y(z)) = \frac{\pi}{4\sqrt{K}} + \frac{3\pi}{4\sqrt{K}} < \frac{\pi}{\sqrt{K}},$$

hence $\exp_y(z) \in V_x$. Because of the curvature bound, $\exp_y |_{B_{T_yV_x}(0, \frac{3\pi}{4\sqrt{K}})}$ is an immersion by Proposition 1.1.14, so we only need to prove that it is injective on $B_{T_yV_x}(0, \frac{\pi}{4\sqrt{K}})$.

Denote by V'_y the space $B_{T_yV_x}(0, \frac{\pi}{2\sqrt{K}})$ endowed with the Riemannian metric induced from V_x (so the exponential map $\exp_y : V'_y \to V_x$ becomes a local isometry) with a distinguished point $\bar{y}$ corresponding to 0 in T_yV_x. Let $z_1, z_2 \in B_{V'_y}(\bar{y}, \frac{\pi}{4\sqrt{K}})$ be such that $\exp_y(z_1) = \exp_y(z_2) = z$ and let $\widetilde{x} \in B_{V'_y}(\bar{y}, \frac{\pi}{4\sqrt{K}})$ be some lift of $\bar{x}$, i.e. any point satisfying $\exp_y(\widetilde{x}) = \bar{x}$. Such a point exists in $B_{V'_y}(\bar{y}' \frac{\pi}{4\sqrt{K}})$ because $d_{V_x}(\bar{x}, y) < \frac{\pi}{4\sqrt{K}}$, but need not be unique. Since $\widetilde{x}, z_1, z_2 \in B_{V'_y}(\bar{y}, \frac{\pi}{4\sqrt{K}})$, $d_{V'_y}(\widetilde{x}, z_1) < \frac{\pi}{2\sqrt{K}}$ and $\exp_{z_1}$ is defined on

$B_{T_z V'_y}(0, \frac{\pi}{2\sqrt{K}})$, hence there exists a geodesic γ_1 in V'_y joining z_1 and $\widetilde{x}$. Similarly, there is a geodesic γ_2 joining z_2 and $\widetilde{x}$. Because $\exp_y$ is a local isometry on V'_y, both $\exp_y(\gamma_1)$ and $\exp_y(\gamma_2)$ are geodesics joining $\bar{x}$ and z inside V_x. But by the construction of the exponential map and the space V_x, all geodesics joining $\bar{x}$ and any other point inside V_x are unique. In particular, $\exp_y(\gamma_1) = \exp_y(\gamma_2)$. We use again the fact that $\exp_y$ is a local isometry around $\widetilde{x}$ to see that both geodesics γ_1 and γ_2 have the same tangent line in $\widetilde{x}$ and the same direction, hence (without loss of generality) γ_1 is a subgeodesic of γ_2. Moreover, because $\exp_y$ does not change the length of geodesics, we have in fact $\gamma_1 = \gamma_2$, hence $z_1 = z_2$. □

Secondly, we check the existence of 'transition maps' which will allow us to perform local constructions on the spaces V_x independently of $x \in M$.

Proposition 3.1.3. *Let $x, y \in M$ be such that $d_M(x, y) < \frac{\pi}{4\sqrt{K}}$. Let also y' be any lift of y to $V_x(\frac{\pi}{4\sqrt{K}})$. Then there exists a locally isometric diffeomorphism $I_{y',x} : V_y(\frac{\pi}{4\sqrt{K}}) \to B_{V_x}(y', \frac{\pi}{4\sqrt{K}})$ such that we have a commutative diagram*

$$\begin{tikzcd}
V_y(\frac{\pi}{4\sqrt{K}}) \arrow[rr, "I_{y',x}"] \arrow[dr, "p_y"'] & & B_{V_x}(y', \frac{\pi}{4\sqrt{K}}) \arrow[dl, "p_x"] \\
& M &
\end{tikzcd}$$

Proof. By Proposition 3.1.2, we know that $\exp_{y'}$ provides a diffeomorphism

$$\exp_{y'} : B_{T_{y'}V_x}(0, \frac{\pi}{4\sqrt{K}}) \to B_{V_x}(y', \frac{\pi}{4\sqrt{K}}) \subset V_x$$

which becomes a local isometry after a change of the Riemannian metric on $B_{T_{y'}V_x}(0, \frac{\pi}{4\sqrt{K}})$. Hence it suffices to show that $V_y(\frac{\pi}{4\sqrt{K}})$ is isometric to $B_{T_{y'}V_x}(0, \frac{\pi}{4\sqrt{K}})$ (with the Riemannian metric induced by $\exp_{y'}$). However, both spaces can be identified with the space of geodesics of length less than $\frac{\pi}{4\sqrt{K}}$ starting from y, with the Riemannian metric induced from M by the map mapping every geodesic to its endpoint. Checking the commutativity of the diagram is straightforward. □

Finally, we establish the lifting property for the spaces V_x with respect to singular simplices with sufficiently small Lipschitz constants. Recall that if X is a metric space and $\gamma : [0, 1] \to X$ then we define the *length* of γ to be

$$L(\gamma) := \sup\{\sum_{i=1}^{n} d_X(\gamma(t_{i-1}), \gamma(t_i)) \ : \ 0 = t_0 < t_1 < ... < t_n = 1, \, n \in \mathbb{N}\},$$

and we say that X is *geodesic* if it is path-connected and for any two points their distance equals the length of the shortest path between them, called *a geodesic*. Note that if γ is a smooth path on a Riemannian manifold then its length defined as above equals the length defined as in Section 1.1.1. Moreover, a complete Riemannian manifold is geodesic as a metric space. In particular, the k-dimensional simplex Δ^k is a geodesic space with diameter $\sqrt{2}$.

We will use the following simple fact.

Lemma 3.1.4. *Let X be a geodesic metric space and let $f : X \to Y$ be a Lipschitz map. Then for every $\varepsilon > 0$*

$$\mathrm{Lip}(f) = \sup\{\frac{d_Y(f(x), f(x'))}{d_X(x, x')} \ : \ x, x' \in X \, , \, 0 < d_X(x, x') < \varepsilon\},$$

where $\mathrm{Lip}(f)$ *is the (optimal) Lipschitz constant of* f*, i.e.*

$$\mathrm{Lip}(f) = \sup_{x \neq x' \in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}.$$

Proof. The '$\geqslant$' inequality is obvious, we need to prove the opposite one. Let $\delta > 0$ and let $x, x' \in X$ be two points for which the expression $\frac{d_Y(f(x),f(x'))}{d_X(x,x')}$ is δ-close to $\mathrm{Lip}(f)$, i.e. such that

$$d_Y(f(x), f(x')) > (\mathrm{Lip}(f) - \delta) d_X(x, x')$$

Let also $\gamma : [0, d_X(x, x')] \to X$ be the geodesic joining x and x'. Subdivide γ into non-trivial subgeodesics $\gamma_1, ..., \gamma_n$ of length less than ε and let $x = x_0, x_1, ..., x_n = x'$ be their subsequent endpoints. Then we have

$$\sum_{i=1}^{n} d_Y(f(x_{i-1}), f(x_i)) \geqslant d_Y(f(x), f(x')) > (\mathrm{Lip}(f)-\delta) d_X(x, x') = (\mathrm{Lip}(f)-\delta) \sum_{i=1}^{n} d_X(x_{i-1}, x_i).$$

Hence for some $i \in \{1, ..., n\}$ we have the inequality

$$d_Y(f(x_{i-1}), f(x_i)) > (\mathrm{Lip}(f) - \delta) d_X(x_{i-1}, x_i)$$

and $0 < d_X(x_{i-1}, x_i) < \varepsilon$. Because δ was arbitrary, the inequality holds. □

Proposition 3.1.5. *Let* $\sigma : \Delta^k \to M$ *be a Lipschitz singular simplex with* $\mathrm{Lip}(\sigma) < \frac{C}{\sqrt{2}} < \frac{\pi}{\sqrt{2K}}$*, let* $y \in \Delta^k$ *and let* $\sigma(y) = x \in M$*. Then there exists a unique Lipschitz lift* $\widetilde{\sigma} : \Delta^k \to V_x(C)$ *of* σ *(i.e.* $\sigma = p_x \circ \widetilde{\sigma}$*) such that* $\widetilde{\sigma}(y) = \bar{x}$*. This lift satisfies also* $\mathrm{Lip}(\widetilde{\sigma}) = \mathrm{Lip}(\sigma)$*.*

Proof. Let $z \in \Delta^k$ and let $I_z : [0, 1] \to \Delta^k$ be the (rescaled) interval connecting y and z, that is $I_z(t) = (1 - t)y + tz$. Let also $\gamma_z = \sigma \circ I_z$. We claim that we can construct a unique path $\widetilde{\gamma}_z : [0, 1] \to V_x(C)$ such that $p_x \circ \widetilde{\gamma}_z = \gamma_z$ and $\widetilde{\gamma}_z(0) = \bar{x}$. Let

$$R = \sup \{r \in [0, 1] \ : \text{ there exists a lift } \widetilde{\gamma}_z^r : [0, r] \to V_x(C) \text{ of } \gamma_z|_{[0,r]} \text{ such that } \widetilde{\gamma}_z^r(0) = \bar{x}\}.$$

We claim that $R = 1$. Note that if we have two lifts $\widetilde{\gamma}_z^s : [0, s] \to V_x$ and $\widetilde{\gamma}_z^t : [0, t] \to V_x$ for $0 \leqslant s \leqslant t \leqslant 1$ satisfying the above conditions then they need to agree on $[0, s]$ because the subset of $[0, s]$ where these two lifts agree is non-empty (because $\widetilde{\gamma}_z^s(0) = \widetilde{\gamma}_z^t(0) = \bar{x}$), open (because p_x is a local diffeomorphism) and closed (because of the continuity of both lifts). Hence we can consider the union of such lifts $\widetilde{\gamma}_z^s$ for $s < R$ to obtain a lift $\widetilde{\gamma}_z'^R : [0, R) \to V_x$ of $\gamma_z|_{[0,R)}$ such that $\widetilde{\gamma}_z'^R(0) = \bar{x}$. To extend it continuously to a lift $\widetilde{\gamma}_z^R : [0, R] \to V_x(C)$ we need to check that

$$\sup_{t \in [0,R)} d_{V_x}(\bar{x}, \widetilde{\gamma}_z'^R(t)) < C,$$

because then the limit $\lim_{t \to R} \widetilde{\gamma}_z'^R(t)$ exists in $V_x(C)$. Fix $0 < t < R$ and consider the path $\widetilde{\gamma}_z^t = \widetilde{\gamma}_z'^R|[0, t]$. Note that because p_x is a local isometry, this path has the same length as $\gamma_z|[0, t]$. Using the fact that $\gamma_z = \sigma \circ I_z$ and that σ is Lipschitz we have

$$d_{V_x}(\bar{x}, \widetilde{\gamma}_z^t(t)) = d_{V_x}(\widetilde{\gamma}_z^t(0), \widetilde{\gamma}_z^t(t)) \leqslant L(\widetilde{\gamma}_z^t) = L(\gamma_z|[0, t]) < (\frac{C}{\sqrt{2}} - \varepsilon) L(I_z) \leqslant C - \varepsilon$$

for some sufficiently small ε depending on σ, but neither on z nor on t. Since $\widetilde{\gamma}_z^t(t) = \widetilde{\gamma}_z'^R(t)$, we have $\sup_{t \in [0,R)} d_{V_x}(\bar{x}, \widetilde{\gamma}_z'^R(t)) \leqslant C - \varepsilon < C$ so we can extend our lift to $\widetilde{\gamma}_z^R : [0, R] \to V_x(C)$. Finally, if $R < 1$ we can use again the fact that p_x is a local diffeomorphism (this time in the neighbourhood of $\widetilde{\gamma}_z^R(R)$) and extend $\widetilde{\gamma}_z^R$ to $\widetilde{\gamma}_z^{R'}$ for some $R' > R$. This fact contradicts the definition of R.

Because the choice of $\widetilde{\gamma}_z$ is unique we can define $\widetilde{\sigma}(z) = \widetilde{\gamma}_z(1)$. Moreover, we can once again use the facts that p_x is a local diffeomorphism and that $[0,1]$ is compact to conclude that $\widetilde{\gamma}_z$ depends continuously on z in the compact-open topology, hence $\widetilde{\sigma}$ as a map $\Delta^k \to V_x(C)$ is continuous.

The last claim to verify is the equality $\mathrm{Lip}(\widetilde{\sigma}) = \mathrm{Lip}(\sigma)$. Note that Δ^k is a geodesic metric space, hence the Lipschitz constants of σ and $\widetilde{\sigma}$ can be computed locally as in Lemma 3.1.4. But $p_x \circ \widetilde{\sigma} = \sigma$ and p_x is a local isometry, hence these 'local' Lipschitz constants are the same. □

By combining the above proposition with Proposition 3.1.3 we obtain a very useful corollary.

Corollary 3.1.6. *Let $\sigma : \Delta^k \to M$ be a Lipschitz singular simplex with $\mathrm{Lip}(\sigma) < \frac{C}{\sqrt{2}} < \frac{\pi}{4\sqrt{2K}}$ such that $\sigma(\Delta^k) \subset B_M(x, \frac{\pi}{4\sqrt{K}})$. Then there exists a Lipschitz lift $\widetilde{\sigma} : \Delta^k \to V_x$ of σ (i.e. $p_x \circ \widetilde{\sigma} = \sigma$) with $\mathrm{Lip}(\widetilde{\sigma}) = \mathrm{Lip}(\sigma)$.*

Moreover, if $y \in \Delta^k$ then the lift is unique up to the choice of $\widetilde{\sigma}(y)$ which can be chosen to be any point $\widetilde{y} \in V_x(\frac{\pi}{4\sqrt{K}})$ such that $p_x \circ \widetilde{\sigma}(\widetilde{y}) = y$. We have then $\widetilde{\sigma}(\Delta^k) \subset B_{V_x}(\widetilde{y}, C)$.

3.1.2. Straight simplices and homotopies

As before, we will assume that M is a connected, complete n-dimensional Riemannian manifold with $\sec(M) < K$, where $K > 0$, and $x \in M$. Let $y, z \in V_x$ be two points such that $y, z \in V_x(\frac{\pi}{8\sqrt{K}})$. By Proposition 3.1.2, there exists a unique shortest geodesic joining them (depending continuously on both endpoints) which we denote by $[y, z]$. Following [25], we can define the *geodesic join* of two maps $f, g : X \to V_x$.

Definition 3.1.7. Let $f, g : Y \to V_x$ be two maps such that $(\mathrm{im}(f) \cup \mathrm{im}(g)) \subset V_x(\frac{\pi}{8\sqrt{K}})$. Then there exists a homotopy $[f, g] : Y \times [0,1] \to V_x$ defined by $(y, t) \mapsto [f(y), g(y)](t)$ called the *geodesic join* (or *geodesic homotopy*) of f and g.

We will often use the following lemma.

Lemma 3.1.8. *Let $f, g : Y \to V_x$ be two maps such that $\mathrm{im}(f) \subset V_x(R_1)$ and $\mathrm{im}(g) \subset V_x(R_2)$ for $R_1, R_2 < \frac{\pi}{8\sqrt{K}}$. Then $\mathrm{im}([f, g]) \subset V_x(R_1 + R_2)$.*

Proof. Suppose there is a point $z = [f, g](y, t)$ such that $d_{V_x}(\bar{x}, z) \geqslant R_1 + R_2$. Then

$$d_{V_x}(z, f(y)) \geqslant d_{V_x}(\bar{x}, z) - d_{V_x}(\bar{x}, f(y)) \geqslant R_2$$

and similarly $d_{V_x}(z, g(y)) \geqslant R_1$. Because z is on the unique minimizing geodesic between $f(y)$ and $g(y)$, we have

$$d_{V_x}(f(y), g(y)) = d_{V_x}(f(y), z) + d_{V_x}(z, g(y)) \geqslant R_1 + R_2.$$

On the other hand,

$$d_{V_x}(f(y), g(y)) \leqslant d_{V_x}(f(y), \bar{x}) + d_{V_x}(\bar{x}, g(y)) < R_1 + R_2.$$

The above contradiction shows that $z \in V_x(R_1 + R_2)$. □

We can consequently define geodesic simplices. Recall that as we identified the standard simplex Δ^k with the subset $\{(z_0, ..., z_k) \in \mathbb{R}^{k+1}_{\geqslant 0} \ : \ \sum_{i=0}^k z_i = 1\}$, we can identify Δ^{k-1} with the subset $\{(z_0, ..., z_k) \in \Delta^k \ : \ z_k = 0\}$.

Definition 3.1.9. The *geodesic simplex* $[x_0, ..., x_k] : \Delta^k \to V_x$ with vertices $x_0, ..., x_k \in V_x(\frac{\pi}{8k\sqrt{K}})$ is defined inductively by the formulas

- $[x_0](\Delta^0) = \{x_0\} \subset V_x$;
- $[x_0, ..., x_k]((1-t)s + t(0, ..., 0, 1)) = [[x_0, ..., x_{k-1}](s), x_k](t)$ for $s \in \Delta^{k-1}$

To prove that the definition is correct it is enough to prove the following lemma.

Lemma 3.1.10. *Let $k \in \mathbb{N}$ and $R < \frac{\pi}{8k\sqrt{K}}$. If $x_0, ..., x_k \in V_x(R)$ then $[x_0, ..., x_k]$ exists and*

$$[x_0, ..., x_k](\Delta^k) \subset V_x((k+1)R).$$

Proof. We prove the statement by induction. For $k = 0$ the existence of a geodesic simplex is obvious and does not require any metric assumptions. For $k > 0$ $[x_0, ..., x_{k-1}]$ exists by the induction hypothesis and $[x_0, ..., x_{k-1}] \subset V_x(kR) \subset V_x(\frac{\pi}{8\sqrt{K}})$. Consider the geodesic join of the map $[x_0, ..., x_{k-1}] : \Delta^{k-1} \to V_x$ and the constant map sending Δ^{k-1} to the point x_k. Obviously this join has the same image in V_x as $[x_0, ..., x_k]$. By Lemma 3.1.8, we get

$$[x_0, ..., x_k](\Delta^k) = [[x_0, ..., x_{k-1}], \{x_k\}](\Delta^k) \subset V_x(kR + R) = V_x((k+1)R).$$

□

The most important fact in this section is a positive curvature analogue of Proposition 2.1 in [25].

Proposition 3.1.11. *Let Y be a compact, smooth manifold (possibly with boundary) and let $f, g : Y \to V_x$ be two Lipschitz maps such that $(\mathrm{im}(f) \cup \mathrm{im}(g)) \subset V_x(C_K)$, where $C_K < \frac{\pi}{8\sqrt{K}}$ is a constant depending only on K. Then $[f, g]$ has the Lipschitz constant bounded by a constant depending only on K and the Lipschitz constants for f and g. Moreover, $[f, g]$ is smooth (C^1) if f and g are smooth (C^1).*

To proceed, we need the following two technical lemmas concerning Riemannian geometry. First is the technical result proved in [25], which can be easily applied in our situation.

Lemma 3.1.12 ([25, Proposition 2.6])**.** *Let M be a complete Riemannian manifold with $\sec(M) < K$, $K > 0$. Then every geodesic k-simplex σ in M such that $\mathrm{diam}(\sigma) < \frac{\pi}{2\sqrt{K}}$ is smooth. Further, there is a constant $L > 0$ such that every geodesic k-simplex σ of diameter less than $\frac{\pi}{4\sqrt{K}}$ satisfies $\|T_x\sigma\| < L$ for every $x \in \Delta^n$.*

Lemma 3.1.13. *Consider a geodesic triangle $[x_0, x_1, x_2]$ in V_x such that $x_0, x_1, x_2 \in V_x(\frac{\pi}{48\sqrt{K}})$. Then there exists a constant D_K, depending only on the curvature bound K, such that for any $t \in [0, 1]$*

$$d_{V_x}([x_0, x_2](t), [x_1, x_2](t)) \leqslant D_K d_{V_x}(x_0, x_1)$$

Proof. If $x_0 = x_1$ there is nothing to prove. If not, consider an extension (in any direction) of $[x_0, x_1]$ to a geodesic of length $\frac{\pi}{24\sqrt{K}}$ and denote the endpoints of this geodesic by x_0', x_1'.

Such a geodesic exists because $B_{V_x}(x_0, \frac{\pi}{24\sqrt{K}}) \subset V_x(\frac{\pi}{8\sqrt{K}})$. Now consider the geodesic triangle $[x_0', x_1', x_2]$. Note that

$$d_{V_x}(x_0', \bar{x}) \leqslant d_{V_x}(x_0', x_0) + d_{V_x}(x_0, \bar{x}) < \frac{\pi}{24\sqrt{K}} + \frac{\pi}{48\sqrt{K}} = \frac{\pi}{16\sqrt{K}}.$$

Similarly, $d_{V_x}(x_1', \bar{x}) < \frac{\pi}{16\sqrt{K}}$, hence by Lemma 3.1.10 we have $[x_0', x_1', x_2] \subset V_x(\frac{3\pi}{16\sqrt{K}})$. We can therefore use Lemma 3.1.12 to conclude that the diffeomorphic simplex map $\sigma : \Delta^2 \to V_x$ from the standard 2-simplex onto $[x_0', x_1', x_2]$ is Lipschitz with constant L independent of σ. Hence

$$\begin{aligned} d_{V_x}([x_0, x_2](t), [x_1, x_2](t)) &\leqslant L \cdot d_{\Delta^2}(\sigma^{-1}([x_0, x_2](t)), \sigma^{-1}([x_1, x_2](t)) \\ &\leqslant L \cdot d_{\Delta^2}(\sigma^{-1}(x_0), \sigma^{-1}(x_1)) \\ &= L\sqrt{2}\frac{d_{V_x}(x_0, x_1)}{\pi/24\sqrt{K}}, \end{aligned}$$

so one can take $D_K = \frac{24L\sqrt{2K}}{\pi}$. $\square$

Proof of Proposition 3.1.11. Put $C_K = \frac{\pi}{48\sqrt{K}}$. To prove smoothness in the case f and g are smooth, one can rewrite $[f, g]$ as

$$[f, g](y, t) = \exp_{f(y)}(t \cdot \exp_{f(y)}^{-1}(g(y))),$$

where we use Proposition 1.1.14 and Proposition 3.1.2 to show that $\exp_{f(y)}$ is invertible on $\operatorname{im} g$.

Now, let $(y, t), (y', t') \in Y \times [0, 1]$. We have

$$\begin{aligned} d_{V_x}([f, g](y, t), [f, g](y', t')) &\leqslant d_{V_x}([f(y), g(y)](t), [f(y), g(y)](t')) \\ &+ d_{V_x}([f(y), g(y)](t'), [f(y'), g(y')](t')) \end{aligned}$$

The first term can be easily estimated as follows

$$d_{V_x}([f(y), g(y)](t), [f(y), g(y)](t')) \leqslant |t - t'| \cdot d_{V_x}(f(y), g(y)) \leqslant |t - t'| \cdot \operatorname{diam}(\operatorname{im}(f) \cup \operatorname{im}(g)).$$

Recall that by assumptions, $(\operatorname{im}(f) \cup \operatorname{im}(g)) \subset V_x(\frac{\pi}{48\sqrt{K}})$. Therefore the second term can be estimated using Lemma 3.1.13 as follows.

$$\begin{aligned} d_{V_x}([f(y), g(y)](t'), [f(y'), g(y')](t')) &\leqslant d_{V_x}([f(y), g(y)](t'), [f(y), g(y')](t')) \\ &+ d_{V_x}([f(y), g(y')](t'), [f(y'), g(y')](t')) \\ &\leqslant D_K(d_{V_x}(g(y), g(y')) + d_{V_x}(f(y), f(y'))) \\ &\leqslant D_K(\operatorname{Lip}(f) + \operatorname{Lip}(g)) d_Y(y, y'). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} d_{V_x}([f, g](y, t), [f, g](y', t')) &\leqslant 2|t - t'|C_K + D_K(\operatorname{Lip}(f) + \operatorname{Lip}(g)) d_Y(y, y') \\ &\leqslant (2C_K + D_K(\operatorname{Lip}(f) + \operatorname{Lip}(g))) d_{Y \times [0,1]}((y, t), (y', t')) \end{aligned}$$

$\square$

Remark 3.1.14. All the facts above could be stated (possibly with some minor changes in the constants used) for any Riemannian manifold V with $\sec(V) < K$ with a distinguished point $\bar{x} \in V$ such that the closure of the open ball $B_V(\bar{x}, R)$ is complete for some R and there exists $r < R$ such that every point in $B_V(\bar{x}, r)$ has injectivity radius at least $\rho > 0$. However, the only examples which are important to us at the moment are the spaces V_x for $x \in M$.

3.1.3. The piecewise straightening itself

Let M be a complete, n-dimensional Riemannian manifold with $\sec(M) < K$, where $0 < K < \infty$, and let $\varepsilon_{n,K} = \frac{C_K}{2(n+1)}$, where C_K is the constant from Proposition 3.1.11. Choose a locally finite family $(F_j)_{j\in J}$ of pairwise disjoint Borel subsets of M together with points $z_j \in F_j$ and Borel maps $s_j : F_j \to V_{z_j}(\varepsilon_{n,K})$ for $j \in J$, such that

- $\bigcup_{j\in J} F_j = M$;
- for every $j \in J$, $\operatorname{diam}(F_j) < \varepsilon_{n,K}$;
- for every $j \in J$, s_j is a section of p_{z_j} (i.e. $p_{z_j} \circ s_j = id : F_j \to F_j$) such that $s_j(z_j) = \bar{z}_j$.

A family with properties described above always exists. We can choose the sets F_j for $j \in J$ using the paracompactness of M, and the sections s_j for $j \in J$ exist because for $x \in F_j$ a lift of a (not necessarily unique) shortest geodesic joining z_j and x has length $< \varepsilon_{n,K}$, hence one can choose $s_j(x)$ to be the endpoint of one of such lifts in a Borel way.

Definition 3.1.15. Let F_j, z_j, s_j for $j \in J$ be as above and let $\pi_U : U \to M$ be a continuous map such that $B_M(z_j, \varepsilon_{n,K}) \subset \operatorname{im}(\pi_U)$. We call a Borel section $s'_j : F_j \to U$ of π_U *admissible* if there exists a continuous map $v_U : V_{z_j}(\varepsilon_{n,K}) \to U$ such that $s'_j = v_U \circ s_j$ and $\pi_U \circ v_U = p_{z_j}$, i.e. it fits into the commutative diagram

$$\begin{tikzcd}
V_{z_j}(\varepsilon_{n,K}) \arrow[r, dashed, "v_U"] \arrow[dr, "p_{z_j}" near start] & U \arrow[d, "\pi_U"] \\
F_j \arrow[u, "s_j"] \arrow[ur, dashed, "s'_j" near start] \arrow[r, hook] & M.
\end{tikzcd}$$

A motivating example is given by the following lemma.

Lemma 3.1.16. *Let $x \in M$ and $x' \in V_x(\frac{\pi}{4\sqrt{K}})$. Then there exists a unique $j \in J$ and a unique admissible section*

$$s_j^{x'} : F_j \to B_{V_x}(x', 2\varepsilon_{n,K})$$

with respect to the map $p_x : V_x \to M$ such that $x' \in s_j^{x'}(F_j)$.

Proof. Let $y = p_x(x')$. Then y is contained in a set F_j for some $j \in J$. By Proposition 3.1.3 we can compose the canonical section s_j with $I^{-1}_{s_j(y),z_j} : B_{V_{z_j}}(s_j(y), \frac{\pi}{4\sqrt{K}}) \to V_y(\frac{\pi}{4\sqrt{K}})$ and obtain an admissible section $s'_j : F_j \to V_y(2\varepsilon_{n,K})$ with respect to p_y such that $s'_j(y) = \bar{y}$. After the composition of this section with $I_{x',x} : V_y(\frac{\pi}{4\sqrt{K}}) \to B_{V_x}(x', \frac{\pi}{4\sqrt{K}})$ we obtain an admissible section $s_j^{x'} : F_j \to B_{V_x}(x', 2\varepsilon_{n,K})$ which satisfies the required conditions.

To see the uniqueness of $s_j^{x'}$, let $s'^{x'}_{j'} : F_{j'} \to B_{V_x}(x', 2\varepsilon_{n,K})$ be another admissible section satisfying the above conditions. Note that $F_j \ni p_x \circ s_j^{x'}(x') = p_x \circ s'^{x'}_{j'}(x') \in F_{j'}$, hence $j = j'$. After composing $s_j^{x'}$ and $s'^{x'}_{j'}$ with $I_{s_j(y),z_j} \circ I^{-1}_{x',x} : B_{V_x}(x', \frac{\pi}{4\sqrt{K}}) \to B_{V_{z_j}}(s_j(y), \frac{\pi}{4\sqrt{K}})$ and using the admissibility of $s'^{x'}_{j'}$, we obtain sections $s_j, s'_j : F_j \to V_{z_j}(\frac{\pi}{4\sqrt{K}})$ and a map $v : V_{z_j}(\varepsilon_{n,K}) \to V_{z_j}$ such that $v \circ s_j(y) = s'_j(y)$ and the following diagram commutes

$$\begin{tikzcd}
V_{z_j}(\varepsilon_{n,K}) \arrow[r, "v"] \arrow[dr, "p_{z_j}" near start] & V_{z_j} \arrow[d, "p_{z_j}"] \\
F_j \arrow[u, "s_j"] \arrow[ur, "s'_j" near start] \arrow[r, hook] & M.
\end{tikzcd}$$

It suffices to show that $v = Id|_{V_{z_j}(\varepsilon_{n,K})}$. But since p_{z_j} is a local isometry, so is v, and hence it is the identity on some neighbourhood of $s_j(y)$. It follows that it must be the identity on a neighbourhood of the geodesic path joining $s_j(y)$ and $\bar{z}_j$, and consequently on every geodesic joining $\bar{z}_j$ to any other point of $V_{z_j}(\varepsilon_{n,K})$. Consequently, $v = Id|_{V_{z_j}(\varepsilon_{n,K})}$. $\square$

Now we turn back to the definition of the piecewise straightening.

Definition 3.1.17. Let $\sigma : \Delta^k \to M$ be a Lipschitz singular simplex. Then we say that σ is *ε-geodesic* (with respect to $(F_j)_{j\in J}$) if $\mathrm{Lip}(\sigma) \leqslant \frac{\varepsilon}{\sqrt{2}}$ and there exists $x \in M$ and a lift $\widetilde{\sigma} : \Delta^k \to V_x$ of σ such that $\widetilde{\sigma}$ is geodesic with vertices in some lifts of the points z_j, $j \in J$.

Note that by Proposition 3.1.3 if $\varepsilon < \frac{\pi}{4\sqrt{K}}$ then the above definition does not depend on the choice of $x \in M$ unless $\sigma(\Delta^k) \subset B_M(x, \frac{\pi}{4\sqrt{K}})$.

Definition 3.1.18. Let $\sigma : \Delta^k \to M$ be a singular simplex and let $S^{(m)}(\sigma) = \sum_i \sigma_i$ be its m-times iterated barycentric subdivision, where $m \in \mathbb{N}$. We say that σ is *(m-)piecewise straight* if every σ_i in $S^{(m)}(\sigma)$ is $\varepsilon_{n,K}$-geodesic (with respect to $(F_j)_{j\in J}$).

We say that a (locally finite) chain $c = \sum_{i\in\mathcal{I}} a_i\sigma_i \in C_*(M)$ is *piecewise straight* if there exists $m \in \mathbb{N}$ such that every σ_i, $i \in \mathcal{I}$, is m-piecewise straight.

Let $\sigma : \Delta^k \to M$ for $k \leqslant n$ be a Lipschitz singular simplex. We define the (m-)*straightening* of σ (with respect to $(F_j)_{j\in J}$) as follows. Choose $m \in \mathbb{N}$ such that each simplex σ_i in $S^{(m)}(\sigma) = \sum_i \sigma_i$ has Lipschitz constant less than $\frac{\varepsilon_{n,K}}{\sqrt{2}}$. Such m exists because the diameters of the subdivided simplices in Δ^k tend to 0, hence also the Lipschitz constants of the simplices in $S^{(m)}(\sigma)$ (see Example 1.2.15). Moreover, we can choose m depending only on n, K and $\mathrm{Lip}(\sigma)$. For every simplex σ_i choose a point $y_i \in \Delta^k$ and let $y'_i = \sigma_i(y_i)$. Then by Corollary 3.1.6, there is a unique lift $\widetilde{\sigma}_i : \Delta^k \to V_{y'_i}(\varepsilon_{n,K})$ of σ_i such that $\widetilde{\sigma}_i(y_i) = \bar{y}'_i$. Denote by $x_{i,0}, ..., x_{i,k}$ its vertices, for $l = 0, ..., k$ let $s'_{i,l} : F_{i,l} \to V_{y'_i}$ be admissible sections containing $x_{i,l}$ in their images constructed by Lemma 3.1.16 and let $z'_{i,l} = s'_{i,l}(z_{i,l})$ for $l = 0, ..., k$. In particular, $z'_{i,0}, ..., z'_{i,k} \in V_{y'_i}(2\varepsilon_{n,K})$, hence the geodesic simplex $[z'_{i,0}, ..., z'_{i,k}]$ exists by Lemma 3.1.10, because

$$2\varepsilon_{n,K} = \frac{2C_K}{2(n+1)} < \frac{\pi}{8(n+1)\sqrt{K}} < \frac{\pi}{8k\sqrt{K}}.$$

Let $\mathrm{str}_{y_i}(\sigma_i) = [z'_{i,0}, ..., z'_{i,k}]$. Define

$$\mathrm{str}_m(\sigma) = (S^{(m)})^{-1}(\sum_i p_{y'_i} \circ \mathrm{str}_{y_i}(\sigma_i)).$$

Moreover, by Lemma 3.1.10 we have

$$[z'_{i,0}, ..., z'_{i,k}] \subset V_{y'_i}(2(k+1)\varepsilon_{n,K}) \subset V_{y'_i}(C_K),$$

so it follows from Proposition 3.1.11 that $[\widetilde{\sigma}_i, [z'_{i,0}, ..., z'_{i,k}]]$ exists and defines a Lipschitz homotopy $\bar{H}_{y_i} : \Delta^k \times I \to V_{y'_i}$ between these simplices, with the Lipschitz constant depending only on m, K and $\mathrm{Lip}(\sigma)$. Define

$$H_m(\sigma) = (S^{(m)} \times Id_I)^{-1}(\sum_i p_{y'_i} \circ \bar{H}_{y_i}(\sigma_i)).$$

To show that str_m and H_m are well defined it suffices to verify that the construction is independent of the choice of $y_i \in \Delta^k$. Indeed, assuming this fact we see that for any $\dot{y}_i \in \Delta^{k-1} \subset \partial_q\Delta^k$ for $q = 0, ...k$, and $\dot{y}'_i = \sigma_i(\dot{y}_i)$ we have

$$\partial_q(p_{y'_i} \circ \mathrm{str}_{y_i}(\sigma_i)) = \partial_q(p_{\dot{y}'_i} \circ \mathrm{str}_{\dot{y}_i}(\sigma_i)) = p_{\dot{y}'_i} \circ \partial_q \mathrm{str}_{\dot{y}_i}(\sigma_i) = p_{\dot{y}'_i} \circ \mathrm{str}_{\dot{y}_i}(\partial_q\sigma_i),$$

where the last equality is a consequence of the fact that the straightening of a face of any singular simplex depends only on this particular face, not on the whole simplex. In particular, if two simplices σ_i and $\sigma_{i'}$ have some common face, their straightenings will also have the same one. This shows that $\sum_i p_{y'_i} \circ \mathrm{str}_{y_i}(\sigma_i)$ lies in the image of $S^{(m)}$, hence (after ordering the vertices of $S^{(m)}(\sigma_i)$) we can choose a preimage in the canonical way. The same proof applies also to H_m.

Now we verify our claim. Let $\dot{y}_i \in \Delta^k$, $\dot{y}'_i = \sigma_i(\dot{y}_i) \in M$ and $\widetilde{y}'_i = \widetilde{\sigma}_i(\dot{y}'_i) \in V_{y'_i}(\varepsilon_{n,K})$. Then Proposition 3.1.3 gives an isometry $I_{\widetilde{y}'_i, y'_i}$ between $V_{\dot{y}'_i}(\frac{\pi}{4\sqrt{K}})$ and $B_{V_{y'_i}}(\widetilde{y}'_i, \frac{\pi}{4\sqrt{K}})$. By Lemma 3.1.8 we have

$$\bar{H}_{y_i}(\Delta^k \times I) \subset V_{y'_i}(C_K + \varepsilon_{n,K}) \subset V_{y'_i}(\frac{3\pi}{16\sqrt{K}}).$$

Because $d_{V_{y'_i}}(\bar{y}'_i, \widetilde{y}'_i) < \varepsilon_{n,K} < \frac{\pi}{16\sqrt{K}}$, the images of $\bar{H}_{y_i}$ and $\bar{H}_{\dot{y}_i}$ stay in $B_{V_{y'_i}}(\widetilde{y}'_i, \frac{\pi}{4\sqrt{K}})$ and $V_{\dot{y}'_i}(\frac{\pi}{4\sqrt{K}})$ respectively. Moreover, $I_{\widetilde{y}'_i, y'_i}$ maps the respective admissible sections $\dot{s}'_{i,l} : F_{i,l} \to V_{\dot{y}'_i}$ to the admissible sections $s'_{i,l} : F_{i,l} \to V_{y'_i}$, hence $\bar{H}_{y_i} = I_{\widetilde{y}'_i, y'_i} \circ H_{\dot{y}_i}$. As a result they are the same after pushing them back on M. This argument applies also to str_{y_i}, since $\mathrm{str}_{y_i} = \bar{H}_{y_i}(-, 1)$.

Let $c = \sum_i a_i \sigma_i$ be a locally finite Lipschitz chain with Lipschitz constant L. We see that we can choose $m \in \mathbb{N}$, depending only on n, L and K, such that $\mathrm{str}_m(\sigma_i)$ is defined for every i, so we can define $\mathrm{str}_m(c)$ simply as $\sum_i a_i \,\mathrm{str}_m(\sigma_i)$. The chain $\mathrm{str}_m(c)$ is Lipschitz because of Proposition 3.1.11 and Lemma 3.1.4, and locally finite, since by construction for any singular simplex $\sigma : \Delta^k \to M$ we have $\mathrm{str}_m(\sigma) \subset B_M(\sigma(\Delta^k), C_K)$, hence for every compact subset $K \subset M$

$$\#\{i \,:\, \mathrm{str}_m(\sigma_i) \cap K \neq \emptyset\} \leqslant \#\{i \,:\, \sigma_i \cap B_M(K, C_K) \neq \emptyset\} < \infty.$$

Note that the straightening defined as above does not define a chain operator $C^{lf,\mathrm{Lip}}_{*\leqslant n}(M) \to C^{lf,\mathrm{Lip}}_{*\leqslant n}(M)$, because we cannot choose m uniformly. However, it allows us to prove a slightly weaker statement. Recall that $C^{lf,<L}_*(M)$ is the chain complex of locally finite singular chains on M consisting of simplices with Lipschitz constant less than L.

Proposition 3.1.19. *For every $L < \infty$, there exists $m \in \mathbb{N}$ such that the operator*

$$\mathrm{str}_m : C^{lf,<L}_{*\leqslant n}(M) \to C^{lf,\mathrm{Lip}}_{\leqslant n}(M)$$

is a well defined chain map homotopic to the inclusion $\iota : C^{lf,<L}_{*\leqslant n} \hookrightarrow C^{lf,\mathrm{Lip}}_*(M)$. Moreover, $|\,\mathrm{str}_m|_1 \leqslant 1$.

Proof. Choose m such that str_m is well defined for any singular simplex $\sigma : \Delta^k \to M$ with $k \leqslant n$ and $\mathrm{Lip}(\sigma) < L$. Then the operators

$$H_m : \mathrm{Lip}^{<L}(\Delta^k, M) \to \mathrm{Lip}^{<L'}(\Delta^k \times I, M),$$

for $k \leqslant n$, defined in this section, send L-Lipschitz simplices to L'-Lipschitz maps for some L' depending only on L, m and n by Proposition 3.1.11 and Lemma 3.1.4. Moreover,

$$H_m(\sigma)(\Delta^k \times I) \subset B_M(\operatorname{im}\sigma, \frac{\pi}{4\sqrt{K}})$$

for any $\sigma : \Delta^k \to M$. Therefore these operators satisfy the assumptions of Lemma 1.2.11 for $C^{(1)}_* = C^{lf,<L}_*(M)$ and $C^{(2)}_* = C^{lf,\mathrm{Lip}}_*(M)$. Note that with the notation from Lemma 1.2.11, $\eta_* = \mathrm{str}_m$, which proves the first part of the statement. The proof that $|\,\mathrm{str}_m|_1 \leqslant 1$ is straightforward. □

Corollary 3.1.20. *Every homology class ξ in $H^{lf,\mathrm{Lip}}_{*\leqslant n}(M)$ can be represented by a piecewise straight chain with vertices in $(z_j)_{j\in J}$. Moreover, the l^1 semi-norm on $H^{lf,\mathrm{Lip}}_{*\leqslant n}(M)$ can be computed on the piecewise straight chains.*

Proof. Let $c = \sum_i a_i\sigma_i \in C^{lf,\mathrm{Lip}}_k(M)$, $k \leqslant n$, be any cycle such that $[c] = \xi$. Then $c \in C^{lf,<L}_k(M)$ for some $L < \infty$. Hence by Proposition 3.1.19 there exists m such that the chain $\mathrm{str}_m(c)$ is well defined, homologuous to c and $|\,\mathrm{str}_m(c)_1| \leqslant |c|_1$. It is also obviously straight and has its vertices in $(z_j)_{j\in J}$. □

Remark 3.1.21. The results above are stated only for $* \leqslant n$. However, for $* > n$ the groups $H^{lf,\mathrm{Lip}}_*(M)$ vanish by Theorem 1.2.19 and the fact that ordinary and locally finite homology of any manifold is trivial in dimensions greater that the dimension of this manifold. Moreover, we could simply modify the constants used in the straightening to work for $* \leqslant N$ for N arbitrarily large. Without loss of generality we will assume in further work that all chains and homology classes are of dimension $* \leqslant n$.

Remark 3.1.22. It is obvious that the straightening procedure depends on the choice of the sets $(F_j)_{j\in J}$, the sections s_j for $j \in J$ and $m \in \mathbb{N}$, which depends on a particular chain which we would like to straighten. However, in most cases these details are of secondary interest, therefore we will just talk briefly about *applying the (piecewise) straightening procedure* meaning applying it with respect to any suitable family $(F_j)_{j\in J}$ and any $m \in \mathbb{N}$ for which the procedure is defined.

Remark 3.1.23. If $\sec(M) \leqslant 0$, then the above construction works for K arbitrary close to 0, hence for constants C_K, $\varepsilon_{n,K}$ arbitrary large. In particular, for any $L < \infty$ we can choose K such that $\mathrm{str}_0(\sigma)$ is defined for every L-Lipschitz singular simplex $\sigma \in C(\Delta^k, M)$.

More convenient approach, but working only for non-positively curved manifolds, is presented in [25]. It uses the fact that by Theorem 1.1.15 the spaces V_x are all isometric to $\widetilde{M}$ and it corresponds to the procedure defined in this section for $K = 0$ (hence $C_K = \varepsilon_{n,K} = \infty$) with the modification that the diameters of F_j for $j \in J$ are bounded by 1. Then there is in fact a chain operator

$$\mathrm{str}_* : C^{lf,\mathrm{Lip}}_*(M) \to C^{lf,\mathrm{Lip}}_*(M),$$

which is chain homotopic to the identity by Lemma 1.2.11.

3.2. Piecewise C^1 homology theories

The straightening procedure described in the previous section is sufficient for some applications, though we need some more complex machinery. One of the key properties of the standard straightening procedure for non-positively curved manifolds is that straightened chains are smooth, because they consist of geodesic simplices. It is important e.g. in the proof of the proportionality principle in the non-positively curved case, which depends on measure homology with C^1 Lipschitz supports, i.e. where 'chains' are Borel measures with finite variation on C^1 singular simplices with C^1-topology, with additional assumption that the support of each 'chain' is contained in L-Lipschitz simplices for some $L < \infty$. Differentiability here is strictly technical, but necessary, because it allows recognising a fundamental cycle by integrating the volume form. However, piecewise straight simplices which we use are only piecewise C^1.

In Section 3.2.1 we define piecewise C^1 simplices and chains and introduce piecewise C^1 homology. In Section 3.2.2 we provide some reasonable topology on these simplices in order to define a corresponding measure homology theory.

3.2.1. Piecewise C^1 homology

Let M be a connected, complete n-dimensional Riemannian manifold with $\sec(M) < K$. Before we continue, let us fix some notation concerning convex polyhedra. Let $V \subset \mathbb{R}^n$ be an affine space and let $\langle , \rangle$ be the truncation of the standard scalar product on $\mathbb{R}^n$ to V.

- for $v \in V$ and $b \in \mathbb{R}$ the *half-space* $H_{v,b} \subset V$ is
$$H_{v,b} = \{x \in V \ : \ \langle x, v\rangle \leqslant b\};$$
- a convex polyhedron $P \subset V$ is an intersection of finite number of half-spaces;
- $\dim P = \min\{\dim W \ : \ P \subset W\,,\, W \subset V \text{ is an affine subspace}\}$;
- P is *non-degenerated* if $\dim P = \dim V$;
- for a convex polyhedron P a map $f : P \to M$ is C^1 if it can be extended to a C^1 map $f' : U \to M$, where $U \subset V$ is some open neighbourhood of $P \subset V$.

Definition 3.2.1. Let $V = \{(x_0, ..., x_k) \in \mathbb{R}^{k+1} \ : \ \sum_{i=0}^{k} x_i = 1\} \supset \Delta^k$. We say that a family $\mathcal{P}$ of non-degenerate convex polyhedra $P \subset \Delta^k$ is Δ^k-*admissible* if it satisfies

- $\bigcup_{P \in \mathcal{P}} P = \Delta^k$;
- $\forall_{P_1, P_2 \in \mathcal{P}}\ P_1 \neq P_2 \Rightarrow \dim P_1 \cap P_2 < k$.

We will denote the family of all Δ^k-admissible families by $\mathscr{P}_k$.

A good example of a Δ^k-admissible family of convex polyhedra is the barycentric subdivision $S\Delta^k$ and more generally the m-fold barycentric subdivision $S^{(m)}\Delta^k$.

For two families $\mathcal{P}_1, \mathcal{P}_2 \in \mathscr{P}_k$ we can define their product as
$$\mathcal{P}_1 \cdot \mathcal{P}_2 = \{P_1 \cap P_2 \ : \ P_1 \in \mathcal{P}_1,\ P_2 \in \mathcal{P}_2\,,\, \dim P_1 \cap P_2 = k\},$$
which is also a Δ^k-admissible family. This product is obviously commutative and associative. Moreover, we can partially order $\mathscr{P}_k$ by
$$\mathcal{P}_1 \leqslant \mathcal{P}_2 \Leftrightarrow \forall_{P_2 \in \mathcal{P}_2} \exists_{P_1 \in \mathcal{P}_1}\ P_2 \subset P_1.$$
Note that with this order every finite set $\{\mathcal{P}_1, ..., \mathcal{P}_m\} \subset \mathscr{P}_k$ has supremum $\mathcal{P}_1 \cdot ... \cdot \mathcal{P}_m$.

Definition 3.2.2. Let $\mathcal{P}$ be a Δ^k-admissible family of convex polyhedra and let $\sigma : \Delta^k \to M$ be a singular simplex. We say that it is $\mathcal{P}$-C^1 if for every $P \in \mathcal{P}$, $\sigma|_P : P \to M$ is of class C^1.

A chain $c \in C_k^{lf}(M)$ is called $\mathcal{P}$-C^1 if it consists of $\mathcal{P}$-C^1 simplices and is *piecewise* C^1 if it is $\mathcal{P}$-C^1 for some $\mathcal{P} \in \mathscr{P}_k$.

Note that if $c_1, c_2 \in C_k^{lf}(M)$ are singular chains such that c_1 is $\mathcal{P}_1$-C^1 and c_2 is $\mathcal{P}_2$-C^1 then $c_1 + c_2$ is $\mathcal{P}_1 \cdot \mathcal{P}_2$-$C^1$, hence finite sums of piecewise C^1 chains are piecewise C^1. Moreover, if $c \in C_k^{lf}(M)$ is $\mathcal{P}$-C^1 then ∂c is $\prod_{q=0}^{k} \partial_q \mathcal{P}$-$C^1$, where
$$\partial_q \mathcal{P} = \{P \cap \partial_q \Delta^k \ : \ P \in \mathcal{P}\,,\, \dim P \cap \partial_q \Delta^k = k - 1\}$$
for $q = 0, ..., k$. In particular, the piecewise C^1 chains form a subcomplex of $C_*^{lf}(M)$. The same is true for $C_*^{lf,\mathrm{Lip}}(M)$ if we consider the piecewise C^1 Lipschitz chains. Therefore we can define *piecewise C^1 locally finite homology* $H_*^{PC^1,lf}(M)$ and *piecewise C^1 locally finite Lipschitz homology* $H_*^{PC^1,lf,Lip}(M)$.

Obviously every piecewise straight chain is piecewise smooth (with respect to some iterated barycentric subdivision) by Lemma 3.1.12. To show that the homology theories defined above are isometric to the corresponding non-C^1 ones, we need the following lemma.

Lemma 3.2.3. *Let $c \in C_k^{PC^1,lf,\mathrm{Lip}}(M)$ be a piecewise C^1 locally finite Lipschitz cycle and let $m \in \mathbb{N}$ be such that $\mathrm{str}_m(c)$ is defined. Then c and $\mathrm{str}_m(c)$ are homologous in $C_k^{PC^1,lf,\mathrm{Lip}}(M)$.*

Proof. The lemma will follow from Lemma 1.2.11 applied to

$$C_*^{(1)} := \{c \in C_*^{PC^1,lf,\mathrm{Lip}}(M) \ : \ \mathrm{str}_m(c) \text{ is defined}\},$$

$C_*^{(2)} = C_*^{PC^1,lf,\mathrm{Lip}}(M)$, and the corresponding operators H_m constructed in Section 3.1.3. The only condition we need to check is that if $c = \sum_i a_i\sigma_i$ is piecewise C^1, then $h(c) = \sum_i a_i P(H_m(\sigma_i))$ is.

Assume that a singular simplex σ is $\mathcal{P}$-C^1. Then the map $H_m(\sigma)|_{P'\times I} : P' \times I \to M$ is C^1 for $P' \in \mathcal{P} \cdot (S^m\Delta^k)$ by Proposition 3.1.11. Moreover, if $P_k = \operatorname{supp} P(\Delta^k \times I)$ then for $\Delta' \in P_k$ the family

$$\mathcal{P}_{m,\Delta'} = \{\Delta' \cap (P \times I) \ : \ P \in \mathcal{P} \cdot (S^m\Delta^k)\,, \dim(\Delta' \cap (P \times I)) = k+1\}$$

is (up to some affine isomorphism) Δ^{k+1}-admissible and $H_m(\sigma)$ is C^1 on every $P \in \mathcal{P}_{m,\Delta'}$, hence $h(\sigma)$ is $\prod_{\Delta'\in P_k}\mathcal{P}_{m,\Delta'}$-$C^1$. In particular, $h(c)$ is piecewise C^1 if c is. □

Proposition 3.2.4. *Let M be a complete Riemannian manifold with $\sec(M) < K < \infty$. Then the map $I_* : H_*^{PC^1,lf,\mathrm{Lip}}(M) \to H_*^{lf,\mathrm{Lip}}(M)$ induced by the inclusion of chains is an isometric isomorphism.*

Proof. The map I_* is onto by Corollary 3.1.20. To see that it is injective consider $c_1, c_2 \in C_*^{PC^1,lf,\mathrm{Lip}}(M)$ which represent the same class in $H_*^{lf,\mathrm{Lip}}(M)$. Then there exists a chain $D \in C_{*+1}^{lf,\mathrm{Lip}}(M)$ such that $\partial D = c_2 - c_1$. We can apply now the piecewise straightening procedure to D to obtain the chain $\mathrm{str}_m(D) \in C_{*+1}^{PC^1,lf,\mathrm{Lip}}(M)$ for some $m \in \mathbb{N}$ such that $\partial\,\mathrm{str}_m(D) = \mathrm{str}_m(c_2) - \mathrm{str}_m(c_1)$. Now apply Lemma 3.2.3 to see that c_1 and c_2 are homologous (in $C_*^{PC^1,lf,\mathrm{Lip}}(M)$) to $\mathrm{str}_m(c_1)$, $\mathrm{str}_m(c_2)$ respectively. It is an isometry on homology by Corollary 3.1.20. □

3.2.2. Piecewise C^1 Measure Homology

Now we turn our attention to chains with finite ℓ^1 norm and the corresponding measure homology theory.

Definition 3.2.5. Let $C_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ be the chain subcomplex of $C_*^{PC^1,lf,\mathrm{Lip}}(M)$ consisting of chains which have finite ℓ^1 norm. We call the homology of this complex *piecewise C^1-ℓ^1 Lipschitz homology* and denote it by $H_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$.

Remark 3.2.6. Note that Lemma 3.2.3 can be also applied to $C_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$, so an analogue of Proposition 3.2.4 for $H_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ is true.

Definition 3.2.7. Let $\mathcal{P} \in \mathscr{P}_k$ be a Δ^k-admissible family and let $\mathcal{P}C^1(\Delta^k, M)$ be the set of singular simplices $\sigma : \Delta^k \to M$ such that $\sigma|_P$ is C^1 for every $P \in \mathcal{P}$. We call it the set of *$\mathcal{P}$-C^1 singular simplices.* We equip it with the topology induced from the embedding onto a closed subspace

$$\mathcal{P}C^1(\Delta^k, M) \to \prod_{P\in\mathcal{P}} C^1(P, M),$$

where $C^1(P, M)$ is the set of C^1 maps $P \to M$ with the topology induced from $C(TP, TM)$ with the compact-open topology. For every $\mathcal{P}_1, \mathcal{P}_2 \in \mathscr{P}_k$ such that $\mathcal{P}_1 \leqslant \mathcal{P}_2$, we have an embedding $\mathcal{P}_1 C^1(\Delta^k, M) \to \mathcal{P}_2 C^1(\Delta^k, M)$ onto a closed subset, because $\mathcal{P}_1 C^1(\Delta^k, M) \subset \prod_{P \in \mathcal{P}_2} C^1(P, M)$ is closed. We denote the direct limit of these spaces with weak topology as $\mathscr{P}C^1(\Delta^k, M)$.

The properties of the above topology on $\mathcal{P}C^1(\Delta^k, M)$ for $\mathcal{P} \in \mathscr{P}_k$ which are crucial to us are the following.

- $\mathcal{P}C^1(\Delta^k, M)$ is a locally compact Hausdorff space.
- For every differential form $\omega \in \Omega^k(M)$ the map

$$I_\omega : \mathcal{P}C^1(\Delta^k, M) \to \mathbb{R} \ , \ f \mapsto \int_{\Delta^k} f^* \omega$$

  is continuous.
- For every $\sigma \in \mathcal{P}C^1(\Delta^k, M)$, the map

$$\mathrm{Isom}^+(M) \to \mathcal{P}C^1(\Delta^k, M) \ , \ g \mapsto g\sigma$$

  is continuous.

The second and the third of the above properties hold because both facts are obviously true for $\prod_{P \in \mathcal{P}} C^1(P, M)$ instead of $\mathcal{P}C^1(\Delta^k, M)$, and the latter can be regarded as a closed subset of the former.

Recall that the *absolute variation* (or the *total variation*) of a measure μ on a measurable space X is

$$\|\mu\| = \sup_\pi \sum_{A \in \pi} |\mu(A)|,$$

where the supremum is taken over all finite families π of disjoint measurable subsets of X. If $\|\mu\| < \infty$, we say that it has *finite variation.* Recall also that the absolute variation provides a norm on the space of measures on X with finite variation.

Definition 3.2.8. Let $\mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ be the chain complex of measures (see Definition 1.2.6) on $\mathscr{P}C^1(\Delta^*, M)$, such that

1. for every measure $\mu \in \mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ there exists $\mathcal{P} \in \mathscr{P}_k$ such that μ is a push-forward of a Borel measure on $\mathcal{P}C^1(\Delta^*, M)$ with finite variation;
2. every measure has *Lipschitz determination*, i.e. there exists $L < \infty$ such that the measure is supported on simplices with Lipschitz constant L.

The obtained homology theory is called *piecewise C^1 measure homology with Lipschitz determination $\mathcal{H}_*^{PC^1,\mathrm{Lip}}(M)$.*

Remark 3.2.9. The space $\mathscr{P}C^1(\Delta^*, M)$ is not locally compact in general, therefore it is a problem with the definition of Borel measures. However, we will say for simplicity that measures in $\mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ are Borel meaning that every such measure is a push-forward of a Borel measure on $\mathcal{P}C^1(\Delta^*, M)$ for some $\mathcal{P} \in \mathscr{P}_k$. Similarly, when integrating over $\mathscr{P}C^1(\Delta^*, M)$, we will understand such an operation as the integration over $\mathcal{P}C^1(\Delta^*, M)$ for some 'sufficiently large' $\mathcal{P} \in \mathscr{P}_k$.

The above homology theory is a variant of Milnor-Thurston homology. We can introduce a semi-norm $\|\cdot\|_1$ on it by taking the infimum of the absolute variations over all measures representing given homology class. An important consequence of the above construction is the following

Proposition 3.2.10. *Let M be a complete Riemannian manifold with* $\sec(M) < K < \infty$. *Then the homology groups $H_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ and $\mathcal{H}_*^{PC^1,\mathrm{Lip}}(M)$ are isometrically isomorphic with respect to the ℓ^1 semi-norm on $H_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ and absolute variation semi-norm on $\mathcal{H}_*^{PC^1,\mathrm{Lip}}(M)$.*

Proof. By interpreting the singular chains with finite ℓ^1 norms as discrete measures with finite variations, we have an obvious inclusion of chains $\iota : C_*^{PC^1,\ell^1,\mathrm{Lip}}(M) \to \mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ which commutes with taking boundaries, hence it is a morphism of chain complexes and induces a homomorphism ι_* on homology.

To show that ι_* is surjective, we will use Lemma 1.2.11 and Remark 1.2.12. Let μ be a measure cycle supported on L-Lipschitz simplices and let $m \in \mathbb{N}$ be such that $\mathrm{str}_m(\sigma)$ is defined for all L-Lipschitz simplices. We apply Lemma 1.2.11 to

$$C_*^{(1)} := \{\mu \in \mathcal{C}_*^{PC^1,\mathrm{Lip}}(M) \,:\, \mathrm{str}_m(\mu) \text{ is defined}\},$$

$C_*^{(2)} = \mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$, and the corresponding operators H_m constructed in Section 3.1.3. Note that the maps $H_m(\sigma)$ depend in a Borel way on σ by the construction of the sets $(F_j)_{j\in J}$ and for any measure $\mu' \in \mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ the measure $P(H_m(\mu'))$ has Lipschitz support by the construction of H_m. Moreover, it is piecewise smooth by the same argument as in the proof of Lemma 3.2.3. Therefore the measure $P(H_m(\mu'))$ is in $\mathcal{C}_{*+1}^{PC^1,\mathrm{Lip}}(M)$ and by Lemma 1.2.11 $\mathrm{str}_m(\mu)$ is homologuous to μ.

Note that by Proposition 3.1.11 and the Lipschitz determination of μ, $S^{(m)}(\mathrm{str}_m(\mu))$ is supported on straight simplices with uniformly bounded Lipschitz constant with vertices in the locally finite set $(z_j)_{j\in J}$. Therefore $S^{(m)}(\mathrm{str}_m(\mu))$ is a locally finite Lipschitz singular chain, hence $\mathrm{str}_m(\mu)$ is. Moreover, $\mathrm{str}_m(\mu)$ is piecewise smooth by Proposition 3.1.11, and $\|\mathrm{str}_m(\mu)\| \leqslant \|\mu\|$. Therefore $\mathrm{str}_m(\mu) \in C_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$, and the surjectivity of ι_* follows.

The injectivity of ι_* can be shown using the similar argument applied to the boundary in $\mathcal{C}_*^{PC^1,\mathrm{Lip}}(M)$ between two cycles in $C_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ and Lemma 3.2.3. Namely, let $\xi \in \mathcal{C}_{k+1}^{PC^1,\mathrm{Lip}}(M)$ be a measure chain homotopy between two singular cycles $c_1, c_2 \in C_k^{PC^1,\ell^1,\mathrm{Lip}}(M)$. Then there exists $m \in \mathbb{N}$ such that $\mathrm{str}_m(\xi) \in C_{k+1}^{PC^1,\ell^1,\mathrm{Lip}}(M)$ exists and provides a chain homotopy between $\mathrm{str}_m(c_1)$ and $\mathrm{str}_m(c_2)$, which are homologuous to c_1 and c_2 respectively by Lemma 3.2.3.

Finally, the fact that ι_* is an isometry is a consequence of the facts that ι is an isometric inclusion and that the straightening procedure does not increase the norm. More precisely, if $c \in C_*^{PC^1,\ell^1,\mathrm{Lip}}(M)$ is a cycle such that $\mathrm{str}_m(\iota(c))$ is defined, then

$$\|[c]\|_1 \leqslant \|[\iota(c)]\| \leqslant \|[\mathrm{str}_m(\iota(c))]\|_1 = \|[c]\|_1.$$

In particular, $\|[c]\|_1 = \|[\iota(c)]\|$. □

Remark 3.2.11. The existence of an isometric isomorphism as above for the 'finite' piecewise C^1 theory $H_*^{PC^1}(M)$ and piecewise C^1 measure homology with compact supports $\mathcal{H}_*^{PC^1}(M)$ can be proved without any curvature assumptions as in [23]. However, the proof given in [23] depends heavily on bounded cohomology and cannot be easily generalised to the locally finite Lipschitz case.

Chapter 4

Applications of the straightening procedure

In this chapter we prove the theorems announced in Section 2.2, using the piecewise straightening. In Section 4.1 we prove Theorem 2.2.7, concerning lower bound of the Lipschitz simplicial volume for negatively curved manifolds. In Section 4.2 we prove the product inequality for the Lipschitz simplicial volume (Theorem 2.2.6), while in Section 4.3 we prove the proportionality principle for the Lipschitz simplicial volume (Theorem 2.2.10).

4.1. Lipschitz simplicial volume of negatively curved manifolds

We take a closer look on the lower estimate of the simplicial volume of negatively curved manifold by the Riemannian volume given by Theorem 2.1.6, and the corresponding inequality for the Lipschitz simplicial volume (Theorem 2.2.7). Namely,

$$\|M\|_{\mathrm{Lip}} \geqslant C_n \operatorname{vol}(M)$$

for closed manifolds M with $\sec(M) \leqslant -1$. The crucial fact used in the proof is the following.

Proposition 4.1.1 ([17, Proposition 1]). *Let* $\sec(M) \leqslant -1$ *be a simply connected complete Riemannian manifold such that* $n = \dim M \geqslant 2$. *Let also* $\sigma : \Delta^n \to M$ *be a geodesic simplex on* M. *Then there exists a constant* C'_n *such that*

$$\operatorname{vol}(\sigma) = \int_{\Delta^n} \sigma^* \operatorname{dvol}_M \leqslant C'_n.$$

Now the proof of Theorem 2.1.6 goes as follows. Given an arbitrary fundamental cycle c, its straightening has not greater ℓ^1 norm. Moreover, if we evaluate it on the volume form, we obtain the volume of M. On the other hand, the evaluation on the volume form yields a sum of volumes of corresponding straight simplices with coefficients, which is bounded from above by $C'_n \cdot |c|_1$ by Lemma 4.1.1.

The proof of the Lipschitz generalisation of 2.1.6 is practically the same. The only problem in the proof that might occur is that the straightening of a locally finite chain need not be locally finite in general. However, if we add the Lipschitz condition, the straightening preserves local finiteness by Remark 3.1.23.

Proof of Theorem 2.2.7. Let $c = \sum_i a_i \sigma_i$ be a locally finite Lipschitz fundamental cycle for M. Then $\operatorname{str}_0(c)$ can be defined by Remark 3.1.23, and is a fundamental cycle such that

$|\operatorname{str}_0(c)|_1 \leqslant |c|_1$. By Theorem 1.2.24 and Lemma 4.1.1, we have

$$\operatorname{vol}(M) = \langle \operatorname{dvol}_M, \operatorname{str}_0(c)\rangle = \sum_i a_i \int_{\Delta^n} \operatorname{str}_0(\sigma)^* \operatorname{dvol}_M \leqslant \sum_i a_i C'_n \leqslant C'_n |\operatorname{str}_0(c)|_1.$$

Because c was arbitrary, we conclude that $\operatorname{vol}(M) \leqslant C'_n \cdot \|M\|$. □

4.2. Product inequality

Recall (Theorem 2.1.5) that if M and N are closed orientable manifolds, then

$$\|M\| \cdot \|N\| \leqslant \|M \times N\| \leqslant \binom{\dim M + \dim N}{\dim M} \|M\| \cdot \|N\|.$$

The second inequality is obtained by simply taking a simplicial approximation of the cross product and can be easily generalised to the locally finite and Lipschitz cases. Namely, if c_1 and c_2 are (locally finite, Lipschitz) fundamental cycles for M and N respectively, then $c_1 \times c_2$ is a (locally finite, Lipschitz) fundamental cycle for $M \times N$ by Proposition 1.2.25 and by the construction of Section 1.2.5,

$$|c_1 \times c_2|_1 \leqslant |c_1|_1 \cdot |c_2|_1 \cdot \binom{\dim M + \dim N}{\dim M}.$$

On the other hand, the first inequality in Theorem 2.1.5 can be established by passing to bounded cohomology and using the duality between the ℓ^1 semi-norm on homology and the ℓ^∞ semi-norm on cohomology. Namely, we have

$$\|M \times N\| = \frac{1}{\|[M]^* \times [N]^*\|_\infty} \geqslant \frac{1}{\|[M]^*\|_\infty} \cdot \frac{1}{\|[N]^*\|_\infty} = \|M\| \cdot \|N\|,$$

where we used Theorem 2.3.1, the fact that the cross-product of two fundamental cocycles is a fundamental cocycle of the product and that by the definition of the cross-product for any two cochains $\phi \in C^*(M)$, $\psi \in C^*(N)$,

$$\|\phi \times \psi\|_\infty \leqslant \|\phi\|_\infty \cdot \|\psi\|_\infty.$$

However, this approach does not generalize directly to the case of non-compact manifolds and the Lipschitz simplicial volume (and in general is false in the non-compact, non-Lipschitz case). Two main problems which arise are a more subtle relation between the ℓ^1 semi-norm on locally finite homology and the ℓ^∞ semi-norm on cohomology with compact supports and the existence of a good product in cohomology with compact supports. However, using Proposition 1.2.26, Proposition 2.3.2 and the piecewise straightening procedure, we are able to generalize it and obtain Theorem 2.2.6.

The proof is a modification of the proof from [25] adapted to the case of bounded positive curvature. As in Section 2.3, by $S_k^{lf,\mathrm{Lip}}(M)$ we denote the family of subsets of $C(\Delta^k, M)$ such that $A \in S_k^{lf,\mathrm{Lip}}$ if and only if it is locally finite and consists of L-Lipschitz simplices for some L, depending on A. Similarly, by $S_k^{lf}(M)$ we denote the family of such subsets without the Lipschitzness condiction. We will also use the notation from Section 1.2.5, namely that $\sigma\rfloor_k$ is the k-dimensional face of σ spanned by the last k vertices, ${}_l\lfloor\sigma$ is the l-dimensional face of σ spanned by the first l vertices and $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the canonical projections.

Definition 4.2.1. Let M and N be two topological spaces, and let $k, l \in \mathbb{N}$. A locally finite set $A \in S_{k+l}^{lf}(M \times N)$ is called (k,l)-*sparse* if

$$A_M := \{\pi_M \circ \sigma\rfloor_k;\ \sigma \in A\} \in S_k^{lf}(M) \quad \text{and} \quad A_N := \{\pi_N \circ {}_l\lfloor\sigma;\ \sigma \in A\} \in S_l^{lf}(N)$$

A locally finite chain $c \in C_{k+l}^{lf}(M \times N)$ is called (k,l)-*sparse* if its support is (k,l)-sparse.

Note that if A is sparse and consists of L-Lipschitz simplices for some $L < \infty$, then the corresponding projections A_M and A_N also consist of L-Lipschitz simplices.

If the Lipschitz simplicial volume of can be computed via sparse cycles, we can easily adapt the proof of Theorem 2.1.5 to the locally finite, Lipschitz case. The following proposition is not stated as such in [25], however, it is actually proved there. We give the proof for completeness.

Proposition 4.2.2. *Let M and N be two complete, oriented manifolds of dimensions m and n respectively such that the Lipschitz simplicial volume of $M \times N$ can be computed via (m,n)-sparse fundamental cycles, i.e.*

$$\|M \times N\|_{\mathrm{Lip}} = \inf\{\|M \times N\|^A \ :\ A \in S_{m+n}^{lf,\mathrm{Lip}}(M \times N),\ A \textit{ is } (m,n)\textit{-sparse}\}.$$

Then

$$\|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}} \leqslant \|M \times N\|_{\mathrm{Lip}}.$$

Proof. Note that because by Proposition 1.2.26 the cross product for Lipschitz compactly supported cohomology is well defined, hence for any two cochains $\phi \in C_{cs,\mathrm{Lip}}^k(M)$ and $\psi \in C_{cs,\mathrm{Lip}}^l(N)$, one has

$$\|\phi \times \psi\|_\infty \leqslant \|\phi\|_\infty \cdot \|\psi\|_\infty.$$

Moreover, for any (k,l)-sparse family $A \in S_{k+l}^{lf,\mathrm{Lip}}(M \times N)$, one has

$$\|\phi \times \psi\|_\infty^A \leqslant \|\phi\|_\infty^{A_M} \cdot \|\psi\|_\infty^{A_N}.$$

In particular, by Proposition 1.2.25 the cross product of fundamental cocycles with Lipschitz compact supports is a (Lipschitz complactly supported) fundamental cocycle of the product, hence for any (m,n)-sparse family $A \in S_{m+n}^{lf,\mathrm{Lip}}(M \times N)$, we have

$$\|[M \times N]_{\mathrm{Lip}}^*\|_\infty^A \leqslant \|[M]_{\mathrm{Lip}}^*\|_\infty^{A_M} \cdot \|[N]_{\mathrm{Lip}}^*\|_\infty^{A_N}.$$

Finally, let $A \in S_{m+n}^{lf,\mathrm{Lip}}$ be an (m,n)-sparse family such that $\|M \times N\|^A \leqslant \|M \times N\|_{\mathrm{Lip}} + \varepsilon$. Then using the duality principle for the Lipschitz simplicial volume (Theorem 2.3.2), we obtain

$$\begin{aligned} \|M \times N\|_{\mathrm{Lip}} + \varepsilon &\geqslant \|M \times N\|^A = \frac{1}{\|[M \times N]_{\mathrm{Lip}}^*\|_\infty^A} \geqslant \frac{1}{\|[M]_{\mathrm{Lip}}^*\|_\infty^{A_M}} \cdot \frac{1}{\|[N]_{\mathrm{Lip}}^*\|_\infty^{A_N}} \\ &= \|M\|^{A_M} \cdot \|N\|^{A_N} \geqslant \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}. \end{aligned}$$

Because ε was arbitrary, we conclude that $\|M \times N\|_{\mathrm{Lip}} \geqslant \|M\|_{\mathrm{Lip}} \cdot \|N\|_{\mathrm{Lip}}$. $\square$

Finally, Theorem 2.2.6 is a corollary from the following proposition, which is a generalization of [25, Proposition 3.20], where it was proved assuming non-positive curvature.

Proposition 4.2.3. *Let M and N be two oriented, connected, complete Riemannian manifolds of dimensions m and n respectively such that $\sec(M), \sec(N) < K$, where by $0 < K < \infty$. Let also $k, l \in \mathbb{N}$. Then for any cycle $c \in C_{k+l}^{lf,\mathrm{Lip}}(M \times N)$ there is a (k,l)-sparse cycle $c' \in C_{k+l}^{lf,\mathrm{Lip}}(M \times N)$ satisfying*

$$|c'|_1 \leqslant |c|_1 \quad \textit{and} \quad c \sim c' \textit{ in } C_{k+l}^{lf,\mathrm{Lip}}(M \times N).$$

In particular, the Lipschitz simplicial volume of $M \times N$ can be computed via sparse fundamental cycles, i.e.

$$\|M \times N\|_{\mathrm{Lip}} = \inf\{\|M \times N\|^A;\ A \in S_{m+n}^{lf,\mathrm{Lip}}(M \times N),\ A \textit{ is } (m,n)\textit{-sparse}\}.$$

Proof. The second statement is a direct consequence of the first one. To prove the first one it is enough to just apply the piecewise straightening procedure, but with the sets $(F_j)_{j \in J}$ chosen more carefully. Choose a family of Borel subsets $(F_j^M)_{j \in J^M}$ of M together with the points $(z_j^M)_{j \in J^M}$ and sections $(s_j^M)_{j \in J}$ with all the properties indicated in the description of the straightening procedure, but with the additional assumption that $\mathrm{diam}(F_j^M) < \frac{\varepsilon_{m+n,K}}{2}$ and $s_j^M : F_j \to B_{V_{z_j^M}}(\widetilde{z}_j^M, \frac{\varepsilon_{m+n,K}}{2})$ for every $j \in J^M$. Similarly choose a family $(F_j^N)_{j \in J^N}$ of Borel subsets of N together with points $(z_j^N)_{j \in J^N}$ and sections $(s_j^N)_{j \in J^N}$ and as the base of the straightening procedure for $M \times N$ take the family $(F_{j_1}^M \times F_{j_2}^N)_{(j_1,j_2) \in J^M \times J^N}$ together with the points $(z_{j_1}^M, z_{j_2}^N)_{(j_1,j_2) \in J^M \times J^N}$ and the sections $(s_{j_1}^M \times s_{j_2}^N)_{(j_1,j_2) \in J^M \times J^N}$. This family is locally finite, satisfying $\mathrm{diam}(F_{j_1}^M \times F_{j_2}^N) < \varepsilon_{m+n,K}$ and $s_{j_1} \times s_{j_2} : (F_{j_1} \times F_{j_2}) \to V_{(z_{j_1}^M, z_{j_2}^N)}(\varepsilon_{m+n,K})$ for every $(j_1, j_2) \in J^M \times J^N$. Hence if $c \in C_k^{lf,\mathrm{Lip}}(M \times N)$ is any locally finite Lipschitz chain it can be straightened with respect to that family. Note also that for any $L < \infty$ and $p \in \mathbb{N}$ the family

$$A_{L,p} := \{\sigma \in C(\Delta^{k+l}, M \times N)\ :\ \mathrm{Lip}(\sigma) \leqslant L;\ \sigma \text{ is a } p\text{-piecewise straight simplex}\}$$

belongs to $S_{k+l}^{lf,\mathrm{Lip}}(M \times N)$ and is (k,l)-sparse by the construction of $(F_{j_1}^M \times F_{j_2}^N)_{(j_1,j_2) \in J^M \times J^N}$ and the Lipschitz condition. To finish the proof note that $c \sim \mathrm{str}_p(c)$ for some $p \in \mathbb{N}$ by Corollary 3.1.20, $|c|_1 \geqslant |\,\mathrm{str}_p(c)|_1$ and $\mathrm{str}_p(c)$ has its support in $A_{L,p}$ for some L, thus it is (k,l)-sparse. □

4.3. Proportionality Principle

Recall (Theorem 2.1.9) that the proportionality principle states that if M and N are closed Riemannian manifolds with isometric universal covers, then

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

We will prove here the corresponding statement for the Lipschitz simplicial volume and complete Riemannian manifolds.

The idea of the original proof, due to Thurston [33], is as follows. Using the common universal cover one can construct a 'smearing map' from the smooth singular chain complex on M into the Milnor-Thurston chain complex. This map does not increase the norm and has the property that it maps real fundamental cycles of M to (Milnor-Thurston) fundamental cycles of N multiplied by $\frac{\mathrm{vol}(M)}{\mathrm{vol}(N)}$. The last step of the proof is an 'isometric' approximation of this generalized cycle by a singular cycle. However, Thurston originally did not finish his proof. It was finished by Löh in [23, 31] by showing that the singular and Milnor-Thurston homology theories are isometrically isomorphic. In the case of locally finite Lipschitz homology, however, different techniques are needed. In the non-positively curved case the proof in [25] was obtained by using the straightening procedure. We generalise this proof for manifolds of sectional curvature bounded from above, using the piecewise straightening.

In the original proof smooth chains and measures were used. They were introduced for a strictly technical reason, namely to recognise the image of the smearing map. In our approach we cannot use C^1 chains and measures, however, piecewise C^1 chains have all required properties. Recall that we are able to evaluate dvol_M not only on Lipschitz chains, but also on Borel measures on $C^1(\Delta^n, M)$ and on piecewise C^1 measures on $\mathscr{P}C^1(\Delta^n, M)$ via the formula

$$\langle \mathrm{dvol}_M, \mu\rangle = \int_M \mu \ \mathrm{dvol}_M := \int_{\mathscr{P}C^1(\Delta^n,M)} \int_{\Delta^n} \sigma^* \ \mathrm{dvol}_M \ d\mu(\sigma),$$

for $\mu \in \mathcal{C}_n^{PC^1,\mathrm{Lip}}(M)$. We will need the following lemma.

Lemma 4.3.1 ([25, Lemma 4.2])**.** *Let M be a Riemannian manifold of finite volume. Then $\pi_1(M)$ is a lattice in $G = \mathrm{Isom}(\widetilde{M})$, i.e. the quotient $\pi_1(M)\backslash G$ admits a finite right-$\pi_1(M)$-invariant (Haar) measure $\mu_{\pi_1(M)\backslash G}$.*

Let U be the common universal cover of M and N with covering maps p_M and p_N respectively, let $G = \mathrm{Isom}^+(U)$ and let $\Lambda = \pi_1(N)$. Denote by $\mu_{\Lambda\backslash G}$ the normalized Haar measure on $\Lambda\backslash G$, i.e. $\mu_{\Lambda\backslash G}(\Lambda\backslash G) = 1$. The following proposition is a combination of [25, Proposition 4.9] and [25, Lemma 4.10] with almost the same proof, which we give for completeness.

Proposition 4.3.2. *Let $\sigma : \Delta^* \to M$ be a piecewise C^1 simplex, and let $\widetilde{\sigma} : \Delta^* \to U$ be a lift of σ to U. Then the push-forward of $\mu_{\Lambda\backslash G}$ under the map*

$$\begin{array}{rcl} \mathrm{smear}_{\widetilde{\sigma}} : \Lambda\backslash G & \to & \mathscr{P}C^1(\Delta^*, N), \\ \Lambda g & \mapsto & p_N \circ g\widetilde{\sigma}, \end{array}$$

does not depend on the choice of the lift of σ and is denoted by μ_σ. There is a well-defined chain map

$$\begin{array}{rcl} \mathrm{smear}_* : C_*^{PC^1,\ell^1,\mathrm{Lip}}(M) & \to & \mathcal{C}_*^{PC^1,\mathrm{Lip}}(N), \\ \sum_\sigma a_\sigma \sigma & \mapsto & \sum_\sigma a_\sigma \mu_\sigma. \end{array}$$

Moreover, for every fundamental cycle $c \in C_n^{PC^1,\ell^1,\mathrm{Lip}}(M)$ we have

$$\langle \mathrm{dvol}_N, \mathrm{smear}_n(c)\rangle = \int_{\mathscr{P}C^1(\Delta^n,N)} \int_{\Delta^n} \sigma^* \ \mathrm{dvol}_N \ d\,\mathrm{smear}_n(c)(\sigma) = \mathrm{vol}(M).$$

Proof. Note first that for any choice of $\widetilde{\sigma}$ the map $\mathrm{smear}_{\widetilde{\sigma}}$ is continuous by the continuity of p_N and the map

$$\begin{array}{rcl} G & \to & \mathscr{P}C^1(\Delta^n, U) \\ g & \mapsto & g\widetilde{\sigma}, \end{array}$$

and by the universal property of the quotient topology. Let $h \in \pi_1(M)$ and denote by $\mu_{\widetilde{\sigma}}$ and $\mu_{h\widetilde{\sigma}}$ the push-forwards of $\mu_{\Lambda\backslash G}$ by $\mathrm{smear}_{\widetilde{\sigma}}$ and $\mathrm{smear}_{h\widetilde{\sigma}}$ respectively. Let also $S \subset \mathscr{P}C^1(\Delta^*, N)$

be a Borel set. We have

$$\begin{aligned}
\mu_{h\widetilde{\sigma}}(S) &= \mu_{\Lambda\backslash G}(\mathrm{smear}_{h\widetilde{\sigma}}^{-1}(S)) = \mu_{\Lambda\backslash G}(\{\Lambda g \,:\, p_N \circ gh\widetilde{\sigma} \in S\}) \\
&= \mu_{\Lambda\backslash G}(\{\Lambda g h^{-1} \,:\, p_N \circ g\widetilde{\sigma} \in S\}) = \mu_{\Lambda\backslash G}(\{\Lambda g \,:\, p_N \circ g\widetilde{\sigma} \in S\}h^{-1}) \\
&= \mu_{\Lambda\backslash G}(\{\Lambda g \,:\, p_N \circ g\widetilde{\sigma} \in S\}) = \mu_{\widetilde{\sigma}}(S),
\end{aligned}$$

where we used the right $\pi_1(M)$-invariance of $\mu_{\Lambda\backslash G}$. Thus μ_σ does not depend on the choice of $\widetilde{\sigma}$.

Now we will prove that smear_* is a well defined chain map. First of all note that if $\sigma : \Delta^n \to M$ is an L-Lipschitz chain, then because p_N is a local isometry, μ_σ is supported on L-Lipschitz simplices in $\mathscr{P}C^1(\Delta^*, N)$. Moreover, $\sum_\sigma a_\sigma \mu_\sigma$ has finite variation because $\|\mu_\sigma\| \leqslant 1$ and measures with finite variations are closed under ℓ^1-finite sums. Thus $\mathrm{smear}_*(c)$ is a well defined measure in $\mathcal{C}_*^{PC^1,\mathrm{Lip}}(N)$. For any simplex $\sigma : \Delta^k \to M$ and $i = 0, ..., k$, we have

$$\mathrm{smear}_{\widetilde{\partial_i \sigma}} = \mathrm{smear}_{\partial_i \widetilde{\sigma}} = \partial_i \,\mathrm{smear}_{\widetilde{\sigma}},$$

which follows from the facts that both p_N and the action by G commute with taking boundary. Therefore

$$\begin{aligned}
\mathrm{smear}_*(\partial\sigma) &= \sum_{i=0}^{k}(-1)^i \,\mathrm{smear}_*(\partial_i\sigma) = \sum_{i=0}^{k}(-1)^i \mu_{\partial_i\sigma} = \sum_{i=0}^{k}(-1)^i (\mathrm{smear}_{\widetilde{\partial_i\sigma}})_* \mu_{\Lambda\backslash G} \\
&= \sum_{i=0}^{k}(-1)^i (\partial_i \circ \mathrm{smear}_{\widetilde{\sigma}})_* \mu_{\Lambda\backslash G} = \sum_{i=0}^{k}(-1)^i \partial_i \mu_\sigma = \partial\mu_\sigma = \partial\,\mathrm{smear}_*(\sigma).
\end{aligned}$$

It remains to show the last equality in the proposition. For a locally finite Lipschitz fundamental cycle $c = \sum_\sigma a_\sigma \sigma$, we have

$$\begin{aligned}
\langle \mathrm{dvol}_N, \mathrm{smear}_n(c)\rangle &= \sum_\sigma a_\sigma \langle \mathrm{dvol}_N, \mu_\sigma\rangle \\
&= \sum_\sigma a_\sigma \int_{\mathscr{P}C^1(\Delta^n,N)} \int_{\Delta^n} \tau^* \,\mathrm{dvol}_N \; d\,\mu_\sigma(\tau) \\
&= \sum_\sigma a_\sigma \int_{\Lambda\backslash G} \int_{\Delta^n} (p_N \circ g\widetilde{\sigma})^* \,\mathrm{dvol}_N \; d\,\mu_{\Lambda\backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Lambda\backslash G} \int_{\Delta^n} (g\widetilde{\sigma})^* \,\mathrm{dvol}_U \; d\,\mu_{\Lambda\backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Lambda\backslash G} \int_{\Delta^n} \widetilde{\sigma}^* \,\mathrm{dvol}_U \; d\,\mu_{\Lambda\backslash G}(g) \\
&= \sum_\sigma a_\sigma \int_{\Delta^n} \widetilde{\sigma}^* \,\mathrm{dvol}_U \\
&= \sum_\sigma a_\sigma \int_{\Delta^n} \sigma^* \,\mathrm{dvol}_M = \langle \mathrm{dvol}_M, c\rangle = \mathrm{vol}(M).
\end{aligned}$$

□

Proof of theorem 2.2.10. We will show that

$$\frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)} \leqslant \frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)},$$

and the opposite inequality will follow by symmetry. If $\|M\|_{\mathrm{Lip}} = \infty$, the inequality is obvious, so we can assume $\|M\|_{\mathrm{Lip}} < \infty$. By Proposition 3.2.4, in this case there exists a fundamental cycle in $C_n^{PC^1,\ell^1,\mathrm{Lip}}(M)$. Let $c = \sum_\sigma a_\sigma \sigma \in C_n^{PC^1,\ell^1,\mathrm{Lip}}(M)$ be a fundamental cycle and consider its image under the smearing map. It follows from Propositions 4.3.2 and 1.2.24 that any singular cycle homologuous to $\mathrm{smear}_n(c)$ represents the fundamental class multiplied by $\frac{\mathrm{vol}(M)}{\mathrm{vol}(N)}$. Moreover, by the construction of the smearing map,

$$\|\,\mathrm{smear}_n(c)\| = \|\sum_\sigma a_\sigma \mu_\sigma\| \leqslant \sum_\sigma |a_\sigma| \cdot \|\mu_\sigma\| = \sum_\sigma |a_\sigma| = |c|_1.$$

By Proposition 3.2.10, there exists a cycle in $C_n^{PC^1\ell^1,\mathrm{Lip}}(N)$ which represents the same homology class as $\mathrm{smear}_n(c)$ with not greater ℓ^1 norm. Because Proposition 3.2.4 implies that the Lipschitz simplicial volume of M can be computed on piecewise C^1 cycles, we obtain

$$\|N\|_{\mathrm{Lip}} \leqslant \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)}\|M\|_{\mathrm{Lip}} \;\Rightarrow\; \frac{\|N\|_{\mathrm{Lip}}}{\mathrm{vol}(N)} \leqslant \frac{\|M\|_{\mathrm{Lip}}}{\mathrm{vol}(M)}.$$

$\square$

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