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Częściowa operada sferyczna

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
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Lipiec 2012

Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data | Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

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Streszczenie

W poniższej pracy definiuję struktury topologicznej i konformnej teorii pola za pomocą operad, PROPów oraz linii wyznacznikowych. Następnie koncentruję się na badaniu łącznej algebry wierzchołkowej, będącej strukturą blisko związaną z konformeną teorią pola. Daje się ona zdefiniować jako algebra nad częściową operadą, której odpowiednie składowe są wiązkami linii wyznacznikowych nad przestrzeniami moduli sfer Riemanna ze sparametryzowanymi punktacjami. Ostatecznie formułuję twierdzenie o izomorfizmie między kategoriami łącznych algebr wierzchołkowych oraz algebr operatorów wierzchołkowcyh. Przedstawiona jest konstrukcja powyższego izomorfizmu, wraz ze szkicem dowodu.

Słowa kluczowe

algebra operatorów wierzchołkowych, konforemna teoria pola, łączna algebra wierzchołkowa, operada, PROP, topologiczna teoria pola

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

81-xx Quantum theory
81T-xx Quantum field theory; realted classical field theories
81T40 Two-dimensional field theories, conformal field theories etc.

18-xx Category theory; homological algebra
18Dxx Categories with structure
18D50 Operads

Tytuł pracy w języku angielskim

Partial sphere operad

Spis treści

Introduction

The main purpose of this work is to formulate and describe in a short way the isomorphism theorem between the categories of vertex operator algebras and vertex associative algebras, which are algebras over operads of determinant lines of Riemann spheres with tubes. The vertex operator algebra (VOA) is an algebraic structure, which consist of a $\mathbb{Z}$-graded module V over Virasoro algebra, the distinguished elements $\mathbf{1} \in V_{(0)}$ and $\omega \in V_{(2)}$ and *the vertex operator*

$$V \otimes V \to V[[x, x^{-1}]]$$

or equivalently

$$Y : v \mapsto Y(v, x) \in End(V)[[x, x^{-1}]].$$

It is in fact rather complicated generalization of an associative, commutative algebra, which is strictly commutative and associative only up to turning some rational function into power series in different ways. On the other hand, the notion of vertex associative algebra is rather geometric. It consist also of a $\mathbb{Z}$-graded vector space with some morphisms between $V^{\otimes n} \to V$ for $n \in \mathbb{N}$. These morphisms are determined by the Riemann spheres with $n+1$ parametrised punctures, one parametrised positively and the other-negatively, together with a point from a line bundle over moduli space of such spheres, called *determinant line bundle*. The operation of sewing two spheres along tubes using parametrisations corresponds to the composition of morphisms.

The isomorphism theorem is important from at least two points of view. The first is that it parially asnswers the question, what is the underlying algebraic structure of the conformal field theory. G. B. Segal defined in [Seg] the conformal field theory in geometric terms; it is an algebra over a PROP of Riemann surfaces, i.e. a vector space with morphisms determined by the Riemann surfaces with parmetrised boundaries. From this description it was not obvious what kind of algebraic structure this construction describes. If the answer was known the problem of constructing conformal field theories, which are objects described geometrically, would be purely algebraic. The second point of view is to understand the vertex operator algebra in a more intuitive geometric terms. In spite of their complicacy, there are some natural and very nontrivial examples of VOAs, the most famous is probably the Moonshine Module, described in [FLM], which is closely related to the Griess-Fischer Monster group.

In the first chapter we introduce the notions of an operad, partial operad and PROP, together with examples. The material is taken from [Vor] and [Hu].

In the second chapter the definitions of topological and conformal field theories are given. The definition of TFT, together with the most of the proof of Theorem (2.1.3), is taken from [Vor2]. The material concerning the determinant lines can be found in [Hu], and the definition of CFT is based on a definition given in [Seg].

In the third chapter we define the partial sphere operad, the vertex associative algebra and the vertex operator algebra and we sketch the proof of an isomorphism theorem. Most of the content of this chapter is based on [Hu], where the isomorphism theorem is formulated and a complete proof is given. In this work we present theoretical stuff necessary to understand the main theorem and we complete its proof in the places which were ommited in [Hu].

Acknowledgements: I would like to thank Prof. Michael Weiss for giving me the idea for writing this work, helping me to understand basic notions related to the topic and indicating the sources. I would also like to thank Prof. Henryk Żołądek for his comments on this work and additional materials and dr Andrzej Weber for his support and help in finding supervisor.

Chapter 1

Operads and PROPs

In this chapter we define notions of an operad and of a PROP. In spite of rather complicated definition, they are rather simple objects introduced to simplify description of much more complex structures. Generally one could think of an operad as a collection of 'boxes with many ($n \geqslant 0$) inputs and one output' such that we can compose two such objects by sewing input of one object with an output of another one, satisfying some axioms on composing the sewing operation and admitting an action of some symmetric groups on inputs. PROP is a slight generalization of an operad: it is a collection of objects with many inputs and many outputs. Good introduction to the operad and PROP theory with many examples is written for example in [Ad] or [Vor]. Another good source is Appendix C in [Hu] where notions of partial and pseudo operads are described.

1.1. Operads

Definition 1.1.1. An *operad* $\mathcal{C}$ is a family of sets $\mathcal{C}(j)$, $j \in \mathbb{N}$ together with:

1. *substitution maps*

$$\begin{aligned} \gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) &\to \mathcal{C}(j_1 + \ldots + j_k) \\ (c; d_1, \ldots, d_k) &\mapsto \gamma(c; d_1, \ldots, d_k) \end{aligned}$$

   for each $k \geqslant 1$, $j_1, \ldots, j_k \in \mathbb{N}$

2. an *identity element* $I \in \mathcal{C}(1)$

3. left action of the symmetric group S_j on $\mathcal{C}(j)$.

Moreover, the following axioms are satisfied:

(i) *Operad-associativity*: for any $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$ $(s = 1, \ldots k)$ and $e_t \in \mathcal{C}(i_t)$ $(t = 1, \ldots, j_1 + \ldots + j_k)$ we have

$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_{j_1+\ldots+j_k}) = \gamma(c; f_1, \ldots, f_k) \tag{1.1}$$

where $f_s = \gamma(d_s; e_{j_1+\ldots+j_{s_1}+1}, \ldots, e_{j_1+\ldots+j_s})$;

(ii) for any $c \in \mathcal{C}(k)$, $d \in \mathcal{C}(l)$, $k, l \in \mathbb{N}$, $k \geqslant 1$,

$$\gamma(c; I, \ldots, I) = c, \tag{1.2}$$
$$\gamma(I; d) = d; \tag{1.3}$$

(iii) for any $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$ $(s = 1, \ldots, k)$, $\sigma \in S_k$,

$$\gamma(\sigma(c); d_1, \ldots, d_k) = \sigma(j_1, \ldots, j_k)\gamma(c; d_{\sigma(1)}, \ldots, d_{\sigma(k)}), \tag{1.4}$$

where $\sigma(j_1, ..., j_k)$ denotes the permutation of $\sum_{s=1}^{k} j_s$ elements which permutes k blocks of letters determined by the partition $(j_1, ..., j_k)$, and for any $\tau_s \in S_{j_s}$, $s = 1, ..., k$,

$$\gamma(c; \tau_1(d_1), ...\tau_k(d_k)) = (\tau_1 \oplus ... \oplus \tau_k)\gamma(c; d_1, ..., d_k), \tag{1.5}$$

where $(\tau_1 \oplus ... \oplus \tau_k)$ denotes the permutation of $\sum_{s=1}^{k} j_s$ elements given as an image of $(\tau_1, ..., \tau_k)$ under the inclusion $S_{j_1} \times ... \times S_{j_k} \to S_{j_1 + ... + j_k}$.

It is easy to see that the axioms (ii) and (iii) describe existence of an 'identity element' I and equivariance of substitution map with respect to the action of symmetric group. The most complicated axiom, operad-associativity, describes an independence of the order of applying substitution maps.

We would like to have also reasonable notion of a morphism between operads.

Definition 1.1.2. A *morphism* ψ between operads $\mathcal{C}$, $\mathcal{D}$ is a collection of maps $\psi_k : \mathcal{C}(k) \to \mathcal{D}(k)$, $k \in \mathbb{N}$ such that

1. $\psi_1(I_{\mathcal{C}}) = I_{\mathcal{D}}$;
2. $\psi_k(\sigma(c)) = \sigma(\psi_k(c))$ for any $c \in \mathcal{C}(k)$, $\sigma \in S_k$;
3. the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{C}(k) \times \mathcal{C}(j_1) \times ... \times \mathcal{C}(j_k) & \xrightarrow{\gamma_{\mathcal{C}}} & \mathcal{C}(j_1 + ... + j_k) \\
\downarrow {\scriptstyle \psi_k \times \psi_{j_1} ... \times \psi_{j_k}} & & \downarrow {\scriptstyle \psi_{j_1 + ... + j_k}} \\
\mathcal{D}(k) \times \mathcal{D}(j_1) \times ... \times \mathcal{D}(j_k) & \xrightarrow{\gamma_{\mathcal{D}}} & \mathcal{D}(j_1 + ... + j_k)
\end{array}$$

Remark 1.1.3. We can require that sets $\mathcal{C}(k)$ in the definition of an operad carry some additional structure, e.g. topological, smooth or analytic. If substitution map and action of symmetric groups preserve this structure, we say such an operads are *topological*, *smooth* or *analytic operads* respectively.

Example 1.1.4 (*Endomorphism operad*)**.** Let X be a set (topological/vector/etc. space) and let

$$\mathcal{C}_X(k) = \{\text{set of (continuous/multilinear/etc.) functions } \underbrace{X \times ... \times X}_{k} \to X\}$$

for $k \geqslant 1$ and $\mathcal{C}_X(0) = X$. We can define an operad structure on collection $\mathcal{C}_X = \{\mathcal{C}_X(k)\}_{k \in \mathbb{N}}$ as follows:

(i) as substitution map define:

$$\gamma(f; g_1, ..., g_k)(x_1, ..., x_{j_1+...+j_k}) = f(g(x_1, ..., x_{j_1}), ..., g(x_{j_1+...+j_{k-1}+1}, ..., x_{j_1+...+j_k})),$$

where $f \in \mathcal{C}_X(k)$, $g_s \in \mathcal{C}_X(j_s)$, $s = 1, ..., k$ and $x_i \in X$, $i = 1, ..., j_1 + ... + j_k$;

(ii) as an identity we take $Id \in \mathcal{C}_X(1) =$ {endomorphisms of X};

(iii) symmetric group acts on $f \in \mathcal{C}_X(k)$ simply as a permutation of arguments;

$$\sigma(f)(x_1, ..., x_k) = f(x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)}).$$

Moreover, if $\mathcal{C}$ is an operad and we have a morphism $\psi : \mathcal{C} \to \mathcal{C}_X$ we say that an operad $\mathcal{C}$ *acts on a space* X (X is a $\mathcal{C}$*-space*).

We should also emphasise two special cases of such operad:

- Let $X = \{*\}$ be a one point set. In this case $\mathcal{C}_X$ is a trivial example of an operad and we call it *symmetric set operad*.
- Let V be a vector space and consider only multilinear functions $\underbrace{V \times ... \times V}_{k} \to V$ (that is, $\mathcal{C}_V(k) = Hom(V^{\otimes k}, V)$). We denote such operad as $\mathcal{E}_V$ and call it *the multilinear endomorphism operad*. If an operad $\mathcal{C}$ acts on a space V by morphism ν, we say that a pair (V, ν) is *an algebra over an operad* $\mathcal{C}$ (a $\mathcal{C}$*-algebra*).

Sometimes a $\mathcal{C}$-algebra is called *a representation of an operad* $\mathcal{C}$, which seems to be more proper name. However, the name $\mathcal{C}$*-algebra* is motivated by examples which show that every such representation is some generalization of an algebra:

Example 1.1.5. Consider an algebra over symmetric set operad S. It is easy to see that such an algebra is equivalent to the commutative, associative algebra with a unit and with product given by an image of $\{*\} \in \mathcal{S}(2)$ in $Hom(V^{\otimes 2}, V)$ and a unit given by an image of $\{*\} \in \mathcal{S}(0)$ in V.

We would also like to present one more example, which is not needed in this work and is slightly different from the other presented examples, but was a motivation for introducing operads.

Example 1.1.6 (*Little cube operad*)**.** Let $\mathcal{L}(0) = \emptyset$ and for $k \geqslant 1$ let $\mathcal{L}_n(k)$ be a set of non-overlapping orientation-preserving embeddings of k n-dimensional cubes into n-dimensional cube. We define I as an identity embedding, the symmetric group acts by permutations of the embeddings and the substitution map

$$\begin{aligned} \gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) &\to \mathcal{C}(j_1 + \ldots + j_k) \\ (c; d_1, \ldots, d_k) &\mapsto \gamma(c; d_1, \ldots, d_k) \end{aligned}$$

as a composition of the embedding of $j_1 + ... + j_k$ cubes into k cubes (represented by $d_1, ..., d_k$) and the embedding of k cubes into one represented by c. It is easy to check that $\mathcal{L}_n$ is an operad. Moreover, on each set $\mathcal{L}_n(k)$ we have compact-open topology, so $\mathcal{L}_n$ is in fact a topological operad. It is known that the action of $\mathcal{L}_n$ on a space with compactly generated topology detects a structure of n-fold loop space.

At the end of this section we would like to give one more remark. Instead of describing substitution map as something that sews simultaneously k objects to all inputs of one object in $\mathcal{C}(k)$, we could describe them as maps sewing at once only one input to one output. More precisely, we can define single substitution map $\circ_i$ by

$$c \circ_i d = \gamma(c; \underbrace{I, ..., I}_{i-1}, d, \underbrace{I, ..., I}_{k-i})$$

for $c \in \mathcal{C}(k)$, $d \in \mathcal{C}(j)$, $1 \leqslant i \leqslant k$. This leads to an alternative definition of an operad.

Definition 1.1.7. An *operad* $\mathcal{C}$ is a family of sets $\mathcal{C}(j)$, $j \in \mathbb{N}$, together with:

1. *substitution maps*

$$\begin{aligned} \circ_i : \mathcal{C}(k) \times \mathcal{C}(j) \to & \mathcal{C}(k+j-1) \\ (c; d) \mapsto & c \circ_i d \end{aligned}$$

   for each $k, j, i \in \mathbb{N}$, $1 \leqslant k$, $1 \leqslant i \leqslant k$;

2. an *identity element* $I \in \mathcal{C}(1)$;

3. left action of the symmetric group S_j on $\mathcal{C}(j)$

such that the following axioms are satisfied:

(i) *Operad-associativity*: for any $c \in \mathcal{C}(k)$ $(k \geqslant 1)$, $d \in \mathcal{C}(j)$ $(k + j - 1 \geqslant 1)$, $e \in \mathcal{C}(l)$, $1 \leqslant i_1 \leqslant k$ and $1 \leqslant i_2 \leqslant k + j - 1$ we have

$$(c \circ_{i_1} d) \circ_{i_2} e = \begin{cases} (c \circ_{i_2} e) \circ_{l+i_1-1} d, & \text{for } i_2 < i_1 \\ c \circ_{i_1} (d \circ_{i_2 - i_1 + 1} e), & \text{for } i_1 \leqslant i_2 < i_1 + j \\ (c \circ_{i_2 - j + 1} e) \circ_{i_1} d, & \text{for } i_1 + j < i_2 \end{cases} \tag{1.6}$$

(ii) for any $c \in \mathcal{C}(k)$, $d \in \mathcal{C}(l)$, $k, l \in \mathbb{N}$, $k \geqslant 1$,

$$c \circ_i I = c, \tag{1.7}$$

$$I \circ_1 d = d; \tag{1.8}$$

(iii) for any $c \in \mathcal{C}(k)$, $d \in \mathcal{C}(j)$, $\sigma \in S_k$, $k \geqslant 1$, $1 \leqslant i \leqslant k$,

$$\sigma(c) \circ_i d = \sigma(\underbrace{1, ..., 1}_{i-1}, j, \underbrace{1..., 1}_{k-i})(c \circ_{\sigma(i)} d) \tag{1.9}$$

and for any $\tau \in S_j$

$$c \circ_i \tau(d) = (\underbrace{1 \oplus ... \oplus 1}_{i-1} \oplus \tau \oplus \underbrace{1 \oplus ... \oplus 1}_{k-i})(c \circ_i d). \tag{1.10}$$

It is easy to see that this definition of an operad is equivalent to the definition 1.1.1 by defining:

$$\gamma(c; d_1, ..., d_k) = (...((c \circ_k d_k) \circ_{k-1} d_{k-1})...) \circ_1 d_1$$

for $c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s)$, $s = 1, ..., k$. All notions described in this section could be easily defined using this alternative definition of an operad, so we will not do that.

1.2. Partial and rescalable operads

In some cases, notion of an operad is too strong, i.e. substituting is not always possible. In such situation we can still have a structure of a partial operad.

Definition 1.2.1. A *partial operad* $\mathcal{P}$ is a collection of sets $\mathcal{P}(k), k \in \mathbb{N}$, which satisfy all the axioms of an operad (in the sense Definition (1.1.1)), but substitution maps γ are only partial functions, i.e. they are defined only on some subsets of $\mathcal{C}(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k)$, $k \geqslant 1$, $j_1, \ldots, j_k \in \mathbb{N}$. It is understood that operad-associativity and permutation axioms holds iff both sides of (1.1), (1.4) and (1.5) exist and both sides of (1.2) and (1.3) in the definition of an operad always exist.

Weakening the definition of an operad, we obtain the following.

Definition 1.2.2. A *partial pseudo-operad* $\mathcal{P}$ is a collection of sets $\mathcal{P}(k)$ satisfying all the axioms of a partial operad (in the sense of definition 1.1.1), except the operad-associativity axiom.

Definition 1.2.3. A *morphism* ψ between partial (pseudo) operads $\mathcal{P}$, $\mathcal{Q}$ is a collection of maps $\psi_k : \mathcal{P}(k) \to \mathcal{Q}(k)$, $k \in \mathbb{N}$, such that:

1. $\psi_1(I_{\mathcal{P}}) = I_{\mathcal{Q}}$;
2. $\psi_k(\sigma(c)) = \sigma(\psi_k(c))$ for any $c \in \mathcal{P}(k)$, $\sigma \in S_k$;
3. image of domains of $\gamma_{\mathcal{P}}$ by ψ is contained in domains of $\gamma_{\mathcal{Q}}$ and the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P}(k) \times \mathcal{P}(j_1) \times ... \times \mathcal{P}(j_k) & \xrightarrow{\gamma_{\mathcal{P}}} & \mathcal{P}(j_1 + ... + j_k) \\ \downarrow{\scriptstyle \psi_k \times \psi_{j_1} ... \times \psi_{j_k}} & & \downarrow{\scriptstyle \psi_{j_1 + ... + j_k}} \\ \mathcal{Q}(k) \times \mathcal{Q}(j_1) \times ... \times \mathcal{Q}(j_k) & \xrightarrow{\gamma_{\mathcal{Q}}} & \mathcal{Q}(j_1 + ... + j_k) \end{array}$$

Remark 1.2.4. One can define the same notions using alternative definition of an operad (1.1.7). However, such two definitions of partial (pseudo) operads are in general not equivalent. Consider a partial (pseudo) operad $\mathcal{P}$ in the sense of definition 1.2.1 with partial substitution maps γ. We can obtain partial single substitution maps $\circ_i$ and using them again partial substitution maps γ'. Maps γ and γ' differ only by the domains; it is obvious that the domains of γ' are contained in the domains of γ, but they are in general not equal. However, we can restrict the domains of maps γ to the domains of γ', and then the equivalence follows. In this case (i.e. domains of γ and γ' are equal) we say that a partial (pseudo) operad $\mathcal{P}$ has *induced domain*. We can therefore say that the notions of a partial (pseudo) operad with induced domain and partial (pseudo) operad obtained by using definition 1.1.7 of an operad are equivalent.

In some cases a given structure is a partial operad, but it is not far from being just an operad, i.e. after applying some minor changes to 'inputs' one can always compose two objects. This leads to the following notions.

Definition 1.2.5. Let $\mathcal{P}$ be a partial operad. A subset $G \in \mathcal{P}(1)$ is called a *rescaling group for* $\mathcal{P}$ if it satisfies following conditions:

1. $I \in G$.

2. $\gamma(g_1; g_2) \in G$ for $g_1, g_2 \in G$ and G is a group with respect to multiplication induced by γ and neutral element I.

3. The substitution maps are defined on $G \times \mathcal{P}(k)$ and $\mathcal{P}(k) \times G^k$ for any $k \in \mathbb{N}$. Moreover, in operad-associativity axiom if $c \in G$, $d_1, ..., d_k \in G$ or $e_1, ..., e_{j_1+...+j_k} \in G$ then both sides of (1.1) exist iff either side exists.

Definition 1.2.6. Let $\mathcal{P}$ be a partial operad with rescaling group G. We say that $c_1, c_2 \in \mathcal{P}(k)$ are *G-equivalent* if there exists $g \in G$ such that $c_2 = \gamma(g; c_1)$.

It is obvious from the definition of rescalable group that G-equivalence forms an equivalence relation on each set $\mathcal{P}(k)$, $k \in \mathbb{N}$.

Definition 1.2.7. We say that a partial operad $\mathcal{P}$ is $(G-)$*rescalable* if there exists a rescaling group G for $\mathcal{P}$ and for any $c \in \mathcal{P}(k)$ and any $d_s \in \mathcal{P}(j_s)$, $s = 1, ..., k$, there exist $d'_s \in \mathcal{P}(j_s)$, $s = 1, ..., k$, such that $d_1, ..., d_k$ are G-equivalent to $d'_1, ..., d'_k$ respectively and $\gamma(c; d'_1, ..., d'_k)$ exists.

There is one important example of a partial pseudo-operad, which leads to generalization of a notion of $\mathcal{C}$-algebra.

Example 1.2.8. Let G be a group and $V = \bigoplus_{M \in \mathcal{A}} V_{(M)}$ be a completely reducible G-module, where $\mathcal{A}$ is a set of equivalence classes of all irreducible G-modules, $V_{(M)}$ is the sum of all G-submodules of V in the class M and $dim(V_{(M)}) < \infty$ for all $M \in \mathcal{A}$. Denote $V' := \bigoplus_{M \in \mathcal{A}} V^*_{(M)}$ and $\bar{V} = V'^* = \prod_{M \in \mathcal{A}} V_{(M)}$ and define $\mathcal{H}^G_V(k) = Hom(V^{\otimes k}, \bar{V})$. We can define a partial pseudo-operad structure on $\mathcal{H}^G_V$ by taking an obvious embedding of V into $\bar{V}$ as I_V^G and an action of S_k on $\mathcal{H}_V^G(k)$ by permuting the arguments of $f \in Hom(V^{\otimes k}, \bar{V})$. In this case it is easier to define partial single substitution maps, but to achieve this, we first define a contraction operator. Given $f \in \mathcal{H}_V^G(k)$ and $g \in \mathcal{H}_V^G(j)$, consider the following series for $v_1, ..., v_{k+j-1} \in V$ and $v' \in V'$:

$$\sum_{M \in \mathcal{A}} \langle v', f(v_1, ..., v_{i-1}, P_M(g(v_i, ..., v_{i+j-1}), v_{i+j}, ..., v_{k+j-1})\rangle$$

where P_M are the obvious projections on the factors $V_{(M)}$. If it converges absolutely for any $v_1, ..., v_{k+j-1} \in V$ and $v' \in V'$, we say that the *contraction of f at the i-th argument and g at the 0-th argument exists.* In this case we can define a *contraction* $f_i *_0 g \in \mathcal{H}_V^G(k+j-1)$. Using this construction, we can define partial single substitution map:

$$\begin{array}{rcl} \circ_i : \mathcal{H}_V^G(k) \times \mathcal{H}_V^G(j) & \to & \mathcal{H}_V^G(k+j-1) \\ (f, g) & \mapsto & f \circ_i g = f_i *_0 g \end{array}$$

where $k, j, i \in \mathbb{N}$, $k \geqslant 1$, $1 \leqslant 1 \leqslant k$. We call partial pseudo-operad $\mathcal{H}_V^G$ a *multilinear endomorphism partial pseudo-operad.* Note that it is only a pseudo-operad; the operad-associativity fails because we cannot expect in general the convergence of multisums corresponding to applying contraction operation several times, therefore the sides of (1.6) may vary.

Definition 1.2.9. Let $\mathcal{P}$ be a partial operad with rescaling group G. A *$\mathcal{P}$-pseudo-algebra* is a pair (V, ν), where $V = \bigoplus_{M \in \mathcal{A}} V_{(M)}$ is a completely reducible G-module with $dim(V_{(M)}) < \infty$ for all $M \in \mathcal{A}$ and ν is a morphism from $\mathcal{P}$ to $\mathcal{H}_V^G$, where $\nu(G) \subset \nu(\mathcal{P}) \subset \mathcal{H}_V^G$ are given representations of G. Note that the image of ν is in fact operad-associative, because $\mathcal{P}$ is so. If $\mathcal{P}$ is G-rescalable, we say that adequate $\mathcal{P}$-pseudo-algebra is just $\mathcal{P}$-algebra.

It is indeed a generalization of an algebra over an operad in finite-dimensional case; note that every operad $\mathcal{C}$ is $\{I\}$-rescalable, where $I \in \mathcal{C}(1)$ forms in fact one-element rescaling group. Then the only irreducible representation of $\{I\}$ is one dimensional, V is a finite sum of one-dimensional spaces and contraction is always defined, hence a substitution map for $\mathcal{H}_V^I$.

1.3. PROPs

A PROP is in fact a simple generalization of an operad; it is an operad which can have more than 1 (or 0) 'outputs'. One could then expect more complicated definition. However, using some category theory, we can define such an object in a much shorter way.

Definition 1.3.1. A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (a *tensor product*), an *identity object* I, and natural isomorphisms $\alpha_{A,B,C}$:

$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, $\lambda_A : A \otimes I \cong A$ and $\rho_A : I \otimes A \cong A$ such that for all A, B, C, D the following diagrams are commutative:

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\
\Big\downarrow {\scriptstyle \alpha_{A \otimes B,C,D}} & & & & \Big\downarrow {\scriptstyle A \otimes \alpha_{B,C,D}} \\
(A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C \otimes D}]{\qquad\qquad\qquad\qquad} & & A \otimes (B \otimes (C \otimes D))
\end{array}$$

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow[\alpha_{A,I,B}]{} & A \otimes (I \otimes B) \\
& {\scriptstyle \rho_A \otimes B} \searrow \quad \swarrow {\scriptstyle A \otimes \lambda_B} & \\
& A \otimes B &
\end{array}$$

We say that monoidal category is *symmetric* if there exist isomorphisms $\gamma_{A.B} : A \otimes B \rightarrow B \otimes A$ and *strict* if all mentioned isomorphisms (α, λ, ρ and γ in the symmetric case) are in fact identities.

Definition 1.3.2. A *PRO* (*PROducts*) is a strict symmetric monoidal category, whose objects are natural numbers $\mathbb{N}$ and tensor product is given by addition, with an identity $0 \in \mathbb{N}$. A *PROP* (*PROducts and Permutations*) is a PRO with a left action of the symmetric group S_m and right action of S_n on a set $Mor(m, n)$, compatible with composition of morphism and tensor product.

It is not difficult to obtain an operad from a PROP: given a PROP $\mathcal{R}$, as $\mathcal{C}(k)$ take $Mor_{\mathcal{R}}(k, 1)$ and define:

$$\gamma(c; d_1, ..., d_k) := c \circ (d_1 \otimes ... \otimes d_k)$$

An operad identity is obviously $Id \in Mor_{\mathcal{R}}(1, 1)$. The operad-associativity holds, for example, because

$$\begin{aligned}
\gamma(\gamma(c; d_1, ..., d_k); e_1, ..., e_{j_1+...+j_k}) &= (c \circ (d_1 \otimes ... \otimes d_k)) \circ (e_1 \otimes ... \otimes e_{j_1+...+j+k}) \\
&= c \circ ((d_1 \otimes ... \otimes d_k) \circ (e_1 \otimes ... \otimes e_{j_1+...+j+k})) \\
&= c \circ ((d_1 \circ (e_1 \otimes ... \otimes e_{j_1})) \otimes ... \\
&\quad ... \otimes (d_k \circ (e_{j_1+...+j_{k-1}+1} \otimes ... \otimes e_{j_1+...+j_k}))) \\
&= \gamma(c; f_1, ..., f_k)
\end{aligned}$$

Where $f_s = \gamma(d_s; e_{j_1+...+j_{s-1}+1}, ..., e_{j_1+...+j_s})$. The last but one equality comes from the fact that tensor product is bifunctor, so it preserves compositions of morphisms, i.e. $(c \otimes d) \circ (e \otimes f) = (c \circ e) \otimes (d \circ f)$ for $c, d, e, f \in Mor(\mathcal{R})$ whenever this equality makes sense.

We have analogous notions of an endomorphism PROP and of an algebra over a PROP.

Example 1.3.3 (*Endomorphism PROP*)**.** Let X be a set (topological/vector/etc. space). Define a PROP $\mathcal{R}_X$ by:

$$Mor_{\mathcal{R}_X}(n,m) = \{\text{set of (continuous/multilinear/etc.) maps } X^n \to X^m\}$$

for $n, m \in \mathbb{N}$, where $X^k = \underbrace{X \times ... \times X}_{k}$ and X^0 is a one point space (a field in the linear case). Composition of morphisms and actions of symmetric groups are obtained in an obvious way. If $\mathcal{R}$ is another PROP and we have a morphism $\psi : \mathcal{R} \to \mathcal{R}_X$ we say that $\mathcal{R}$ *acts on a space* X.

Example 1.3.4. Let V be a vector space and consider only multilinear maps $V^n \to V^m$ (that is, $Mor_{\mathcal{R}_V}(n,m) = Hom(V^{\otimes n}, V^{\otimes m})$). We denote such PROP as $\mathcal{E}_V$ and call it *the multilinear endomorphism PROP*. If a PROP $\mathcal{R}$ acts on a space V by a morphism ν then we say that a pair (V, ν) is *an algebra over a PROP* $\mathcal{R}$ (a $\mathcal{R}$-*algebra*).

Chapter 2

Riemann surfaces and field theories

In this chapter we define important example of a PROP, in which morphisms are surfaces (with topological or complex structure) with parametrized boundaries and composition of morphisms, is defined by sewing two surfaces along boundaries. Such PROP leads to the definition of topological field theory, more information can be found for example in [Vor2]. In order to define conformal field theory, we need to define PROP using not only surfaces with complex structures, but also a determinant line bundle over a moduli space of surfaces. A definition of mentioned PROP is given, for example, in [Vor] and the theory of determinant lines is described in appendix D in [Hu].

2.1. Topological field theory

Definition of the topological field theory using PROPs is rather easy. Consider a PROP $\mathcal{T}$ where we identify object $k \in \mathbb{N}$ with the one-dimensional compact manifold with k labelled connected components (i.e. k labelled circles) and let:

$$Mor_{\mathcal{T}}(n, m) = \{\text{cobordisms between } n \text{ and } m \text{ labelled circles, up to a homeomorphism}\}$$

Morphisms in this PROP are in fact homeomorphism classes of topological surfaces with $n + m$ labelled boundary components. If we have two surfaces Σ_1 and Σ_2 in $Mor_{\mathcal{T}}(n, m)$ and $Mor_{\mathcal{T}}(m, l)$ respectively, we can define the composition as a cobordism $\Sigma_1 \# \Sigma_2 \in Mor_{\mathcal{T}}(n, l)$ obtained by sewing Σ_1 and Σ_2 along m boundary circles, using the labeling to decide how the circles need to be sewn. Because topological type of obtained cobordism does not depend on the way we sew single circles, this operation is well defined. Symmetric groups acts on Σ by relabelling corresponding circles in the boundary, and the identity is a trivial cobordism $I \times S^1 \in Mor_{\mathcal{T}}(1, 1)$. Having PROP $\mathcal{T}$, we can introduce the following.

Definition 2.1.1. A *topological field theory* is an algebra over a PROP $\mathcal{T}$

This definition gives very simple description of the TFT. What interests us next is the underlying algebraic structure. In this case the answer is also not complicated; it turns out that every TFT in the sense of the definition above is equivalent to a commutative Frobenius algebra defined below.

Definition 2.1.2. *A Frobenius algebra* over a field k is finite-dimensional, unital, associative algebra A over k equipped with a functional $\mathrm{tr} : A \to k$ (a *trace*) such that the bilinear form $\langle , \rangle : A \times A \to k$ defined as $\langle a, b\rangle := \mathrm{tr}(ab)$ is non-degenerate. We say that A is *commutative* if it is commutative as an algebra.

Note that commutativity implies also symmetry of the bilinear form $\langle , \rangle$.

Theorem 2.1.3 (Folklore)**.** *Categories of topological field theories based on a vector spaces over k and commutative Frobenius algebras over k are isomorphic.*

Proof. Let (V, ν) be a TFT in the sense of Definition 2.1.1. We define product $V \times V \to V$ by the ν-image of an element of $Mor_{\mathcal{T}}(2, 1)$ shown in the figure 2.1:

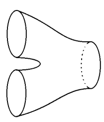


Figure 2.1: $a \otimes b \mapsto ab$

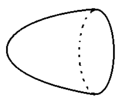


Figure 2.2: $1 \mapsto e$

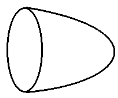
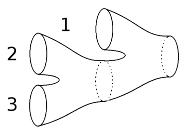
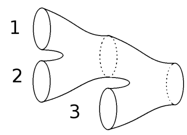


Figure 2.3: $a \mapsto \mathrm{tr}(a)$

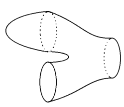
Because there exists a homeomorphism from surface defined by Figure 2.1 to itself which interchange the labels of the input circles, we see, that this product is commutative. It is also associative by the homeomorphism:



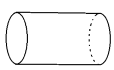
$a(bc)$ $\cong$ $(ab)c$



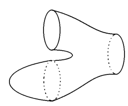
Unit is defined by an image of $1 \in k$ under a function $k \to V$ defined by a surface shown in Figure 2.2. It is indeed a unit: we have homeomorphisms:



$e \cdot a$ $\cong$ a



a $\cong$ $a \cdot e$



We define the trace $\mathrm{tr} : V \to k$ by an image of surface from Figure 2.3. We need only to show that bilinear form obtained by sewing the surfaces form Figures 2.1 and 2.3 is non-degenerate and V is finite-dimensional. Note that because of TFT structure we have a symmetric coproduct $V \to V \otimes V$ represented by a surface from Figure 2.4:

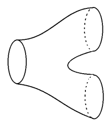


Figure 2.4: $a \mapsto \sum_{i=1}^{n} a_i \otimes a_i'$

To prove both statements, consider the homeomorphism:

Figure 2.5:

$a \otimes (\sum_i^n e_i \otimes e_i') \mapsto \sum_{i=1}^{n} \langle a, e_i \rangle e_i'$ $\cong$ a

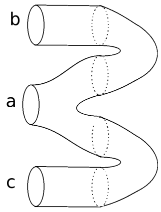
From the equality $a = \sum_{i=1}^{n} \langle a, e_i \rangle e_i'$ which holds for every $a \in V$ we see that every element of V is a linear combination of elements $e_1', ..., e_n'$, hence V is finite-dimensional. Moreover, the map $V \to V^*$ induced by a bilinear form $\langle , \rangle$ has trivial kernel, so it is an isomorphism (if $\langle b, a \rangle = 0$ for every $a \in V$, then $b = \sum_{i=1}^{n} \langle b, e_i \rangle e_i' = 0$). It follows that V is a commutative Frobenius algebra and, using a morphism $\psi : (V, \nu) \to (W, \mu)$ of two TFTs, we obtain an obvious linear map of associated vector spaces $V \to W$ which preserves the product and trace, because they were constructed as an images of morphisms in $\mathcal{T}$.

Before we continue the proof in the opposite direction we shall show that the coproduct $V \to V \otimes V$ shown on the figure 2.4 can be described in terms of bilinear form and dual spaces as follows. Firstly, there is an isomorphism $V \cong V^*$ given by $V \ni a \mapsto \langle a, \cdot \rangle \in V^*$. Note also that, because V is finite-dimensional, the obvious inclusion $V^* \otimes V^* \hookrightarrow (V \otimes V)^*$ is an isomorphism. Our coproduct is in fact a composition

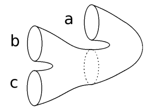
$$V \cong V^* \stackrel{(\cdot)^*}{\to} (V \otimes V)^* \cong V^* \otimes V^* \cong V \otimes V$$

Where $(\cdot) : V \otimes V \to V$ is just the product. To show this, let $\alpha : V \to V \otimes V$ be a coproduct given by the surface from Figure 2.4. Because of non-degeneracy of $\langle \cdot, \cdot \rangle$ we only need to show that for every $a, b, c \in V$ we have $\alpha^*(a)(b \otimes c) = \langle a, bc \rangle$, where $\alpha^* : V \to (V \otimes V)^*$ is a composition of α with isomorphisms $V \otimes V \cong V^* \otimes V^* \cong (V \otimes V)^*$. It can be shown using the following homeomorphism:

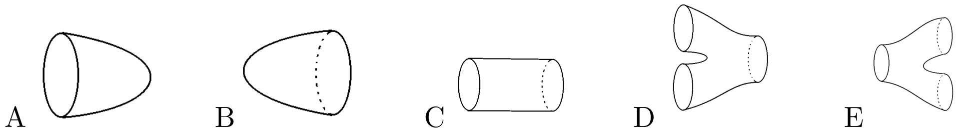
Figure 2.6:



$\alpha^*(a)(b \otimes c)$ $\cong$ $\mathrm{tr}(a(bc)) = \langle a, bc \rangle$



Assume now that we have a Frobenius algebra A. We will construct a TFT based on A as a vector space. To construct a morphism from $\mathcal{T}$ to $\mathcal{E}_A$ observe that any surface with boundary can be cut into the following parts:



We define corresponding morphisms as:

- A: trace $\mathrm{tr} : A \to k$
- B: $1 \mapsto e$, where $e \in A$ is a unit
- C: $Id : A \to A$
- D: product $(a, b) \mapsto ab$
- E: coproduct $V \cong V^* \xrightarrow{(\cdot)^*} (V \otimes V)^* \cong V^* \otimes V^* \cong V \otimes V$

The fact, that homomorphism does not depend on the way we cut corresponding surface is implied by associativity, commutativity and construction of a coproduct (which gives the equalities associated with homeomorphism shown on Figure 2.6, hence also on Figure 2.5) and coassociativity of this coproduct. Given morphism $A \to B$ of Frobenius algebras, it can be easily extended to the morphism of associated TFTs, so we have a functor from the category of Frobenius Algebras to the category of TFTs (over k). The fact that both constructed functors are inverse to each other is obvious from the construction, so we have the desired isomorphism. □

2.2. Conformal field theory

We would like to generalize our definition to the conformal case, i.e. to define a conformal field theory as an algebra over a PROP of complex cobordisms between circles, up to conformal isomorphisms. However, to do this we need some additional structure. First of all it is no longer true that a surface obtained by sewing two surfaces along circle(s) is independent from the homeomorphism of the circles, we cannot even choose every homeomorphism, because they need to preserve holomorphic structures on the sewing surfaces. Next thing is that in conformal case we should take care of 'conformal anomaly': morphism given by a surface which is sewn from two other is the composition of corresponding morphisms only up to some scalar factor, depending on *central charge* of a theory. Therefore we need to give more precise definition of a PROP of Riemann surfaces and introduce determinant line bundle. From now on all vector spaces will be over $\mathbb{C}$.

2.2.1. PROP of Riemann surfaces

Our goal in this section is to give precise definition of a PROP of Riemann surfaces $\mathcal{R}$. Because it is not clear how to sew two surfaces along boundaries, in order to define morphism in a surface PROP we need some additional structure. To define $Mor_{\mathcal{R}}$, we take closed Riemann surfaces with $n+m$ punctures (distinguished points) with parametrised neighbourhoods instead of surfaces with $n+m$ boundary components.

Definition 2.2.1. A *Riemann surface with n negatively-oriented and m positively-oriented punctures* is a tuple $(\Sigma; x_1, ..., x_n; y_1, ..., y_m; (U_1, \phi_1), ..., (U_n, \phi_n); (V_1, \psi_1), ..., (V_m, \psi_m))$, where:

- Σ is closed Riemann surface.
- $x_i \in \Sigma$, $i = 1, ..., n$, are distinguished negatively-oriented points together with open neighbourhoods $x_i \in U_i \subset \Sigma$, $i = 1, ..., n$, and maps $\phi_i : U_i \to \mathbb{C}$ biholomorphic on their images such that for each $i = 1, ..., n$, the image of ϕ_i contains unit circle (i.e. $\{z : |z| \leqslant 1\} \subset \phi_i(U_i))$ and $\phi_i(x_i) = 0$.
- $y_j \in \Sigma$, $j = 1, ..., m$, are distinguished positively-oriented points together with open neighbourhoods $y_j \in V_j \subset \Sigma$, $j = 1, ..., m$, and maps $\psi_j : V_j \to \hat{\mathbb{C}}$ biholomorphic on their images such that for each $j = 1, ..., m$, $\{z : |z| \geqslant 1\} \subset \psi_j(V_j))$ and $\psi_j(y_j) = \infty$.
- Sets U_i, $i = 1, ..., n$, may have non-empty intersection, but the preimages of certain closed disks under the maps ϕ_i, ψ_j are disjoint, i.e. sets $\phi_i^{-1}(\{z : |z| \leqslant 1\})$, $i = 1, ..., n$, $\psi_i^{-1}(\{z : |z| \geqslant 1\})$, $j = 1, ..., m$ are pairwise disjoint. The only exceptions from this rule are the 'identity' surface $\mathcal{S} = (\hat{\mathbb{C}}; 0; \infty; (\mathbb{C}, z); (\hat{\mathbb{C}} \setminus 0, z))$ (we denote here by z a standard map $\hat{\mathbb{C}} \ni z \mapsto z \in \hat{\mathbb{C}}$) and disjoint sums of other surfaces with $\mathcal{S}$.

One can think of a Riemann surface with $n+m$ parametrized punctures as of a Riemann surface with $n+m$ parametrised boundary components. Therefore the above construction with boundary components actually sewn with disks may seem unnatural, but it allows us to provide identity in a natural way. This point of view will be also useful in the next chapter, where the partial sphere operad is defined.

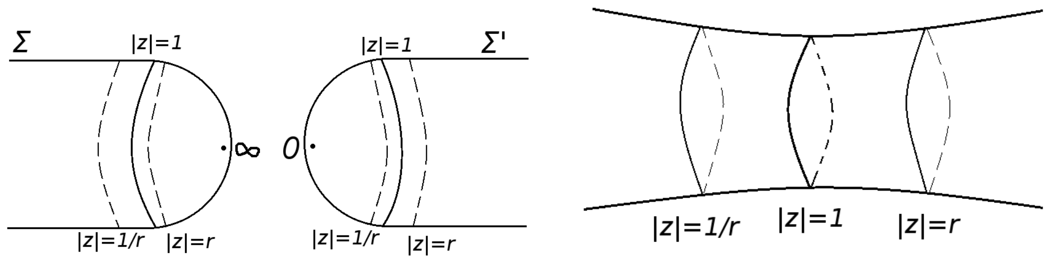
Sometimes we will say for short that a Riemann surface with n negatively-oriented and m positively-oriented punctures is just a *Riemann surface of type* (n, m).

Definition 2.2.2. We say that two Riemann surfaces of type (n, m): $(\Sigma; x_1, ..., x_n; y_1, ..., y_m; (U_1, \phi_1), ..., (U_n, \phi_n); (V_1, \psi_1), ..., (V_m, \psi_m))$, $(\Sigma'; x'_1, ..., x'_n; y'_1, ..., y'_m; (U'_1, \phi'_1), ..., (U'_n, \phi'_n); (V'_1, \psi'_1), ..., (V'_m, \psi'_m))$ are *isomorphic*, if there exists biholomorphic map $F : \Sigma \to \Sigma'$ such that $F(x_i) = x'_i$, $i = 1, ..., n$, $F(y_i) = y'_i$, $j = 1, ..., m$, and $\phi'_i \circ F = \phi_i$, $i = 1, ..., n$, $\psi'_j \circ F = \psi_j$, $j = 1, ..., m$, on some neighbourhoods of $x_1, ..., x_n$, $y_1, ..., y_n$ respectively.

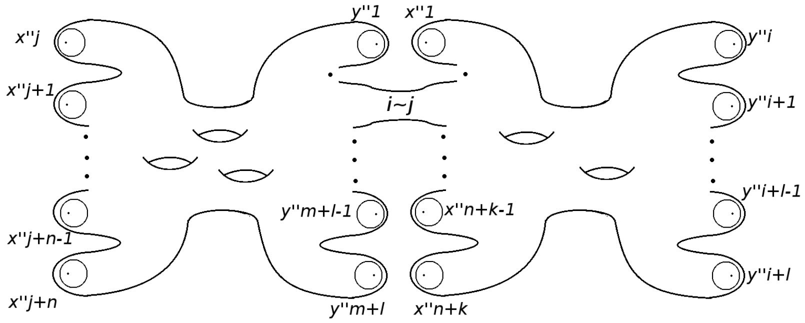
Remark 2.2.3. Note that the isomorphism class of a Riemann surface of type (n, m) is independent of a choice of neighbourhoods $U_1, ..., U_n$ and $V_1, ..., V_n$. The definition above guaranties that, because of the identity theorem, the corresponding maps $\phi'_i \circ F$ and $\psi'_j \circ F$ agree on some neighbourhoods of the preimages of $\{z : |z| \leqslant 1\}$ and $\{z : |z| \geqslant 1\}$ respectively.

In order to give a definition of a PROP $\mathcal{R}$, we only need to describe compositions of morphisms, i.e. sewing operation. Suppose we are given two Riemann surfaces $(\Sigma; x_1, ..., x_n; y_1, ..., y_m; (U_1, \phi_1), ..., (U_n, \phi_n); (V_1, \psi_1), ..., (V_m, \psi_m))$, $(\Sigma'; x'_1, ..., x'_k; y'_1, ..., y'_l; (U'_1, \phi'_1), ..., (U'_k, \phi'_k); (V'_1, \psi'_1), ..., (V'_l, \psi'_l))$ of types (n, m), (k, l) respectively, where $m \geqslant 1$ and $k \geqslant 1$. Given $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant k$, we can obtain new surface $(\Sigma''; x''_1, ..., x''_{n+k-1}; y''_1, ..., y''_{m+l-1}; (U''_1, \phi''_1), ..., (U''_{n+k-1}, \phi''_{n+k-1}); (V''_1, \psi''_1), ..., (V''_{m+l-1}, \psi''_{m+l-1}))$ of type $(n + k - 1, m + l - 1)$ in the following way:

- Choose $1 < r$ such that $\{z : |z| > 1/r\} \subset \psi_i(V_i)$ and $\{z : |z| < r\} \subset \phi'_j(U'_j)$ and let
$$\Sigma'' = (\Sigma \setminus \psi_i^{-1}(\{z : |z| \geqslant r\})) \sqcup (\Sigma' \setminus \phi_j'^{-1}(\{z : |z| \leqslant 1/r\}))/\sim$$
where the relation $\sim$ identify points $x \in \Sigma$, $y \in \Sigma' \Leftrightarrow \psi_i(x) = \phi'_j(y)$, whenever this equality makes sense (it can happen actually only when $x \in \psi_i^{-1}(\{z : 1/r < |z| < r\})$ and $y \in \phi_j'^{-1}(\{z : 1/r < |z| < r\})$). The new surface Σ'' is compact because it is homeomorphic to the connected sum of the underlying topological surfaces Σ, Σ'. It is also easy to see that the new surface Σ'' has complex structure, because on parts $\Sigma \setminus \psi_i^{-1}(\{z : |z| \geqslant r\})$ and $\Sigma' \setminus \phi_j'^{-1}(\{z : |z| \leqslant 1/r\})$ it has structures induced by complex structures of Σ and Σ' respectively and on the intersection (i.e. $\psi_i^{-1}(\{z : 1/r < |z| < r\}) = \phi_j'^{-1}(\{z : 1/r < |z| < r\})$) both structures agree.



- Let $x''_s = \begin{cases} x'_s & \text{for } 0 < s < j \\ x_{s-j+1} & \text{for } j \leqslant s < j + n \\ x'_{s-n+1} & \text{for } j + n \leqslant s < n + k \end{cases}$ and $y_t = \begin{cases} y_t & \text{for } 0 < t < i \\ y'_{t-i+1} & \text{for } i \leqslant t < i + l \\ y_{t-l+1} & \text{for } i + l \leqslant t < l + m \end{cases}$



- Maps and neighbourhoods (U''_i, ϕ''_i), $1 \leqslant i < m+l$ and (V''_j, ψ''_j), $1 \leqslant j < n+k$, are the same as maps and neighbourhoods of the corresponding points in Σ and Σ', maybe after reducing the neighbourhoods because of deleting open sets $\psi_i^{-1}(\{z : |z| \geqslant r\})$ and $\phi_j'^{-1}(\{z : |z| \leqslant 1/r\})$. Because deleted sets are contained in the preimages of certain open disks, all new maps contain disks $\{z : |z| \leqslant 1\}$ or $\{z : |z| \geqslant 1\}$ in their images. Preimages of interiors of these disks are also disjoint because of the same reason.

It is easy to see that the whole construction is independent of the choice of constant r. We often denote this new surface as $\Sigma_i \#_j \Sigma'$. Moreover, if we have surface Σ of type (n, m), where $n, m > 0$, we can obtain in an analogous way new surface $\Sigma_{i\sim j}$ ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$) by sewing positively and negatively oriented punctures of Σ. It is always well defined except in the case of the 'identity' surface $\mathcal{S}$. To see what happens, consider the underlying topological situation: in fact we try to sew both sides of a circle (degenerated surface with two boundary components). As a result the surface $\mathcal{S}_{1\sim 1}$ is in fact an open annulus $\{z : 1/r < |z| < r\}$, so it is neither independent of r, nor closed. The reason is that we do not assume that preimages of a closed disks $\{z : |z| \leqslant 1\}$ and $\{z : |z| \geqslant 1\}$ have empty intersection in $\mathcal{S}$. It means that, if we consider the underlying topological situation with certain open disks removed, this is the degenerate case. However, the sewing operation for one surface will be always well defined in our applications.

Now we are ready to give proper definition of a PROP $\mathcal{R}$:

Definition 2.2.4. *PROP of Riemann surfaces* $\mathcal{R}$ is a PROP where $Mor_{\mathcal{R}}(n, m)$ is the set of equivalence classes of closed Riemann surfaces of type (n, m), the identity is the surface $\mathcal{S} = (\hat{\mathbb{C}}; 0; \infty; (\mathbb{C}, z); (\hat{\mathbb{C}} \setminus 0, z))$, and tensor product of morphisms is given by taking disjoint union of two surfaces. Given two surfaces Σ, Σ' of types (n, m) and (m, l) respectively, we define the composition of morphisms:

$$\Sigma' \circ \Sigma = (\Sigma_m \#_m \Sigma')_{(m-1)\sim(m-1),(m-2)\sim(m-2),\ldots,1\sim 1}$$

Remark 2.2.5. In fact the PROP $\mathcal{R}$ is topological, i.e. space $Mor_{\mathcal{R}}(n, m)$ is topological for every $n, m \in \mathbb{N}$ with one connected component for every topological type of surfaces. We only sketch the construction here, the details can be found in [Seg]. Denote by $\mathcal{M}_{g,k}$ the moduli space of all closed surfaces of genus g with k distinguished points and prescribed tangent vectors at these points; it can be given the topological structure. One can construct the space $\mathcal{B}_{g,n,m}$ of all genus g surfaces of type (n, m) by constructing a fibration over $\mathcal{M}_{g,n+m}$, whose fiber over $([\Sigma]; x_1, ..., x_{n+m}; \xi_1, ..., \xi_{n+m})$ is the set of all holomorphic embeddings of $n + m$ closed disks into Σ satisfying all conditions described in the definition of PROP $\mathcal{R}$ and such that their derivatives in 0 are $\xi_1, ..., \xi_{n+m}$. This construction can be also improved to give the space $\mathcal{B}_{g,n,m}$ a holomorphic structure. In the case of genus-zero there is a simple method to give the space $\mathcal{B}_{0,n,m}$ the structure of infinite dimensional complex Banach manifold; for more details see [Hu]. Moreover, one can show that the composition of morphisms induce a holomorphic map

$$\sqcup_{g_1,g_2\in\mathbb{N}}\mathcal{B}_{g_1,n,m} \times \mathcal{B}_{g_2,m,k} \to \sqcup_{g\in\mathbb{N}}\mathcal{B}_{g,n,k}$$

for $n, m, k \in \mathbb{N}$.

2.2.2. Determinant lines

In this section we define the notion of a determinant line over a Riemann surface and show that PROP $\mathcal{R}$ can be enriched in order to carry this additional structure. All the proofs are technical and usually long, so we give here only sketches of corresponding constructions. All the details can be found in appendix D in [Hu] and in [Seg].

In fact a determinant line is not a straightforward notion. It may refer to a finite dimensional vector space, a Fredholm operator between Hilbert spaces or a Riemann surface. In all listed cases we have the functor from these categories to the category of graded lines. The determinant line of an object is, briefly speaking, the image of this object under corresponding functor. We begin therefore with the definition of graded line.

Definition 2.2.6. *A graded line* $(L, \deg L)$ is a one dimensional vector space together with an element of $\mathbb{Z}_2$ called the *degree*.

Graded lines form a category $\mathcal{G}$ in a natural way. In fact it is a symmetric monoidal category, because given two lines L_1, L_2 we can define the tensor product of these lines $L_1 \otimes L_2$ of the degree $\deg L_1 + \deg L_2$. Symmetry of graded lines is given by a natural isomorphism $\mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ defined as:

$$L_1 \otimes L_2 \ni v_1 \otimes v_2 \mapsto (-1)^{\deg L_1 \deg L_2} v_2 \otimes v_1 \in L_2 \otimes L_1$$

Next we define the functors from several categories to the category of graded lines. Before we proceed, we need to make remark about isomorphisms of graded lines. Because all one dimensional vector spaces are isomorphic, every two graded lines are isomorphic whenever they have the same degree. We will say that two graded lines are *canonically isomorphic*, if the isomorphism comes from a natural transformation of a functors from some category to $\mathcal{G}$.

Definition 2.2.7. let V be a finite dimensional vector space of dimension n. A *determinant line* Det V of V is a graded line $\bigwedge^n V$ of degree $(n \mod 2)$.

It is obvious that the assignment defined above can be extended to a functor to $\mathcal{G}$ from the category with finite dimensional vector spaces as objects and linear isomorphisms as morphisms. We extend also this definition to a (formal) 0-dimensional vector spaces by $\bigwedge^0 V = \mathbb{C}$.

Now let H_1 and H_2 be Hilbert spaces.

Definition 2.2.8. A bounded linear operator $F : H_1 \to H_2$ is called a *Fredholm operator*, if its kernel and cokernel are finite dimensional. The *index* of a Fredholm operator is the number $\mathrm{Index}F = dim\mathrm{Ker}F - dim\mathrm{Coker}F$.

Definition 2.2.9. Let $F_1 : H_1 \to H_2$ and $F_2 : H_3 \to H_4$ be two Fredholm operators. An *equivalence from* F_1 *to* F_2 is a pair of bounded linear isomorphisms $f_1 : H_1 \to H_3$, $f_2 : H_2 \to H_4$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1 & \xrightarrow{F_1} & H_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ H_3 & \xrightarrow{F_2} & H_4 \end{array}$$

Definition 2.2.10. A *determinant line* of a Fredholm operator F is a graded line:

$$\mathrm{Det}\ F = \mathrm{Det}\ (\mathrm{Ker}F)^* \otimes \mathrm{Det}\ (\mathrm{Coker}F)$$

with the degree $\mathrm{Index}F \mod 2$.

We can extend this definition to have a functor to $\mathcal{G}$ from the category of Fredholm operators as objects and equivalences as morphisms. Note that if F is an isomorphism we have $\mathrm{Det}\ F = \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ (because $\bigwedge^0 \mathrm{Ker}F = \bigwedge^0 \mathrm{Coker}F = \mathbb{C}$). We state one of the most important properties of the determinant lines of Fredholm operators.

Proposition 2.2.11 (Snake lemma)**.** *Let H_i, $i = 1, ..., 6$, be Hilbert spaces. Consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & 0 \\ & & {\scriptstyle F_1}\downarrow & & {\scriptstyle F_2}\downarrow & & {\scriptstyle F_3}\downarrow & & \\ 0 & \longrightarrow & H_4 & \longrightarrow & H_5 & \longrightarrow & H_6 & \longrightarrow & 0 \end{array}$$

where F_1, F_2, F_3 are bounded linear operators, and the horizontal lines are exact. If two of the operators F_1, F_2, F_3 are Fredholm, so is the third. Moreover, there is a canonical isomorphism Det $F_1 \otimes$ Det $F_3 \cong$ Det F_2.

Now we turn again to the case of Riemann surfaces. We define the determinant line of a surface $\mathcal{S}$ to be just $\mathbb{C}$ of degree 0. From now on we will consider here only non-degenerated surfaces. If $(\Sigma; x_1, ..., x_n; y_1, ..., y_m; (U_1, \phi_1), ..., (U_n, \phi_n); (V_1, \psi_1), ..., (V_m, \psi_m))$ is a Riemann surface with punctures, we denote by Σ^c compact Riemann surface with boundary obtained from Σ by deleting the images of open disks $\{z : |z| < 1\}$ under

$\phi_1^{-1}, ..., \phi_n^{-1}$ and $\{z : |z| > 1\}$ under $\psi_1^{-1}, ..., \psi_m^{-1}$. Then we have the Cauchy-Riemann operator $\overline{\partial} : \Omega^0(\Sigma^c) \to \Omega^{0,1}(\Sigma^c)$, where $\Omega^0(\Sigma^c)$ is the space of smooth functions on Σ^c and $\Omega^{0,1}(\Sigma^c)$ is the space of $(0,1)$-forms on Σ^c (i.e. forms that locally can be written as $f(x,y)d\overline{z}$, where $z = x + iy$).

Let $C_1, C_2, ..., C_n$ be a negatively oriented connected components of $\partial\Sigma^c$ (i.e. preimages of set $\{z : |z| = 1\}$ under $\phi_1, ..., \phi_n$) and $D_1, ..., D_m$ be a positively oriented components of $\partial\Sigma^c$. We denote by $\Omega^0_+(C_i)$, $i = 1, ..., n$, the set of smooth functions on C_i (viewed as smooth functions on S^1 via the parametrizations ϕ_i) such that their Fourier expansions are of the form $\sum_{n\in\mathbb{Z}_-} a_n e^{2\pi n\theta i}$, where θ is the usual parametrisation of S^1 by angles. Analogously, if $D_1, ..., D_m$ are positively oriented connected components of $\partial\Sigma^c$, we denote by $\Omega^0_+(D_j)$, $j = 1, ..., m$ the set of smooth functions on D_j such that their Fourier expansions are of the form $\sum_{n\in\mathbb{N}} a_n e^{2\pi n\theta i}$. Finally, we define $\Omega^0_+(\partial\Sigma^c) = \oplus_{i=1}^n \Omega^0_+(C_i) \oplus_{j=1}^m \Omega^0_+(D_j)$. In an analogous way we can define the space $\Omega^0_-(\partial\Sigma^c)$ as consisting functions on $\partial\Sigma^c$ which are of the form $\sum_{n\in\mathbb{N}} a_n e^{2\pi n\theta i}$ on negatively oriented components and $\sum_{n\in\mathbb{Z}_-} a_n e^{2\pi n\theta i}$ on positively oriented ones. We denote by $\mathrm{pr} : \Omega^0(\Sigma^c) \to \Omega^0_+(\partial\Sigma^c)$ the composition of a restriction operator $\Omega^0(\Sigma^c) \to \Omega^0(\partial\Sigma^c)$ with a projection $\Omega^0(\partial\Sigma^c) \to \Omega^0_+(\partial\Sigma^c)$. We would like to define the determinant line of a surface Σ^c as the determinant line of the operator

$$\overline{\partial} \oplus \mathrm{pr} : \Omega^0(\Sigma^c) \to \Omega^{0,1}(\Sigma^c) \oplus \Omega^0_+(\partial\Sigma^c)$$

However, it is not possible to do it explicitly, because the spaces $\Omega^0(\Sigma^c)$ and $\Omega^{0,1}(\Sigma^c) \oplus \Omega^0_+(\partial\Sigma^c)$ are not Hilbert, therefore we need to extend them to some reasonable Hilbert spaces.

Definition 2.2.12. Let U be an open subset of $\mathbb{R}^n$. A *Sobolev space* $H_{(k)}(U)$, $k \in \mathbb{R}$, $k \geqslant 0$, is the space

$$H_{(k)}(U) = \{f \in L^2(U) : (1 + |\xi|^2)^{\frac{k}{2}} \mathcal{F}(f) \in L^2(\mathbb{R}^n)\}$$

where $\mathcal{F}$ is the Fourier transform (i.e. $\mathcal{F}(f)(\xi) = (2\pi)^{-k/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx$, at least for functions belonging to $L^1(U)$). It is a Hilbert space with the product

$$\langle f, g\rangle = \langle (1 + |\xi|^2)^{\frac{k}{2}} \mathcal{F}(f), (1 + |\xi|^2)^{\frac{k}{2}} \mathcal{F}(g)\rangle_{L^2}$$

We can also define $H_{(k)}(\overline{\mathbb{R}}^n_+)$ to be $H_{(k)}(\mathbb{R}^n)/H_{(k)}(\mathbb{R}^n_-)$ with quotient topology (where $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$).

Definition 2.2.13. Let M be a smooth compact manifold (possibly with boundary). We define a *Sobolev space* $H_{(k)}(M)$ as the set of all functions f s.t. for every chart (U, ϕ) and every function $\psi \in C^\infty_{comp}(U)$ (functions in $C^\infty(U)$ with compact support) the function $\phi_*(\psi f) \in H_{(k)}(\mathbb{R}^n)$, where $\phi_* = (\phi^{-1})^*$ (respectively $H_{(k)}(\overline{\mathbb{R}}^n_+)$, if the chart contains boundary).

In fact, these spaces also carry Hilbert space structure. The details on the defined Sobolev spaces can be found, for example, in [Hö]. Using them, we have the following result.

Proposition 2.2.14. *For any $k \geqslant 1$ the operator $\overline{\partial} \oplus pr$ can be extended uniquely to the Fredholm operator*

$$H_{(k)}(\overline{\partial} \oplus pr) : H_{(k)}(\Sigma^c) \to (H_{(k-1)}(\Sigma^c) \otimes \Omega^{0,1}(\Sigma^c)) \oplus H_{(k-1/2)}(\partial\Sigma^c)$$

The kernel of this extension is equal to the kernel of $\overline{\partial} \oplus pr$ and the cokernel is canonically isomorphic to the cokernel of $\overline{\partial} \oplus pr$.

In fact the kernel of the operator $\overline{\partial} \oplus \mathrm{pr}$, hence also of $H_{(k)}(\overline{\partial} \oplus \mathrm{pr})$, can be easily described: it is the set of constant functions if Σ is of type $(n, 0)$, $n \in \mathbb{N}$, and 0 otherwise.

If Σ is of type $(0, 0)$, $\ker \overline{\partial} \oplus \mathrm{pr} = \ker \overline{\partial}$. Let $f \in \ker \overline{\partial}$. The condition that $f \in \ker \overline{\partial}$ means that f is holomorphic. Because f is defined on a closed surface it is constant because of the maximum principle.

Suppose now that Σ is of type $(n, 0)$, $n \geqslant 1$ and $f \in \ker \overline{\partial} \oplus \mathrm{pr}$. Then, using the parametrisations $\phi_1, ..., \phi_n$ we see, that on every circle C_i, $i = 1, ..., n$, f can be written as $\sum_{n\in\mathbb{N}} a_{i,n} z^n$, therefore it can be extended (using map ϕ_i and identity theorem) to a function on Σ by setting $f \circ \phi_i^{-1}(z) = \sum_{n\in\mathbb{N}} a_{i,n} z^n$. Hence it must be constant. We can give the same proof in the case of Σ of type (n, m) for $m \geqslant 1$, however, in this case $f \in \ker \overline{\partial} \oplus \mathrm{pr}$ must be zero because it is of the form $\sum_{n\in\mathbb{Z}_-} a_{j,n} z^n$ on every set $\psi_j^{-1}(\{z : |z| \geqslant 1\})$.

We are now ready to give a proper definition of Det $_\Sigma$.

Definition 2.2.15. A *determinant line of a surface* Σ is the determinant line of the Fredholm operator $H_{(k)}(\overline{\partial} \oplus \mathrm{pr})$, i.e. $\mathrm{Det}\ \Sigma := \mathrm{Det}\ H_{(k)}(\overline{\partial} \oplus \mathrm{pr})$.

Let $Hol(\Sigma^c)$ be the set of holomorphic functions defined on a surface Σ^c. In case $(n, m) \neq (0, 0)$ we have the following alternative description of determinant line over Σ:

Proposition 2.2.16. *For any $s \geqslant 1$ the operator*

$$\pi_{\Sigma^c} : Hol(\Sigma^c) \to \Omega^0_+(\partial\Sigma^c)$$

can be extended to a Fredholm operator $\overline{\pi}_{\Sigma^c(s)} : \overline{Hol}_{(s)}(\Sigma^c) \to H_{(s-1/2)}$, where $\overline{Hol}_{(s)}(\Sigma^c)$ is the completion of $Hol(\Sigma)$ in $H_{(s)}(\Sigma^c)$. The kernel and cokernel of this extension are canonically isomorphic to the kernel and cokernel of π_{Σ^c} respectively. Moreover, there is a canonical isomorphism $Det\ \Sigma \cong Det\ \pi_{\Sigma^c}$.

Proof. The key of the proof is the fact that the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hol(\Sigma^c) & \longrightarrow & \Omega^0(\Sigma^c) & \longrightarrow & \Omega^{0,1}(\Sigma^c) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\Sigma^c}} & & \downarrow{\scriptstyle \overline{\partial}\oplus\mathrm{pr}} & & \downarrow{\scriptstyle Id_{\Omega^{0,1}(\Sigma^c)}} & & \\ 0 & \longrightarrow & \Omega^0_+(\partial\Sigma^c) & \longrightarrow & \Omega^{0,1}(\Sigma^c) \oplus \Omega^0_+(\partial\Sigma^c) & \longrightarrow & \Omega^{0,1}(\Sigma^c) & \longrightarrow & 0 \end{array}$$

induces the commutative diagram of the corresponding Sobolev spaces. Then one can prove that the extension $\overline{\pi}_{\Sigma^c(s)}$ has the desired properties and, using the snake lemma, we obtain the canonical isomorphism $\mathrm{Det}\ \Sigma \cong \mathrm{Det}\ \pi_{\Sigma^c} \otimes \mathrm{Det}\ Id_{\Omega^{0,1}(\Sigma^c)} \cong \mathrm{Det}\ \pi_{\Sigma^c}$. $\square$

One of the most important properties of determinant lines of Riemann surfaces is the existence of sewing isomorphism. We have the following result.

Theorem 2.2.17. *For any Σ, Σ' of types (n,m), (k,l) respectively $(m,k \geqslant 1)$ and $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, we have the canonical isomorphism*

$$l^{i,j}_{\Sigma,\Sigma'} : Det\ \Sigma \otimes Det\ \Sigma' \to Det\ \Sigma_i \#_j \Sigma'.$$

Proof. The above theorem is a consequence of the following three lemmas.

Lemma 2.2.18. *The rows of the following commutative diagram are exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hol(\Sigma^c_i \#_j \Sigma'^c) & \longrightarrow & Hol(\Sigma^c \sqcup \Sigma'^c) & \xrightarrow{\Delta_{\Sigma^c,\Sigma'^c}} & Hol(S) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{\Sigma_i \#_j \Sigma'}} & & \downarrow{\scriptstyle \overline{\pi}_{\Sigma^c,\Sigma'^c}} & & \downarrow{\scriptstyle Id_{Hol(S)}} & & \\
0 & \longrightarrow & \Omega^0_+(\partial(\Sigma^c_i \#_j \Sigma'^c)) & \longrightarrow & \Omega^0_+(\partial(\Sigma^c_i \#_j \Sigma'^c)) \oplus Hol(S) & \longrightarrow & Hol(S) & \longrightarrow & 0
\end{array}$$

where $S = D_i = C'_j \subset \Sigma_i \#_j \Sigma'$, $\Delta_{\Sigma^c,\Sigma'^c}(f) = f|_{D_i} - f|_{C'_j}$ for $f \in Hol(\Sigma^c \sqcup \Sigma'^c)$ and $\overline{\pi}_{\Sigma^c,\Sigma'^c}(f) = (f|_{\partial(\Sigma^c_i \#_j \Sigma'^c)}) \oplus \Delta_{\Sigma^c,\Sigma'^c}(f)$.

Lemma 2.2.19. *The operator $\pi_{\Sigma \sqcup \Sigma'} - \overline{\pi}_{\Sigma^c,\Sigma'^c}$ is of a trace-class.*

Lemma 2.2.20. *Let H_1, H_2 be Hilbert spaces. If $F : H_1 \to H_2$ is a Fredholm operator and $T : H_1 \to H_2$ is a trace-class operator, there is a canonical isomorphism $Det\ F \cong Det\ (F + T)$ such that if $T' : H_1 \to H_2$ is another trace-class operator then the following diagram of canonical isomorphisms is commutative:*

$$\begin{array}{ccc}
Det\ (F + T + T') & \longrightarrow & Det\ (F + T) \\
\downarrow & & \downarrow \\
Det\ (F + T') & \longrightarrow & Det\ F
\end{array}$$

Indeed, from the lemma 2.2.18 we have an isomorphism Det $\Sigma_i \#_j \Sigma' \cong$ Det $\overline{\pi}_{\Sigma^c,\Sigma'^c}$. The lemmas 2.2.19 and 2.2.20 give an isomorphism $Det \overline{\pi}_{\Sigma^c,\Sigma'^c} \cong$ Det $\pi_{\Sigma \sqcup \Sigma'}$. Finally, Det $\pi_{\Sigma \sqcup \Sigma'} \cong$ Det $\Sigma \otimes$ Det Σ' because of the obvious commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hol(\Sigma^c) & \longrightarrow & Hol(\Sigma^c \sqcup \Sigma'^c) & \longrightarrow & Hol(\Sigma'^c) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{\Sigma}} & & \downarrow{\scriptstyle \pi_{\Sigma^c \sqcup \Sigma'^c}} & & \downarrow{\scriptstyle \pi_{\Sigma'^c}} & & \\
0 & \longrightarrow & \Omega^0_+(\partial \Sigma^c) & \longrightarrow & \Omega^0_+(\partial(\Sigma^c \sqcup \Sigma'^c)) & \longrightarrow & \Omega^0_+(\partial \Sigma^c) & \longrightarrow & 0
\end{array}$$

where $Hol(\Sigma^c \sqcup \Sigma'^c) \cong Hol(\Sigma^c) \oplus Hol(\Sigma'^c)$ and $\Omega^0_+(\partial(\Sigma^c \sqcup \Sigma'^c)) \cong \Omega^0_+(\partial\Sigma^c) \oplus \Omega^0_+(\partial\Sigma'^c)$.
□

Theorem 2.2.21. *Suppose Σ, Σ' and Σ'' are Riemann surfaces of types (n,m), (k,l) and (p,q) respectively ($m,l,k,p \geqslant 1$) and $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, $1 \leqslant s \leqslant l$, $1 \leqslant t \leqslant p$. Then the following diagram of canonical isomorphism is commutative:*

$$\begin{array}{ccc} Det\ \Sigma \otimes Det\ \Sigma' \otimes Det\ \Sigma'' & \longrightarrow & Det\ \Sigma_i \#_j \Sigma' \otimes Det\ \Sigma'' \\ \downarrow & & \downarrow \\ Det\ \Sigma \otimes Det\ \Sigma'_s \#_t \Sigma'' & \longrightarrow & Det\ \Sigma_i \#_j \Sigma'_s \#_t \Sigma'' \end{array}$$

Finally, we can repeat our reasoning with slight modifications to conclude that there are also canonical isomorphisms Det $\Sigma \cong$ Det $\Sigma_{(i_1 \sim j_1),\ldots,(i_k \sim j_k)}$ whenever this formula makes sense. Hence we obtain

Theorem 2.2.22. *Let Σ, Σ' be of types (n,m), (m,l) respectively. Then there is a canonical isomorphism $l_{\Sigma,\Sigma'} : Det\ \Sigma \otimes Det\ \Sigma' \cong Det\ \Sigma' \circ \Sigma$, where*

$$\Sigma' \circ \Sigma = (\Sigma_m \#_m \Sigma')_{(m-1)\sim(m-1),(m-2)\sim(m-2),\ldots,1\sim 1}.$$

We would like to make one more remark, which will help us in defining CFT, and will be also useful later. Recall that if Σ is not of the type $(0,0)$ the kernel of the map $\pi_{\Sigma^c} : Hol(\Sigma^c) \to \Omega^0_+(\partial\Sigma^c)$ is 0 if Σ has at least one positively oriented puncture and $\mathbb{C}$ (constant functions) otherwise. If Σ is of genus 0, the cokernel of this map is also known.

Lemma 2.2.23. *If Σ is of genus 0 and is not of the type $(0,0)$, the cokernel of the map π_{Σ^c} is 0.*

As a consequence we have the following fact:

Proposition 2.2.24. *If $m \geqslant 1$, $\sqcup_{[\Sigma]\in\mathcal{B}_{0,n,m}} Det\ [\Sigma]$ is in canonical bijection with $\mathcal{B}_{0,n,m} \times \mathbb{C}$.*

Remark 2.2.25. The above proposition shows, in fact, that Det forms a trivial holomorphic line bundle over $\mathcal{B}_{0,n,m}$. One can show also that if we denote by Det $\mathcal{B}_{0,n,m}$ the mentioned bundle, the sewing map Det $\mathcal{B}_{0,n,m} \otimes$ Det $\mathcal{B}_{0,m,l} \to$ Det $\mathcal{B}_{0,n,l}$ for $n,m,l \in \mathbb{N}$ defined as $([\Sigma],\lambda) \otimes ([\Sigma'],\lambda') \mapsto (\Sigma' \circ \Sigma, l_{\Sigma,\Sigma'}(\lambda \otimes \lambda'))$ is holomorphic.

Moreover, because the bijection in the above proposition is canonical, we obtain a canonical flat connection on $\mathcal{B}_{0,n,m}$. We can ask whether this connection is compatible with the sewing map, i.e. the canonical connection on Det $\mathcal{B}_{0,n,l}$ on the image of the sewing map is equal to the push-forward of sum of canonical connections on Det $\mathcal{B}_{0,n,m}$ and Det $\mathcal{B}_{0,m,l}$. It turns out to be not true in general. However, if one restrics the question to the case where all surfaces are just Riemann spheres $\hat{\mathbb{C}}$ and prametrizations are given only by dilations and rotations (i.e. they are of the form $z \mapsto az$, $a \in \mathbb{C}^\times$), the canonical connections are actually compatible with the sewing map. More generally, it is true as long as one restricts attention to the Riemann spheres with arbitrary fixed (up to dilations and rotations) parametrisations. The details can be found in [Hu].

2.2.3. The definition of a conformal field theory

Now we are ready to give the definition of a conformal field theory. The definition may differ from the definitions given in the literature, the closest definition is given for example in [Seg]. Although in our definition we forget about some structure mentioned in the definition given in [Seg], we will point out the differences between these two definitions and indicate how to complete the definition given here.

We begin with the definition of an underlying PROP.

Definition 2.2.26. A *PROP of Riemann surfaces with determinant lines of central charge c* ($c \in \mathbb{Z}$) $\mathrm{Det}^c\mathcal{R}$ is the PROP where the morphisms $Mor_{\mathcal{H}}(n,m)$ are pairs (Σ, λ), where Σ is a Riemann surface of type (n,m) and $\lambda \in \mathrm{Det}^c\Sigma$, the c-th tensor power of $\mathrm{Det}\ \Sigma$ (where Det^{-1} is Det^*). We define the composition of morphism by:

$$(\Sigma', \lambda') \circ (\Sigma, \lambda) = (\Sigma' \circ \Sigma, l^c_{\Sigma,\Sigma'}(\lambda \otimes \lambda'))$$

where $l^c_{\Sigma,\Sigma'} = l_{\Sigma,\Sigma'} \otimes ... \otimes l_{\Sigma,\Sigma'} : \mathrm{Det}^c\Sigma \otimes \mathrm{Det}^c\Sigma' \cong \mathrm{Det}^c\Sigma' \circ \Sigma$ is the tensor power of c canonical isomorphisms $l_{\Sigma,\Sigma'}$.

One can actually extend the definition to the case of complex parameter c but we will not do this here. However, there is a good description of a Det^c in the case of genus-zero surfaces, which we will present it in the next chapter.

For our purposes it is convenient to give one more definition. We see that every set of morphisms in the PROP $\mathrm{Det}^c\mathcal{R}$ has a $\mathbb{C}$-bundle structure. In particular, the multiplication by elements of $\mathbb{C}$ is defined. It is obviously also well defined for every $\mathbb{C}$-linear vector space. Hence we can give the following definition.

Definition 2.2.27. Let k be a field, and $\mathcal{P}$ a PROP such that every set of morphisms $Mor_{\mathcal{P}}(n,m)$, $n,m \in \mathbb{N}$, is equipped with a k-bundle structure. We say that an algebra (V, ψ) over a PROP $\mathcal{P}$ is *fiberwise linear*, if the $\mathcal{P}$-algebra structure is compatible with the bundle structure, in the sense that for $(h, \lambda) \in Mor_{\mathcal{P}}(n,m)$, $\psi((h, d \cdot \lambda)) = d\psi((h, \lambda))$, where $d \in k$ and λ is the element of the fiber over h.

Now we are ready to give the definition of CFT:

Definition 2.2.28. Let V be a topological vector space over $\mathbb{C}$. A *conformal field theory of central charge* $2c$ ($c \in \mathbb{Z}$) is the continuous fiberwise linear algebra (V, ψ) over a PROP $\mathrm{Det}^c\mathcal{R}$.

The continuity assumption means that all the maps $\psi_{n,m}$ defining a morphism $\psi : \mathrm{Det}^c\mathcal{R} \to \mathcal{E}_V$ are continuous. It is in fact quite weak assumption; if we could define a suitable holomorphic structures on all sets of morphisms in PROPs $\mathrm{Det}^c\mathcal{R}$ (this is satisfied by remark 2.2.5) and $\mathcal{E}_V$, we would demand that the maps $\psi_{n,m}$ are holomorphic.

The natural question is what is the underlying algebraic structure. However, it is much more difficult question than in the case of TFT. There is even a problem with constructing nontrivial examples of CFTs. In the next chapter we give a partial answer to this question. It shows the complexity of a possible general answer, but on the other hand it shows connections with other areas of mathematics, rather unexpected when looking only on the geometric construction above.

Remark 2.2.29. We can correct the above definition to be compatible with the definition given in [Seg]. To do this we need to add the assumptions that we have a hermitian product on V and all the operators in the image of ψ (except the identities) are of a trace-class. Another axiom is that we have also the morphism $Mor_{\mathrm{Det}^c\mathcal{R}}(n,m) \to Mor_{\mathrm{Det}^c\mathcal{R}}(n-k,m-k)$ for $k \leqslant n,m$, induced by the sewing k boundary components in Σ^c, where Σ is of type (n,m) (i.e. an operation $\Sigma \mapsto \Sigma_{(i_1\sim j_1),\ldots,(i_k\sim j_k)})$, and this morphism lifts to a morphism $Hom(V^{\otimes n}, V^{\otimes m}) \to Hom(V^{\otimes n-k}, V^{\otimes m-k})$ obtained by taking the partial trace of $Hom(V^{\otimes k}, V^{\otimes k})$. The last one is that we have a morphism $Mor_{\mathrm{Det}^c\mathcal{R}}(n,m) \to Mor_{\mathrm{Det}^c\mathcal{R}}(m,n)$ induced by the reversing complex structures on the underlying Riemann surfaces, which lifts to the morphism $Hom(V^{\otimes n}, V^{\ otimesm}) \to Hom(V^{\otimes m}, V^{\otimes n})$ obtained by taking the adjoint homomorphisms.

Remark 2.2.30. The original definition in [Seg], instead of using the determinant lines, was given by assuming that the images of ψ lie not in the spaces $Hom(V^{\otimes n}, V^{\otimes m})$, but in the projective spaces of these. Either approach leads to the definition where in fact the image of composition of morphisms is defined only up to some scalar factor, but the determinant lines give us in most cases better control of the situation. In general it may not be true, but we will see in the next chapter that because Det forms a trivial line bundle over $\mathcal{B}_{0,n,1}$ we have in this case a canonical section of this bundle, therefore we can compare the results obtained by composition of morphisms more precisely. Moreover, the determinant lines (and complex powers of these) are the only bundles that can extend our base PROP to give a reasonable algebra, where the compositions of morphism are defined only up to some scalar factor. This is a consequence of Mumford-Segal theorem; for more precise formulation and proof see [Seg] or [Hu].

Chapter 3

Partial sphere operad

In this chapter we define the partial sphere operad and formulate an isomorphisms theorem which states that every fiberwise linear algebra over the partial operad of spheres with determinant lines is equivalent to a vertex operator algebra. Partial sphere operad is just the operad obtained by taking the genus-zero part from the PROP $\mathcal{R}$ of Riemann surfaces with exactly one positive puncture. However, we also weaken the assumption that all the parametrisations must contain closed disks in their images. Because of this fact the sewing operation is defined only partially, but this operad turns out to be $\mathbb{C}^{\times}$-rescalable. Good description of such operad and the isomorphism theorem is given in [Hu]. Vertex operator algebras, on the other hand, are rather complicated generalizations of the commutative associative algebras. They were originally introduced to explain the nature of Griess-Fischer Monster group, but vertex operators were previously used by physicists. It turned out that this group can be presented as the group of automorphisms of a Moonshine module, which has a structure of a vertex operator algebra. This result is presented in [FLM]. Later on more examples were found. For more details, see [BF].

3.1. Construction of an operad

As mentioned above, the partial sphere operad is obtained by taking genus-zero parts of $Mor_{\mathcal{R}}(n,1)$ ($n \in \mathbb{N}$) from the PROP $\mathcal{R}$ of Riemann surfaces and by weakening the assumption that the images of parametrisations around the punctures need to contain closed disks. We formulate this more precisely.

Definition 3.1.1. The *sphere with tubes* of type $(n,1)$, $n \in \mathbb{N}$, is a tuple $\mathcal{Q} = (\mathcal{S}; p_0, p_1, ..., p_n; (\phi_0, U_0), ..., (\phi_n, U_n))$, where:

- $\mathcal{S}$ is a Riemann surface of genus zero;
- $p_0, ..., p_n \in \mathcal{S}$, where all these points are different;
- $\mathcal{S} \supset U_i$, $i = 0, ..., n$ are open neighbourhoods of p_i, $i = 1, ..., n$ respectively, such that they do not contain any other distinguished point (i.e. $p_0, ..., p_{i-1}, p_{i+1}, ..., p_n \notin U_i$);

- $\phi_i : U_i \to \hat{\mathbb{C}}$, $i = 0, ..., n$ are biholomorphic maps on their images such that $\phi_0(p_0) = \infty$ and $\phi_i(p_i) = 0$ for $i \geqslant 1$.

The set of all spheres with tubes of type $(n, 1)$ will be denoted as $K(n)$.

We define the isomorphism between two such spheres analogously as in Definition (2.2.2), so the isomorphism class of two spheres with tubes in fact does not depend on the neighbourhoods U_i, $i = 1, ..., n$. Therefore we will omit writing down these sets unless it will be necessary.

To obtain the operad structure, we only need to define the substitution maps. In this case it will be much easier to define single substitution maps. Let $\mathcal{Q} = (\mathcal{S}_1; p_0, p_1, ..., p_n; (\phi_0, U_0), ..., (\phi_n, U_n))$, $\mathcal{Q}' = (\mathcal{S}_2; q_0, q_1, ..., q_m; (\psi_0, V_0), ..., (\psi_m, V_m))$ be two spheres with tubes of types $(n, 1)$ and $(m, 1)$ respectively, where $n \geqslant 1$. Let $1 \leqslant i \leqslant n$ and assume that there exists a constant $r > 0$ such that $\{z : |z| \leqslant r\} \subset \phi_i(U_i)$ and $\{z : |z| \geqslant r\} \subset \psi_0(V_0)$. Then we say that *the spheres with tubes $\mathcal{Q}$, $\mathcal{Q}'$ can be sewn along the i-th tube of $\mathcal{Q}$*. We denote their composition as $\mathcal{Q}_i\infty_0\mathcal{Q}' = (\mathcal{S}_3; p_0, p_1, ..., p_{i-1}, q_1, ..., q_m, p_{i+1}, ..., p_n; (\phi_0, U_0'), ..., (\phi_{i-1}, U_{i-1}'), (\psi_1, V_1'), ..., (\psi_m, V_m'), (\phi_{i+1}, U_{i+1}'), ..., (\phi_n, U_n'))$. We construct $\mathcal{S}_3$ as follows: choose r_1, r_2 such that $r_1 < r < r_2$ and $\{z : |z| < r_2\} \subset \phi_i(U_i)$ and $\{z : |z| > r_1\} \subset \psi_0(V_0)$ and let

$$\mathcal{S}_3 = (\mathcal{S}_1 \setminus \phi_i^{-1}(\{z : |z| \leqslant r_1\})) \sqcup (\mathcal{S}_2 \setminus \psi_0^{-1}(\{z : |z| \geqslant r_2\}))/\sim$$

where the relation $\sim$ identify points $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2 \Leftrightarrow \phi_i(x) = \psi_0(y)$, whenever this equality makes sense (it can happen actually only when $x \in \phi_i^{-1}(\{z : r_1 < |z| < r_2\})$ and $y \in \psi_0^{-1}(\{z : r_1 < |z| < r_2\})$). The new surface $\mathcal{S}_3$ is compact and of genus zero because it is homeomorphic to the connected sum of the underlying topological spheres $\mathcal{S}_1$, $\mathcal{S}_2$. It is also easy to see that the new surface $\mathcal{S}_3$ has complex structure, because on parts $\mathcal{S}_1 \setminus \phi_i^{-1}(\{z : |z| \leqslant r_1\})$ and $\mathcal{S}_2 \setminus \psi_0^{-1}(\{z : |z| \geqslant r_2\})$ it has structures induced by complex structures of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively and on the intersection (i.e. $\phi_i^{-1}(\{z : r_1 < |z| < r_2\}) = \psi_0^{-1}(\{z : r_1 < |z| < r_2\})$) both structures agree. Moreover, the isomorphism class of $\mathcal{S}_3$ does not depend on the choice of r, r_1 and r_2. Finally, the sets $U_0', ..., U_{i-1}', V_1', ..., V_m', U_{i+1}', ..., U_n'$ are the sets $U_0, ..., U_{i-1}, V_1, ..., V_m, U_{i+1}, ..., U_n$, maybe reduced because of the deletion of open sets $\phi_i^{-1}(\{z : |z| \leqslant r_1\})$ and $\psi_0^{-1}(\{z : |z| \geqslant r_2\})$ from $\mathcal{S}_1$, $\mathcal{S}_2$ respectively.

Note also that the element $K(1) \ni I = (\hat{\mathbb{C}}; \infty, 0; (z, \hat{\mathbb{C}} \setminus \{0\}), (z, \mathbb{C}))$ (where by z we denote the identity map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) acts as an identity for this operation; it can be sewn with any other sphere with tubes and $I_1\infty_0\mathcal{Q} \cong \mathcal{Q}$ for every $\mathcal{Q}$ and $\mathcal{Q}_i\infty_0 I \cong \mathcal{Q}$ for $\mathcal{Q}$ of type $(n, 1)$, $n \geqslant i \geqslant 1$.

Now we are ready to define the operad.

Definition 3.1.2. *The partial sphere operad* $\mathcal{K}$ is the partial operad where $\mathcal{K}(n) = K(n)$, the identity is the element $I \in K(1)$, the group S_n acts on $K(n)$ by permuting the negatively oriented punctures and the partial single substitution map is

$$\begin{aligned} \circ_i : \mathcal{K}(n) \times \mathcal{K}(m) &\to \mathcal{K}(n + m - 1) \\ (\mathcal{Q}; \mathcal{Q}') &\mapsto \mathcal{Q}_i\infty_0\mathcal{Q}' \end{aligned}$$

for $1 \leqslant i \leqslant n$, whenever $\mathcal{Q}$ and $\mathcal{Q}'$ can be sewn along the i-th tube of $\mathcal{Q}$.

Our next task is to show that this operad is in fact $\mathbb{C}^\times$-rescalable. We would also like to have reasonable definition of meromorphic functions on $K(n)$. To achieve this, however, we need better description of the sets $K(n)$. We start by recalling the well known fact, that all genus-zero Riemann surfaces are in fact holomorphically isomorphic to $\hat{\mathbb{C}}$. Moreover we can make this statement more strict.

Proposition 3.1.3. *Let $1 \geqslant n \in \mathbb{N}$, $1 \leqslant j \leqslant n$ and let $\mathcal{Q}$ be a sphere with tubes of type $(n, 1)$. Then $\mathcal{Q}$ is isomorphic to the sphere with tubes of the form*

$$(\hat{\mathbb{C}}; \infty, z_1, ..., z_{n-1}, z_n; (\phi_0, B_\infty^{r_0}), (\phi_1, B_{z_1}^{r_1}), ..., (\phi_n, B_0^{r_n}))$$

where $z_j = 0$, $z_i \in \mathbb{C}^\times$ for $i = 1, ..., n$, $i \neq j$, B_z^r is the open disk of radius r centered at z and

$$\tfrac{d(1/\phi_0(z))}{d(1/z)}|_{z=\infty} = \lim_{z\to\infty} \tfrac{1/\phi_0(z)}{1/z} = \lim_{z\to\infty} \tfrac{z}{\phi_0(z)} = \lim_{w\to 0} \tfrac{1/\phi_0(1/w)}{w} = 1,$$

$$\tfrac{d\phi_i(z)}{dz}|_{z=z_i} = \lim_{z\to z_i} \tfrac{\phi_i(z)}{z-z_i} \neq 0 \quad i = 1, ..., n.$$

Proof. It is obviously enough to give the proof in the case $j = n$. Let $\mathcal{Q} = (\mathcal{S}; p_0, p_1, ..., p_n; (\psi_0, U_0), ..., (\psi_n, U_n))$. We can assume that $\mathcal{S} = \hat{\mathbb{C}}$. Moreover, using the conformal isomorphism of $\hat{\mathbb{C}}$: $z \mapsto \frac{z-p_n}{z-p_0}$ which maps p_0 to ∞ and p_n to 0, we can actually assume that $p_0 = \infty$ and $p_n = 0$. Let $a = \lim_{z\to\infty} \frac{z}{\psi_0(z)}$ ($a \neq 0$, because ψ_0 is a biholomorphic map). We see that by the conformal isomorphism of $\hat{\mathbb{C}}$: $z \mapsto \frac{z}{a}$ the sphere with tubes $\mathcal{Q}$ is isomorphic to $(\hat{\mathbb{C}}; z_0, z_1, ..., z_n; (\phi_0, B_{z_0}^{r_0}), (\phi_1, B_{z_1}^{r_1}), ..., (\phi_n, B_{z_n}^{r_n}))$, where $z_0 = \infty, z_1 = \frac{p_1}{a}, ..., z_{n-1} = \frac{p_{n-1}}{a}, z_n = 0$ and $\phi_i = \psi_i \circ (z \mapsto \frac{z}{a})^{-1}$ for $i = 0, ..., n$. We have also

$$\lim_{z\to\infty} \frac{z}{\phi_0(z)} = \lim_{z\to\infty} \frac{z}{\psi_0(az)} = \lim_{z\to\infty} \frac{z/a}{\psi_0(z)} = \frac{a}{a} = 1.$$

Obviously, $\frac{d\psi_i(z)}{dz}|_{z=z_i} \neq 0$, for $i = 1, ..., n$, because all maps ψ_i are biholomorphic. □

We call the above form *the canonical form* of a sphere with tubes. If $j = n$, we say that the canonical form is *standard*. In the case $n = 0$ we can actually improve our result and state that $\frac{1}{\phi_0(1/z)} = z + \sum_{j=3}^\infty a_j z^j$, without $j = 2$ term. It is an easy technical fact. We will assume from now on that $n \geqslant 1$, although all the results are true also for $n = 0$ (maybe after some slight modifications) and proofs are straightforward.

The description above has also the following uniqueness property.

Proposition 3.1.4. *Let $\mathcal{Q} = (\hat{\mathbb{C}}; \infty, z_1, ..., z_{n-1}, 0; \phi_0, \phi_1, ..., \phi_n)$ and $\mathcal{Q}' = (\hat{\mathbb{C}}; \infty, z_1', ..., z_{n-1}', 0; \phi_0', \phi_1', ..., \phi_n')$ be two spheres with tubes of type $(n, 1)$ in the (standard) canonical form, where $n \geqslant 1$. Then they are isomorphic if and only if $z_i = z_i'$ for $i = 1, ..., n-1$ and $\phi_i = \phi_i'$ for $i = 0, ..., n$ on some neighbourhoods of $z_i = z_i'$.*

Proof. If the all above equalities holds then the spheres are obviously isomorphic. Conversely, suppose there is a conformal isomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $F(0) = 0$, $F(\infty) = \infty$, $F(z_i) = z'_i$ for $i = 1, ..., n-1$ and $\phi = \phi' \circ F$. We know that all conformal automorphisms of the sphere are Möbius transformations $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Because $F(0) = 0$ and $F(\infty) = \infty$, F must be of the form $F(z) = az$ for some constant $a \in \mathbb{C}$. We have also

$$1 = \lim_{z \to \infty} \frac{z}{\phi_0(z)} = \lim_{z \to \infty} \frac{z}{\phi'_0 \circ F(z)} = \lim_{z \to \infty} \frac{z}{\phi'_0(az)} = \frac{1}{a}.$$

Hence $a = 1 \Rightarrow F = Id$. □

This description allows us to define the space $K(n)$ precisely enough to describe the meromorphic functions on $K(n)$. Note that if the $(\hat{\mathbb{C}}; \infty, z_1, ..., z_{n-1}, 0; \phi_0, \phi_1, ..., \phi_n)$ is a sphere with tubes in the standard canonical form, functions $\frac{1}{\phi_0(1/z)}, \phi_1(z+z_1), ..., \phi_{n-1}(z+z_{n-1}), \phi_n(z)$ can be expanded in the neighbourhood of 0 as a power series and are of the forms: $\frac{1}{\phi_0(\frac{1}{z})} = z + \sum_{j \in \mathbb{Z}_+ + 1} a_j^{(0)} z^j$, $\phi_i(z + z_i) = \sum_{j \in \mathbb{Z}_+} a_j^{(i)} z^j$ for $i = 1, ..., n$, where $a_0^{(i)} \neq 0$ for $i = 1, ..., n$. Hence $K(n)$ is the subspace of $H = (\mathbb{C}^\times)^{n-1} \times ((\mathbb{C}^\times) \times \mathbb{C}^\infty)^{n+1}$ such that if $H \ni \mathcal{S} = (z_1, ..., z_{n-1}, (a_1^{(0)}, ..., a_m^{(0)}, ...), ..., (a_1^{(n)}, ..., a_m^{(n)}, ...))$ then $z_i \neq z_j$ for $i \neq j$ and the power series $\sum_{j \in \mathbb{Z}_+} a_j^{(i)} z^j$ converge absolutely for $i = 0, ..., n$.

Definition 3.1.5. We say that a function $f : K(n) \to \mathbb{C}$ is *meromorphic*, if it is a rational function with respect to the variables z_i for $i = 1, ..., n-1$, $a_1^{(i)}$ for $i = 1, ..., n$ and $a_j^{(i)}$ for $i = 0, ..., n$, $j \in \mathbb{Z}_+ + 1$, with $z_i = 0$, $z_i = \infty$ $i = 1, ..., n-1$, $z_i = z_j$, $1 \leqslant i < j \leqslant n-1$, $a_1^{(i)} = 0$, $a_1^{(i)} = \infty$, $i = 1, ..., n$ and $a_j^{(i)} = \infty$, $i = 0, ..., n$, $j \in \mathbb{Z}_+ + 1$, as the only possible poles. We denote the set of such functions as $Mer(K(n))$.

We need to mention here one more technical tool, which allows us to handle the sewing operation. Suppose $\mathcal{Q}$ and $\mathcal{Q}'$ are two spheres with tubes of types $(n, 1)$, $(m, 1)$ $(m \geqslant 1)$ respectively, such that $\mathcal{Q}$ and $\mathcal{Q}'$ can be sewn along the i-th tube of $\mathcal{Q}$. Given the spheres $\mathcal{Q}$ and $\mathcal{Q}'$ in canonical forms, how can we obtain the canonical form of $\mathcal{Q}_0\infty_i\mathcal{Q}'$? What we need is an unique isomorphism $F : \mathcal{Q}_0\infty_i\mathcal{Q}' \mapsto \hat{\mathbb{C}}$ that defines the canonical form for $\mathcal{Q}_i\infty_0\mathcal{Q}'$. Let $\mathcal{Q} = (\hat{\mathbb{C}}; \infty, z_1, ..., z_{n-1}, 0; (\phi_0, B_\infty^{r_0}), (\phi_1, B_{z_1}^{r_1}), ..., (\phi_n, B_0^{r_n}))$ and $\mathcal{Q}' = (\hat{\mathbb{C}}; \infty, z'_1, ..., z'_{m-1}, 0; (\psi_0, B_\infty^{r'_0}), (\psi_1, B_{z'_1}^{r'_1}), ..., (\psi_m, B_{z'_m}^{r'_m}))$. Choose $r > 0$ such that $\phi_i(B_{z_i}^{r_i}) \supset \{z : |z| \leqslant r\}$ and $\psi_0^{-1}(B_\infty^{r_0}) \supset \{z : |z| \geqslant r\}$. What we need are two biholomorphic maps $F^{(1)}$ and $F^{(2)}$, defined on some neighbourhoods of $\hat{\mathbb{C}} \setminus \phi_i^{-1}(\{z : |z| < r\})$ and $\hat{\mathbb{C}} \setminus \psi_0^{-1}(\{z : |z| > r\})$ respectively, such that the following equation is satisfied for w in the neighbourhood of $\phi_i^{-1}(\{z : |z| = r\})$:

$$F^{(1)}(w) = F^{(2)}(\psi_0^{-1} \circ \phi_i(w)) \tag{3.1}$$

together with boundary conditions $F^{(1)}(\infty) = \infty$, $F^{(2)}(0) = 0$ and $\lim_{w \to \infty} \frac{F^{(1)}(w)}{w} = 1$ (the resulting sphere will be in canonical form with $z_{i+j-1} = 0$, where $1 \leqslant j \leqslant m$ and $z'_j = 0$).

The equation (3.1) is called the *sewing equation*. Obviously it has unique solution because of the proposition 3.1.4. In fact this equation can be solved, and this task is one of the the most important and complicated part of the proof of the isomorphism theorem. Here we mention only one special case, in which $\phi_i(w) = a(w - z_i)$; then

$$F^{(1)}(w) = \tfrac{1}{a}\psi_0^{-1}(a(w - z_i)) \quad , \quad F^{(2)}(w) = \tfrac{1}{a}w \tag{3.2}$$

This is well defined solution indeed, because in this case $F^{(1)}(w) = \frac{1}{a}\psi_0^{-1} \circ \phi_i(w)$. The map $\phi_i(z) = a(w - z_i)$ is well defined on the neighbourhood of $\hat{\mathbb{C}} \setminus \phi_i^{-1}(\{z : |z| < r\})$ and the image of this set is some neighbourhood of $\{z : |z| \geqslant r\}$, where ϕ_0^{-1} is defined.

Remark 3.1.6. In fact functions $F^{(1)}$, $F^{(2)}$ depend analytically on ϕ_i, ψ_0 (i.e. coefficients of $\phi_i(z) - z_i$ and $\frac{1}{\psi_0(1/z)}$). Therefore it can be shown that the sewing maps $K(n) \times K(m) \to K(n+m-1)$ are 'holomorphic' in the sense that they induce the morphisms of meromorphic functions $Mer(K(m+n-1)) \to Mer(K(n) \times K(m))$. Moreover, if we put better structures on spaces $K(n)$ (for example infinite-dimensional Banach manifold, see appendix B in [Hu]) it can be shown that these maps are holomorphic in the more usual sense, i.e. they preserve given holomorphic structure.

If we have the solutions $F^{(1)}$ and $F^{(2)}$ of the sewing equation, we can write down explicitly the canonical form of $\mathcal{Q}_i\infty_0\mathcal{Q}'$. Suppose $\mathcal{Q} = (\hat{\mathbb{C}}; \infty, z_1, ..., z_{n-1}, 0; \phi_0, \phi_1, ..., \phi_n)$ and $\mathcal{Q}' = (\hat{\mathbb{C}}; \infty, z'_1, ..., z'_m; \phi'_0, \phi'_1, ..., \phi'_m)$. Then the sphere $\mathcal{Q}_i\infty_0\mathcal{Q}'$ is in the form

$$\begin{aligned}(\hat{\mathbb{C}}; \infty, F^{(1)}(z_1), ..., F^{(1)}(z_{i-1}), F^{(2)}(z'_1), ..., F^{(2)}(z'_m), F^{(1)}(z_{i+1}), ..., F^{(1)}(z_n); \phi_0 \circ (F^{(1)})^{-1},\\ ..., \phi_{i-1} \circ (F^{(1)})^{-1}, \psi_1 \circ (F^{(2)})^{-1}, ..., \psi_m \circ (F^{(2)})^{-1}, \phi_{i+1} \circ (F^{(1)})^{-1}, ..., \phi_n \circ (F^{(1)})^{-1}).\end{aligned} \tag{3.3}$$

Obviously, if $z'_j = 0$ then the $(i + j - 1)$-th puncture of the new sphere is 0.

Now we are ready to prove the following proposition.

Proposition 3.1.7. *An operad $\mathcal{K}$ is $\mathbb{C}^\times$-rescalable.*

Proof. Let $G \subset K(1)$ be the set of spheres of the form $K_a := (\hat{\mathbb{C}}; \infty, 0; z, az)$, where $a \in \mathbb{C}^\times$. We will show first that G is a rescaling group in $\mathcal{K}$ and it is isomorphic to $\mathbb{C}^\times$. Obviously $I = K_1 \in G$. Let $a_1, a_2 \in \mathbb{C}^\times$. Using formula (3.2) we see that the solutions for sewing equations for $K_{a_1 1}\infty_0 K_{a_2}$ are functions $F^{(1)}(w) = \frac{1}{a_1}(a_1 w) = w$ and $F^{(2)}(w) = \frac{1}{a_1}w$. Because $K_{a_1 1}\infty_0 K_{a_2}$ is of the form as in (3.3), we obtain

$$K_{a_1 1}\infty_0 K_{a_2} \cong (\hat{\mathbb{C}}; \infty, 0; z, a_2(w \mapsto \frac{1}{a_1}w)^{-1}(z)) = (\hat{\mathbb{C}}; \infty, 0; z, a_1 a_2 z) = K_{a_1 a_2}.$$

Hence G is a group and $G \cong \mathbb{C}^\times$ by the obvious isomorphism $K_a \mapsto a$. The substitution maps are also defined on $G \times K(n)$ and $K(n) \times G^k$, because image of every parametrisation of a neighbourhood of a puncture in the sphere in G contains disk of arbitrary large radius. Finally, it is obvious that both sides in the operad-associativity axiom (1.1) exist if either side exists and $c \in G$ or $d_1, ..., d_k \in G$ or $e_1, ..., e_{j_1+...+j_k} \in G$. Therefore G is the rescaling group for $\mathcal{K}$.

To show that $\mathcal{K}$ is G-rescalable, consider the sphere with tubes $\mathcal{Q} = (\hat{\mathbb{C}}; \infty, z_1, ..., z_n; \phi_0, ..., \phi_n)$. We use again formula (3.2) to obtain corresponding maps for $K_{a1}\infty_0\mathcal{Q}$: $F^{(1)}(w) = \frac{1}{a}\phi_0^{-1}(aw)$, $F^{(2)}(w) = \frac{1}{a}w$. In particular, new parametrization around point $z_0' = \infty$ is

$$\phi_0'(z) = (z \mapsto z)^{-1} \circ (F^{(1)})^{-1}(z) = (F^{(1)})^{-1}(z) = \frac{1}{a}\phi_0(az).$$

Note that $\lim_{z\to\infty} z/\phi_0(z) = 1$, so ϕ_0 is close to the identity map at ∞. Therefore ϕ_0' can contain arbitrary large disk D in its image when $|a|$ is large enough. Moreover, because $F^{(2)}(w) = \frac{1}{w}$, a can be chosen such that all other punctures $z_1', ..., z_n'$ are in the arbitrary small neighbourhod of 0, in particular they will not be contained in the preimage of the disk D. Therefore if $\mathcal{Q}'$ is another sphere with tubes of type $(1, m)$ and $1 \leqslant j \leqslant m$ then there always exists a number $a \in \mathbb{C}^\times$ such that spheres $\mathcal{Q}'$ and $K_{a1}\infty_0\mathcal{Q}$ can be sewn along the j-th tube of $\mathcal{Q}'$. Hence $\mathcal{K}$ is G-rescalable. □

At the end of this section we would like to introduce the notion of a vertex associative algebra. It is a genus-zero analogue of the CFT. To define it, however, we need to define determinant line bundle over spaces $K(n)$ and describe the sewing operations, because by taking genus-zero part of an operad obtained from the CFT one obtain only the suboperad of $\mathcal{K}$.

Denote this suboperad by $\hat{\mathcal{K}}$, i.e. $\hat{K}(n)$ is the subset of $K(n)$ isomorphic to $\mathcal{B}_{0,1,n}$. We know from Proposition (2.2.24) and Remark (2.2.25) that Det forms a trivial holomorphic line bundle over every set $\hat{K}(n)$. Therefore we can extend this bundle over every set $K(n)$ to obtain a trivial line bundle Det $K(n)$. Moreover, the isomorphism Det $\hat{K}(n) \cong \mathcal{B}_{0,n,1} \times \mathbb{C}$ is canonical, hence we have a canonical flat connection on Det $\hat{K}(n)$, which can be obviously extended to the flat connection on Det $K(n)$. Using these connections, by a parallel transport we obtain *canonical sections* of Det $K(n)$.

Our next task is to define the sewing map $\circ_i$: Det $K(n) \times$ Det $K(m) \to$ Det $K(n + m - 1)$, where $1 \leqslant i \leqslant n$. Let $\mathcal{Q} \in K(n)$ and $\mathcal{Q}' \in K(m)$ be two spheres with tubes which can be sewed along the i-th tube of $\mathcal{Q}$. There exist elements $K_{a_0}, ..., K_{a_n}$ and $K_{b_0}, ..., K_{b_m}$ of the rescaling group G such that the spheres

$$\mathcal{B} = K_{a_0 1}\infty_0(...(\mathcal{Q}_1\infty_0 K_{a_1})_2\infty_0 K_{a_2})...)_n\infty_0 K_{a_n})$$
$$\mathcal{B}' = K_{b_0 1}\infty_0(...(\mathcal{Q}_1'\infty_0 K_{b_1})_2\infty_0 K_{b_2})...)_m\infty_0 K_{b_m})$$

can be sewn along i-th tube of $\mathcal{B}$ and $\mathcal{B} \in \hat{K}(n)$, $\mathcal{B}' \in \hat{K}(m)$. Now let $\lambda \in$ Det $\mathcal{Q}$ and $\lambda' \in$ Det $\mathcal{Q}'$. Using canonical sections of Det, these elements determine uniquely elements $\rho \in$ Det $\mathcal{B}$ and $\rho' \in$ Det $\mathcal{B}'$. We can apply canonical isomorphism $l_{\mathcal{B},\mathcal{B}'}$ to obtain an element $l_{\mathcal{B},\mathcal{B}'}(\rho \otimes \rho') \in$ Det $\mathcal{B}_i\infty_0\mathcal{B}'$. Now we use again canonical section of Det to get the element $l^i_{\mathcal{Q},\mathcal{Q}'}(\lambda \otimes \lambda') \in$ Det $\mathcal{Q}_i\infty_0\mathcal{Q}'$. Finally, we define the sewing map as

$$\begin{aligned}\circ_i : \text{Det } K(n) \times \text{Det } K(m) \to& \text{Det } K(n + m - 1)\\ ((\mathcal{Q}, \lambda), (\mathcal{Q}', \lambda')) \mapsto& (\mathcal{Q}_i\infty_0\mathcal{Q}', l^i_{\mathcal{Q},\mathcal{Q}'}(\lambda \otimes \lambda'))\end{aligned}$$

Because of Remark (2.2.25) this definition does not depend on the choice of numbers $a_0, ..., a_n, b_0, ..., b_m \in \mathbb{C}^\times$. Therefore we can define the operad $\mathrm{Det}^c\mathcal{K}$, $c \in \mathbb{Z}$, in an obvious way. We can define also the meromorphic functions on $\mathrm{Det}\mathcal{K}$.

Definition 3.1.8. We say that a function $f : \mathrm{Det}^c K(n) \to \mathbb{C}$ is *meromorphic*, if it a polynomial with respect to the variable λ (the last coordinate in $(\mathcal{Q}, \lambda) \in \mathrm{Det}^c K(n)$) with coefficients in $Mer(K(n))$. We denote the set of such functions as $Mer(\mathrm{Det}^c K(n))$.

Before we give the last definition in this chapter, recall that all irreducible $\mathbb{C}^\times$-modules are isomorphic to $\mathbb{C}$ and the action of $\mathbb{C}^\times$ is given by $(a, x) \mapsto a^{-k}x$, where $a \in \mathbb{C}^\times$, $x \in \mathbb{C}$ and $k \in \mathbb{Z}$ is an arbitrary integer. Therefore every completely reducible $\mathbb{C}^\times$-module V can be written as $V = \bigoplus_{k\in\mathbb{Z}} V_{(k)}$, where $a \in \mathbb{C}^\times$ acts on $V_{(k)}$ by a multiplication by a^{-k}.

Definition 3.1.9. The *vertex associative algebra* of rank $2c$ ($c \in \mathbb{Z}$) (V, ν) is the $\mathbb{C}^\times$-rescalable fiberwise linear $\mathrm{Det}^c\mathcal{K}$-algebra such that:

1. $\dim V_{(k)} < \infty$ for every $k \in \mathbb{Z}$.
2. (*Positive energy*) $V_{(k)} = 0$ for k sufficiently small.
3. Let $n \in \mathbb{N}$, $v' \in V' = \bigoplus_{k\in\mathbb{Z}} V^*_{(k)}$ and $v_1, ..., v_n \in V$. Then the function
$$(\mathcal{Q}, \lambda) \mapsto \langle v', \nu((\mathcal{Q}, \lambda))(v_1 \otimes ... \otimes v_n)\rangle$$
is meromorphic on $\mathrm{Det}\, K(n)$ in the sense of Definition (3.1.8). If z_i, z_j, $i \neq j$, $1 \leqslant i, j \leqslant n$, are i-th and j-th punctures of $\mathcal{Q}$ (we use convention $z_n = 0$) then for any $v_i, v_j \in V$ there exists a number $N(v_i, v_j)$ such that for any $v' \in V'$ and $v_k \in V$, $k = 1, ..., n$, $k \neq i, j$, the order of the pole $z_i = z_j$ is less than $N(v_i, v_j)$.
4. If $P(z) = (\mathbb{C}; \infty, z, 0; w, w - z, w) \in K(2)$ and $v' \in V^*_{(k)}$ $v_1 \in V_{(l)}$, $v_2 \in V_{(m)}$, then the function
$$z \mapsto \langle v', \nu(P(z))(v_1 \otimes v_2)\rangle$$
is of the form $a_{-(l+m-k)}z^{-(l+m-k)}$ locally around 0.

Remark 3.1.10. The last condition in the definition of vertex associative algebra is optional and was not originally given in [Hu], but guarantees that all vertex operators obtained by applying isomorphism theorem have gradation compatible with the definition of vertex algebra given in [BF]. Weakening the assumption about gradation in the definition of vertex algebra, we can skip this condition.

Remark 3.1.11. It seems that more proper name for a notion just defined would be something like 'geometric vertex (operator) algebra'. However, geometric vertex operator algebra was already defined in [Hu]. In fact, it is essentially the same notion as the associative vertex algebra, but defined without using operads and determinant lines. One would not make big mistake when using these notions alternately.

In fact, we did not make use of an additional structure given by determinant lines so far. In the general case of Riemann surfaces of various genus it may not give much new information indeed. However, in the case of an algebra over $\mathrm{Det}^c\mathcal{K}$, because there exist canonical sections of $\mathrm{Det}^c K(n)$ for $n \in \mathbb{N}$, we can choose one coherent family $(\mu_n)_{n\in\mathbb{N}}$ of these and interpret the homomorphism $\nu((\mathcal{Q}, \psi_n(\mathcal{Q})))$ as the 'real' homomorphism associated to the sphere with tubes $\mathcal{Q}$ of type $(n, 1)$. This way we get control over the 'conformal anomaly', i.e. we can compute the difference between the homomorphism associated with a sphere $\mathcal{Q}_i\infty\mathcal{Q}'$ and the composition of homomorphisms associated to spheres $\mathcal{Q}$ and $\mathcal{Q}'$, when sewing them along the i-th tube of $\mathcal{Q}$.

We can make this statement more precise by choosing canonical sections $(\mu_n)_{n\in\mathbb{N}}$. Let $J = (\hat{\mathbb{C}}; \infty; z) \in K(0)$ and let $\mu_0(J)$ be any fixed nontrivial element of Det $K(0)$. It obviously defines the canonical section μ_0 of Det $K(0)$. We define the section μ_1 of Det $K(1)$ as the unique nontrivial section such that

$$l^1_{\mathcal{S},\mathcal{S}}(\mu_1(\mathcal{S}) \otimes \mu_1(\mathcal{S})) = \mu_1(\mathcal{S}).$$

The section μ_2 is defined as the unique section such that

$$l^1_{P(z),J}(\mu_2(P(z)) \otimes \mu_0(J)) = \mu_1(\mathcal{S}),$$

where $P(z) = (\hat{\mathbb{C}}; \infty, z, 0; w, w - z, w)$. In general we define sections μ_n, $n \geqslant 3$, to be the uniquely determined ones by the element

$$l^{n-1}_{...,P(1)}(...(l^3_{...,P(n-3)}(l^2_{P(n-1),P(n-2)}(\mu_2(P(n-1)) \otimes \mu_2(P(n-2)))) \otimes \mu_2 P(n-3)) \otimes ...)$$

of Det $(...((P(n-1)_2\infty_0 P(n-2))_3\infty_0 P(n-3))...)_{n-1}\infty_0 P(1) =$ Det $(\hat{\mathbb{C}}; \infty, n-1, n-2, ..., 0; w, w-n+1, ..., w)$.

If we define sections $(\mu_n)_{n\in\mathbb{N}}$ as above, we have the following proposition. The proof can be found in [Hu].

Proposition 3.1.12. *The i-th substitution map $\circ_i : Det^c K(n) \times Det^c K(m) \to Det^c K(n+m-1)$ is given by*

$$\circ_i : ((\mathcal{Q}, a_1\mu_n(\mathcal{Q})), (\mathcal{Q}', a_2\mu_m(\mathcal{Q}'))) \mapsto (\mathcal{Q}_i\infty_0\mathcal{Q}', a_1 a_2 e^{2c\Gamma((a^{(i)}_j)_{j\in\mathbb{N}},(b^{(0)}_j)_{j\in\mathbb{N}})}\mu_{n+m-1}(\mathcal{Q}_i\infty_0\mathcal{Q}'))$$

where the parametrisations around i-th point of $\mathcal{Q}$ and 0-th point of $\mathcal{Q}'$ are given by $\phi_i(z - z_i) = \sum_{j=0}^{\infty} a^{(i)}_j z^j$ and $\frac{1}{\psi_0(1/z)} = \sum_{j=0}^{\infty} b^{(0)}_j z^j$ respectively and $\Gamma((a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}})$ is some (described in [Hu]) canonical power series of variables a_j and b_j, $j \in \mathbb{N}$, which converges if the sequences $(a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}}$ represent the parametrisations of the holes which can be sewn.

Corollary 3.1.13. *We can define vertex associative algebra of rank $2c$ for every $c \in \mathbb{C}$ as a $\mathbb{C}^\times$-rescalable algebra over Det $\mathcal{K}$ satisfying all conditions from Definition (3.1.9), with substitution maps:*

$$\circ_i : ((\mathcal{Q}, a_1\mu_n(\mathcal{Q})), (\mathcal{Q}', a_2\mu_m(\mathcal{Q}'))) \mapsto (\mathcal{Q}_i\infty_0\mathcal{Q}', a_1 a_2 e^{2c\Gamma((a^{(i)}_j)_{j\in\mathbb{N}},(b^{(0)}_j)_{j\in\mathbb{N}})}\mu_{n+m-1}(\mathcal{Q}_i\infty_0\mathcal{Q}')).$$

3.2. Vertex operator algebras

In this section we define vertex operator algebra. We will give here only brief definition; reader can found more details and nontrivial examples for example in [BF].

Before we give the definition, we need to introduce few technical facts and definitions. First two are the definitions of specific Lie algebras.

Definition 3.2.1. The *Witt algebra* is a complex Lie algebra of meromorphic vector fields on the Riemann sphere with poles only at 0 and ∞. That is, they are generated by $L_n = -z^{n+1}\frac{\partial}{\partial z}$ for $n \in \mathbb{Z}$ and the Lie bracket is given by $[L_m, L_n] = (m-n)L_{m+n}$.

Definition 3.2.2. The *Virasoro algebra of central charge c* ($c \in \mathbb{C}$) is the central extension of Witt algebra, i.e. it is spanned by L_n, $n \in \mathbb{Z}$ and d, such that

$$\begin{array}{lcl} [L_n, d] & = & 0 \\ [L_m, L_n] & = & (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}d \end{array}$$

for $n, m \in \mathbb{Z}$. We denote this algebra by Vir_c.

We would also like to have some control over formal power series in many variables. Let $S \subset \mathbb{C}[x_1, ..., x_n]$ be a collection of homogeneous linear polynomials. Denote by $\mathbb{C}[x_1, ..., x_n]_S$ the ring of rational functions obtained by localizing polynomial ring $\mathbb{C}[x_1, ..., x_n]$ with respect to a set of products of polynomials from S. We would like to define reasonable maps $\iota_{i_1,..,i_n} : \mathbb{C}[x_1, ..., x_n]_S \to \mathbb{C}[[x_1, x_1^{-1} ..., x_n, x_n^{-1}]]$ for any permutation $(i_1, ..., i_n) \in S_n$. Because $\mathbb{C}[x_1] \subset \mathbb{C}[x_1, x_1^{-1}] \subset \mathbb{C}[[x_1, x_1^{-1}]]$, ι_1 is simply an inclusion. We can define ι_{12} to be an operator turning the element of $\mathbb{C}[x_1, x_2]_S$ into Laurent power series with respect to x_2. The coefficients are obviously the elements of $\mathbb{C}[x_1, x_1^{-1}]$, so this operator is well defined. We define ι_{21} analogously. In general assume that $\iota_{i_1...i_{n-1}}$ are already defined for all permutations $(i_1...i_{n-1}) \in S_n$. Let

$$f(x_1, ..., x_n) = \frac{g(x_1, ..., x_n)}{\prod_{k=1}^{m}(\sum_{j=2}^{n} a_{kj}x_{i_j}) \prod_{k=1}^{l}(\sum_{j=1}^{n} b_{kj}x_{i_j})}$$

be an element of $\mathbb{C}[x_1, ..., x_n]_S$, where $g(x_1, ..., x_n) \in \mathbb{C}[x_1, ..., x_n]$ and $b_{k1} \neq 0$ for $k = 1, ..., l$. We can turn every term $1/(\sum_{j=1}^{n} b_{kj}x_{i_j})$ into power series with respect to the variable $\sum_{j=2}^{n} b_{kj}x_{i_j}$. If we multiply all these series, we obtain power series $h(x_1, ..., x_n)$. Denote by $g_t(x_{i_2}, ..., x_{i_n})$ the coefficient of $x_{i_1}^t$ in $g(x_1, ..., x_n)h(x_1, ..., x_n)$. We set

$$\iota_{i_1...i_n} f(x_1, ..., x_n) = \sum_{t \in \mathbb{Z}} \iota_{i_2...i_n}\left(\frac{g_t(x_{i_1}, ..., x_{i_n})}{\prod_{k=1}^{m}(\sum_{j=2}^{n} a_{kj}x_{i_j})}\right)x_{i_1}^t.$$

Now we are ready to give the definition of vertex operator algebra.

Definition 3.2.3. The *vertex operator algebra of central charge c* is a $\mathbb{Z}$-graded Vir-module such that $\dim V_{(k)} < \infty$ for $k \in \mathbb{Z}$, together with the *vacuum* $\mathbf{1} \in V_{(0)}$, the *Virasoro element* $\omega \in V_{(2)}$ and a linear map

$$\begin{array}{lcl} V & \to & (End(V))[[x, x^{-1}]] \\ v & \mapsto & Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \text{ (where } v_n \in End(V)) \end{array}$$

($Y(v,x)$ is called the *vertex operator associated with* v) such that if $v \in V_{(m)}$ then the operator v_n is of degree $(m-n-1)$, satisfying the following axioms:

1. (*positive energy*) $V_{(k)} = 0$ for k sufficiently small;
2. $Y(\mathbf{1}, x) = Id$ (the identity endomorphism);
3. (*creation property*) $Y(v,x)\mathbf{1} \in V[[x]]$ and $v_{-1}\mathbf{1} = v$ (that is, $\lim_{x\to 0} Y(v,x)\mathbf{1} = v$);
4. $dv = cv$ (where d is one of the generators of Vir) and $L_0 v = kv$, where $v \in V_{(k)}$;
5. $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}$ (where we treat the element $L_n \in Vir_c$ as an endomorphism of V), i.e. $L_n = \omega_{n+1}$;
6. (*derivative property* or *translation axiom*) $\frac{d}{dx} Y(v,x) = Y(L_{-1}v, x)$;
7. (*rationality of products and commutativity*) if $v' \in V'$, $v, v_1, v_2 \in V$ then the power series $\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle$ lies in the image of ι_{12}, and
$$(\iota_{12})^{-1}\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle = \frac{g(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t} = (\iota_{21})^{-1}\langle v', Y(v_2,x_2)Y(v_1,x_1)v\rangle,$$
where $g(x_1,x_2) \in \mathbb{C}[x_1,x_2]$ and $r,s,t \in \mathbb{N}$;
8. (*rationality of iterates*) if $v' \in V'$, $v, v_1, v_2 \in V$ then the two power series
$$\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle,\ \langle v', Y(v_1,x_0+x_2)Y(v_2,x_2)v\rangle$$
lie in the image of ι_{20} and ι_{02} respectively, and
$$(\iota_{20})^{-1}\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle = \frac{k(x_0,x_2)}{x_0^r x_2^s (x_0+x_2)^t} = (\iota_{02})^{-1}\langle v', Y(v_1,x_0+x_2)Y(v_2,x_2)v\rangle$$
where $k(x_0,x_2) \in \mathbb{C}[x_1,x_2]$ and $r,s,t \in \mathbb{N}$;
9. (*associativity*) for $v' \in V'$ and $v, v_1, v_2 \in V$, we have
$$(\iota_{12})^{-1}\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle = ((\iota_{20})^{-1}\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle)|_{x_0=x_1-x_2}.$$

Remark 3.2.4. The assumption that the vertex operator v_n is of degree $(k-n-1)$ whenever $v \in V_{(k)}$ is in fact optional, although in most cases it is satisfied. One needs to assume only that for any $v, u \in V$, $u_n v = 0$ for n sufficiently large, the formal series $\langle v', Y(v_1,x_1)Y(v_2,x_2)v\rangle$ involves only finitely many negative powers of x_2 and finitely many positive powers of x_1 and the formal series $\langle v', Y(Y(v_1,x_0)v_2,x_2)v\rangle$ and $\langle v', Y(v_1,x_0+x_2)Y(v_2,x_2)v\rangle$ involve only finitely many negative powers of x_0 and finitely many positive powers of x_2.

The commutativity and associativity axioms state in fact that the vertex operator algebra is some generalization of commutative, associative algebra, where various composition of operators may vary up to different ways of turning rational function into formal series. From the definition it is obvious that any nontrivial examples of VOA are rather complicated, although they exist and many of them can be found in [BF]. We will give here only the trivial one:

Example 3.2.5. Let A be an associative, commutative $\mathbb{C}$-algebra with unit $\mathbf{1}$. Then A has a natural structure of a vertex operator algebra of central charge 0, where all elements of A are of degree 0, the Vir_0-module structure is trivial, the vacuum is $\mathbf{1}$ and the Virasoro element is $\mathbf{0} \in A$. The vertex operator is given obviously by

$$Y(v,x)u = vu,$$

so $v_n = 0$ for $n \neq -1$ and v_{-1} is just a multiplication by v. All axioms of VOA are trivially satisfied.

Moreover, the converse is also true: if V is strictly associative and commutative VOA then it is an associative and commutative algebra with gradation and derivation (in the sense that it is an operator $d : V \to V$ of degree 1 such that $d(vu) = d(v)u + vd(u)$) given by L_{-1} and the action of Vir_c is given. Details can be found again in [BF].

3.3. Isomorphism theorem

In this section we will finally construct the isomorphism between categories of vertex associative algebras and vertex operator algebras, announced in previous chapters. The proof is too long for this work and is for the most part very technical; it can be found in [Hu]. We will, however, present a few shorter and easier steps of it.

Let us state the isomorphism theorem.

Theorem 3.3.1. *Let $c \in \mathbb{C}$. Then the category of vertex associative algebras of rank c and the category of vertex operator algebras of central charge c are isomorphic.*

We will start from the more difficult part of construction, i.e. given vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, we will define vertex associative algebra (V, ν^Y). To do this, we need to describe the parametrisations of neighbourhoods of punctures on the sphere with tubes in a different way. Before we start, recall that if V is a vector space and $F : V \to V$ is a linear endomorphism, then we define $e^F = \exp(F)$ to be

$$e^F(v) = \exp(F)(v) = \sum_{k=0}^{\infty} \frac{F^k}{k!}(v).$$

Let $(\alpha_j)_{j=1}^{\infty}$ be any sequence of complex numbers. We define the sequence $(E_j(\alpha))_{j=1}^{\infty}$ as

$$z + \sum_{j=1}^{\infty} E_j(\alpha) z^{j+1} = \exp((\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z}))z = \sum_{k=0}^{\infty} \frac{1}{k!}(\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z$$

where we treat $z^j \frac{\partial}{\partial z}$, $j \in \mathbb{N}$, as a linear operator $\mathbb{C}[[z]] \to \mathbb{C}[[z]]$. We have the following fact.

Proposition 3.3.2. *The map $E : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is a bijection. In particular, for any power series of the form*

$$f(z) = z + \sum_{j=1}^{\infty} \alpha_j z^{j+1}$$

we have

$$f(z) = (\exp(\sum_{j=1}^{\infty} E_j^{-1}(\alpha) z^{j+1} \frac{\partial}{\partial z})z.$$

Proof. We will prove that $E_j(\alpha) = \alpha_j + r_j(\alpha_1, ..., \alpha_{j-1})$, where $r_j \in \mathbb{C}[x_1, ..., x_{j-1}]$, $j \in \mathbb{Z}_+$. Observe that $(\prod_{j=1}^{k}(z^{p_j} \frac{\partial}{\partial z}))z = cz^q$, where $c \in \mathbb{C}^\times$ and $q = (\sum_{j=1}^{k} p_j) - k + 1$. In particular, if $p_j \geqslant 2$ for $j = 1, ..., k$, then $q > k$. We have

$$\begin{aligned} z + \sum_{j=1}^{\infty} E_j(\alpha) z^{j+1} &= \sum_{k=0}^{\infty} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z \\ &= z + (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})z + \sum_{k=2}^{\infty} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z \\ &= z + \sum_{j=1}^{\infty} \alpha_j z^{j+1} + \sum_{k=2}^{\infty} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z. \end{aligned}$$

We see that $E_1(\alpha) = \alpha_1$ because the series $\sum_{k=2}^{\infty} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z$ has the form $\sum_{i=3}^{\infty} \beta_i z^i$. Moreover, $E_n(\alpha) = \alpha_n + r_n(\alpha)$, where r_n depends only on $\sum_{k=2}^{n} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z$ because the series $\sum_{k=n+1}^{\infty} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z$ are of the form $\sum_{i=n+2}^{\infty} \beta_i z^i$. Finally, $r_n(\alpha)$ is a polynomial of variables $\alpha_1, ..., \alpha_{n-1}$, for if the coefficient of z^m in $\sum_{k=2}^{n} \frac{1}{k!} (\sum_{j=1}^{\infty} \alpha_j z^{j+1} \frac{\partial}{\partial z})^k z$ is dependent on α_l, $l \geqslant n$, then $m \geqslant l+1 > n$.

Now consider the infinite system of equations

$$E_j(\beta) = \alpha_j$$

for $j \in \mathbb{Z}_+$, where $\beta = (\beta_j)_{j=1}^{\infty}$ is the unknown sequence. Because E_j has the form indicated above, we can solve this equation inductively. This approach proves also that the solution is unique. Therefore we can define $E_j^{-1}(\alpha)$ to be the unique solution of this equation. The conclusion that $E(E^{-1}(\alpha)) = E^{-1}(E(\alpha))$ and

$$f(z) = (\exp(\sum_{j=1}^{\infty} E_j^{-1}(\alpha) z^{j+1} \frac{\partial}{\partial z})z$$

is obvious. □

We need to extend our construction to the case $f(z) = \sum_{j=0}^{\infty} a_j z^{j+1}$, where $a_0 \in \mathbb{C}^\times$, but is not necessarily equal to 1. In this case we define $\bar{E} : \mathbb{C}^\times \times \mathbb{C}^\infty \to \mathbb{C}^\times \times \mathbb{C}^\infty$ as

$$\bar{E}(\alpha_0, \alpha) = \alpha_0(1, E(\alpha))$$

where $\alpha = (\alpha_j)_{j\in\mathbb{Z}_+}$. In this case we also have the inverse $\bar{E}^{-1} : \mathbb{C}^\times \times \mathbb{C}^\infty \to \mathbb{C}^\times \times \mathbb{C}^\infty$ which satisfies

$$f(z) = \alpha_0(\exp(\sum_{j=1}^{\infty} \bar{E}_j^{-1}(\alpha/\alpha_0)z^{j+1}\frac{\partial}{\partial z})z$$

for $f(z) = \sum_{j=0}^{\infty} \alpha_j z^{j+1}$

Suppose now we are given the sphere with tubes of type $(n, 1)$ in the standard canonical form $\mathcal{Q} = (\mathbb{C}; n; \infty; z_1, ..., z_{n-1}, 0; (a_j^{(0)})_{j\in\mathbb{Z}_+}, (a_j^{(1)})_{j\in\mathbb{N}}, ..., (a_j^{(n)})_{j\in\mathbb{N}})$. Using the construction described above, we can describe this sphere as

$$\mathcal{Q} = \mathcal{Q}(n; z_1, ..., z_{n-1}; (A^{(0)}), (a_0^{(1)}, A^{(1)}), ..., (a_0^{(n)}, A^{(n)})),$$

where $A^{(i)} = (A_j^{(i)})_{j\in\mathbb{Z}_+} = (\bar{E}_j^{-1}(a^{(i)}/a_0^{(i)}))_{j\in\mathbb{Z}_+}$, $i = 0, ..., n$ (where we assume $a_0^{(0)} = 1$). We will write for short $\mathcal{Q} = \mathcal{Q}(n, z, a, A)$ if there will be no need to be more precise.

Before we give the definition of vertex associative algebra (V, ν^Y) associated with vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, let us introduce few final notational conventions. Let V be a $\mathbb{Z}$-graded Vir-module such that $L_0(v) = kv$ if $v \in V_{(k)}$. For a sequence $(A_j)_{j\in\mathbb{Z}_+}$ we define two operators $L^+(A), L^-(A) : V \to \bar{V} = \prod_{k\in\mathbb{Z}} V_{(k)}$ by

$$L^+(A) = \textstyle\sum_{j=1}^{\infty} A_j L_j, \qquad L^-(A) = \sum_{j=1}^{\infty} A_j L_{-j}.$$

These operators are always well defined because it is easy to check that each operator L_n, $n \in \mathbb{Z}$, has the degree $(-n)$. We can also define the adjoint operators $L'^+(A), L'^-(A) : V' \to \bar{V}' = \prod_{k\in\mathbb{Z}} V_{(k)}^*$. Moreover, for $a \in \mathbb{C}$ we define operators $a^{L_0} : V \to V$ as

$$a^{L_0} v = a^k v$$

for $v \in V_{(k)}$.

Finally, given vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ we define the map $\nu_n^Y : \mathrm{Det}\ K(n) \to \mathcal{H}_V^{\mathbb{C}^\times}(n)$ as

$$\langle v', \nu_n^Y((\mathcal{Q}(n, z, a, A), \mu_n(\mathcal{Q})))(v_1 \otimes ... \otimes v_n)\rangle =$$

$$= \iota_{1...n-1}^{-1}\langle e^{-L'^+(A^{(0)})}v', (\prod_{i=1}^{n-1} Y(e^{-L^+(A^{(i)})}(a_0^{(i)})^{-L_0}v_1, x_1))e^{-L^+(A^{(n)})}(a_0^{(n)})^{-L_0}v_n\rangle|_{x_j=z_j, j=1,...,n-1}$$

for $n \geqslant 1$ and

$$\langle v', \nu_n^Y((\mathcal{Q}(0; A^{(0)}), \mu_0(\mathcal{Q})))\rangle = \langle e^{-L'^+(A^{(0)})}v', \mathbf{1}\rangle$$

for $n = 0$. In particular, we have

$$\langle v', \nu_0^Y((\mathcal{Q}(0; \mathbf{0})), \mu_0(\mathcal{Q}(0; \mathbf{0})))\rangle = \langle v', \mathbf{1}\rangle,$$

$$\langle v', \nu_1^Y(K_a, \mu_1(K_a))(v)\rangle = \langle v', \nu_1^Y(\mathcal{Q}(1; \mathbf{0}, (a, \mathbf{0})), \mu_1(K_a))(v)\rangle = \langle v', a^{-L_0}v\rangle,$$

$$\langle v', \nu_2^Y(P(z), \mu_2(P(z)))(v_1 \otimes v_2)\rangle = \langle v', \nu_2^Y(\mathcal{Q}(2; z; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})), \mu_2(P(z)))(v_1 \otimes v_2)\rangle$$
$$= \langle v', Y(v_1, x)v_2\rangle|_{x=z}.$$

If it will not lead to any ambiguity, we will write simply $\nu(\mathcal{Q})$ instead of $\nu((\mathcal{Q}, \mu_n(\mathcal{Q})))$.

Proposition 3.3.3. (V, ν_Y) *is a vertex associative algebra.*

Proof. The most difficult part of the proof that (V, ν^Y) is a vertex associative algebra is checking that it forms an algebra over $\mathrm{Det}^c\mathcal{K}$, especially the part that if two spheres with tubes can be sewn, the corresponding contraction of maps $V \to \bar{V}$ exists. For details, see [Hu]. We will check all other axioms in Definition (3.1.9) of vertex associative algebra:

1. $\dim V_{(k)} < \infty$ for every $k \in \mathbb{Z}$: it is satisfied from the definition of VOA;

2. Positive energy axiom: also follows from the definition of VOA;

3. Meromorphicity: the fact that the function

$$(\mathcal{Q}, \lambda) \mapsto \langle v', \nu_n^Y((\mathcal{Q}(n, z, a, A), \lambda))(v_1 \otimes ... \otimes v_n)\rangle$$

is meromorphic is implied by the rationality of VOA. For the rest of the proof we can assume without loss of generality that $\mathcal{Q}$ is of the form $\mathcal{Q} = (n; z_1, ..., z_{n-1}; \mathbf{0}, (1, \mathbf{0}), ..., (1, \mathbf{0}))$ and $i < j$. Suppose first that $j < n$. Using commutativity and associativity of VOA we obtain

$$\begin{aligned} &\langle v', \nu_n^Y(\mathcal{Q})(v_1 \otimes ... \otimes v_n)\rangle = \iota^{-1}_{1...n-1}\langle v', Y(v_1, x_1)...Y(v_{n-1}, x_{n-1})v_n\rangle|_{x_k = z_k, k=1,...,n-1} \\ &= \iota^{-1}_{1...(i-1)ij(i+1)...(j-1)(j+1)...n-1}\langle v', Y(v_1, x_1)...Y(v_i, x_i)Y(v_j, x_j)Y(v_{i+1}, x_{i+1})... \\ &\qquad \cdot Y(v_{j-1}, x_{j-1})Y(v_{j+1}, x_{j+1})...Y(v_{n-1}, x_{n-1})v_n\rangle|_{x_k = z_k, k=1,...,n-1} \\ &= \iota^{-1}_{1...(i-1)ij(i+1)...(j-1)(j+1)...n-1}\langle v', Y(v_1, x_1)...Y(Y(v_i, x_0)v_j, x_j)Y(v_{i+1}, x_{i+1})... \\ &\quad \cdot Y(v_{j-1}, x_{j-1})Y(v_{j+1}, x_{j+1})...Y(v_{n-1}, x_{n-1})v_n\rangle|_{x_k = z_k, k=1,...,n-1, k \neq i, x_0 = z_i - z_j}. \end{aligned}$$

We see that, because there exists a constant $N(v_i, v_j)$ such that $(v_i)_k v_j = 0$ for $k > N(v_i, v_j)$, the order of a pole $z_i = z_j$ is bounded by $N(v_i, v_j)$. In the case $j = n$ by commutativity we obtain

$$\begin{aligned} &\langle v', \nu_n^Y(\mathcal{Q})(v_1 \otimes ... \otimes v_n)\rangle = \iota^{-1}_{1...n-1}\langle v', Y(v_1, x_1)...Y(v_{n-1}, x_{n-1})v_n\rangle|_{x_k = z_k, k=1,...,n-1} \\ &= \iota^{-1}_{1...(i-1)(i+1)...(n-1)i}\langle v', Y(v_1, x_1)...Y(v_{i-1}, x_{i-1})Y(v_{i+1}, x_{i+1})...Y(v_{n-1}, x_{n-1}) \\ &\qquad\qquad \cdot Y(v_i, x_i)v_n\rangle|_{x_k = z_k, k=1,...,n-1}. \end{aligned}$$

Hence the order of the pole $z_i = z_j$ is again bounded by a constant $N(v_i, v_j)$.

4. Let $P(z) = \mathcal{Q}(2; z; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0}))$ and assume that $v' \in V^*_{(k)}, v_1 \in V_{(l)}, v_2 \in V_{(m)}$. Then we have

$$\begin{aligned} \langle v', \nu_2(P(z))(v_1 \otimes v_2)\rangle &= \langle v', Y(v_1, x)v_2\rangle|_{x=z} = \langle v', \sum_{n \in \mathbb{Z}} (v_1)_n v_2 x^{-n-1}\rangle|_{x=z} \\ &= \langle v', (v_1)_{l+m-k-1} v_2\rangle z^{-(l+m-k)}. \end{aligned}$$

□

In the other direction suppose we are given vertex associative algebra (V, ν). Our task is to obtain the vertex operator algebra $(V, Y_\nu, \mathbf{1}_\nu, \omega_\nu)$. We define

$$\mathbf{1}_\nu = \nu_0(\mathcal{Q}(0; \mathbf{0})),$$

$$\omega_\nu = -\frac{d}{d\epsilon}\nu_0(\mathcal{Q}(0; (0, \epsilon, 0, ..., 0, ...)))|_{\epsilon=0}$$

and $Y_\nu(v_1, x) = \sum_{n\in\mathbb{Z}}(v_1)_n x^{-n-1}$ by

$$Y_\nu(v_1, x)v_2|_{x=z} = \nu(P(z))(v_1 \otimes v_2).$$

In particular,

$$\langle v', (v_1)_n v_2\rangle = Res_z z^n \langle v', \nu(P(z))(v_1 \otimes v_2)\rangle.$$

Proposition 3.3.4. *$(V, Y_\nu, \mathbf{1}_\nu, \omega_\nu)$ has a structure of a vertex operator algebra.*

Proof. Before we check all the axioms of VOA, we need to show that the elements $\mathbf{1}_\nu, \omega_\nu$ are in $V_{(0)}$ and $V_{(2)}$ respectively (note that they are defined as some elements of $\bar{V}$) and V carries the structure of Vir-module.

Using the solution (3.2) of the sewing equation for $K_a \in K(1)$ and $\mathcal{Q}(0; \mathbf{0}) \in K(0)$ we obtain

$$(a) \cdot \mathbf{1}_\nu = \nu_0(K_{a1}\infty_0\mathcal{Q}(0; \mathbf{0})) = \nu_0(\mathcal{Q}(0; \mathbf{0})) = a^0\mathbf{1}_\nu,$$

hence $\mathbf{1}_\nu \in V_{(0)}$.

To prove that $\omega_\nu \in V_{(2)}$ it is enough to show that $\langle v', \nu_1(K_a)(\omega_\nu)\rangle = a^{-2}\langle v', \omega_\nu\rangle$ for $a \in \mathbb{C}$ and $v' \in V'$. Let $A(\epsilon) = \mathcal{Q}(0; (0, \epsilon, 0, ..., 0, ...))$. By meromorphicity axiom we know that the function $f(\epsilon) = \langle v', \nu_0(A(\epsilon))\rangle$ is a polynomial, hence $f(\epsilon) = \sum_{j=1}^n a_j \epsilon^j$. The expression

$$\langle v', -\frac{d}{d\epsilon}\nu_0(A(\epsilon))\rangle|_{\epsilon=0} = -\frac{d}{d\epsilon}\langle v', \nu_0(A(\epsilon))\rangle|_{\epsilon=0}$$

corresponds to taking the $-a_1$ coefficient of polynomial $f(\epsilon)$. Applying the solution (3.2) of the sewing equation to $K_{a1}\infty_0 A(\epsilon)$ we obtain an element of $K(0)$ with parametrization of neighbourhood of ∞ equal to $\tilde{\psi}_0 : w \mapsto (\frac{1}{a}\psi_0^{-1}(aw))^{-1} = \frac{1}{a}\psi_0(aw)$, where ψ_0 is the parametrization around ∞ in $A(\epsilon)$. Thus we have

$$\frac{1}{\tilde{\psi}_0(1/z)} = \frac{1}{(1/a)\psi_0(a/z)} = a \cdot \exp(\epsilon(\frac{z}{a})^3\frac{d}{d(z/a)})\frac{z}{a} = \exp(\epsilon a^{-2}z^3\frac{d}{dz})z,$$

hence $K_{a1}\infty_0 A(\epsilon) = A(\epsilon a^{-2})$. We conclude that

$$\begin{aligned}\langle v', \nu_1(K_a)(\omega_\nu)\rangle &= \langle v', \nu_1(K_a)_1 \circ_0 -\frac{d}{d\epsilon}\nu_0(A(\epsilon))|_{\epsilon=0}\rangle = -\frac{d}{d\epsilon}\langle v', \nu_0(K_{a1}\infty_0 A(\epsilon))\rangle|_{\epsilon=0} \\ &= -\frac{d}{d\epsilon}\langle v', \nu_0(A(\epsilon a^{-2}))\rangle|_{\epsilon=0} = -a^{-2}\frac{d}{d\epsilon}\langle v', \nu_0(A(\epsilon))\rangle|_{\epsilon=0} = a^{-2}\langle v', \omega_\nu\rangle.\end{aligned}$$

Finally, we define $(d)(v) = cv$ (where $d \in Vir_c$ is one of the generators) and L_n, $n \in \mathbb{Z}$, by:

$$\begin{aligned} L_n(v) &= -\frac{\partial}{\partial A^{(0)}_{-n}}\nu_1(\mathcal{Q}(1;(A^{(0)}),(a^{(1)}_0,A^{(1)})))|_{A^{(0)},A^{(1)}=\mathbf{0},a^{(1)}_0=1} \quad \text{for } n<0,\\ L_0(v) &= -\frac{\partial}{\partial a^{(1)}_0}\nu_1(\mathcal{Q}(1;(A^{(0)}),(a^{(1)}_0,A^{(1)})))|_{A^{(0)},A^{(1)}=\mathbf{0},a^{(1)}_0=1},\\ L_n(v) &= -\frac{\partial}{\partial A^{(1)}_{n}}\nu_1(\mathcal{Q}(1;(A^{(0)}),(a^{(1)}_0,A^{(1)})))|_{A^{(0)},A^{(1)}=\mathbf{0},a^{(1)}_0=1} \quad \text{for } n>0. \end{aligned}$$

It is not obvious why these operators satisfy all the relations in Vir_c. It fact it requires some complicated computations, which can be found in [Hu].

Now we will check (or at least sketch the proof) all the axioms of vertex operator algebra:

1. Positive energy: It is obvious from the definition of $(V, Y_\nu, \mathbf{1}_\nu, \omega_\nu)$.

2. We have

$$\begin{aligned} Y_\nu(\mathbf{1}_\nu,x)v|_{x=z} &= \nu_2(P(z))(\mathbf{1}\otimes v) = (\nu_2(P(z))_1 \circ_0 \nu_0(\mathcal{Q}(0;\mathbf{0})))(v)\\ &= \nu_1(P(z)_1\infty_0\mathcal{Q}(0;\mathbf{0}))(v) = \nu_1(\mathcal{Q}(1;\mathbf{0},(1,\mathbf{0})))(v) = \nu_1(K_1)(v) = v, \end{aligned}$$

   hence $Y_\nu(\mathbf{1}_\nu, x) = Id$;

3. Creation property: We have

$$\begin{aligned} Y_\nu(v,x)\mathbf{1}_\nu|_{x=z} &= \nu_2(P(z))(v\otimes \mathbf{1}_\nu) = (\nu_2(P(z))_2 \circ_0 \nu_0(\mathcal{Q}(0;\mathbf{0})))(v)\\ &= \nu_1(P(z)_2\infty_0\mathcal{Q}(0;\mathbf{0}))(v). \end{aligned}$$

   Using the solution (3.2) of the sewing equation we obtain $F^{(0)}(w) = w, F^{(1)}(w) = w$, hence $P(z)_2\infty_0\mathcal{Q}(0;\mathbf{0}) = B(z) \in K(1)$ is a sphere with tubes at ∞ and z with standard parametrizations around these punctures. We can define $B(z)$ this way for every $z \in \mathbb{C}$. Thus, because of meromorphicity axiom, the function

$$g(z) = \langle v', \nu_1(A(z))(v)\rangle$$

   is a polynomial in z, hence $Y(v,x)\mathbf{1}_\nu \in V[[x]]$. Moreover

$$\lim_{z\to 0}\langle v', \nu_1(A(z))(v)\rangle = \langle v', \nu_1(\mathcal{Q}(1;\mathbf{0},(1,\mathbf{0})))(v)\rangle = \langle v', v\rangle$$

   hence $\lim_{x\to 0} Y(v,x)\mathbf{1}_\nu = v$;

4. Obviously $dv = cv$. For $v \in V_{(k)}$ we have also

$$\begin{aligned} L_0(v) = -\frac{\partial}{\partial a^{(1)}_0}\nu_1(\mathcal{Q}(1;(A^0),(a^{(1)}_0,A^{(1)})))|_{A^{(0)},A^{(1)}=\mathbf{0},a^{(1)}_0=1} &= -\frac{\partial}{\partial a}\nu_1(K_a)v|_{a=1}\\ = -\frac{\partial}{\partial a}a^{-k}v|_{a=1} = ka^{-k-1}v|_{a=1} &= kv; \end{aligned}$$

5. Let $A(\epsilon) = \mathcal{Q}(0;(0,\epsilon,0,...,0,...)) \in K(0)$. We define the linear functional

$$\mathcal{L}_I(z)F = (\frac{d}{d\epsilon}F(P(z)_1\infty_0 A(\epsilon)))|_{\epsilon=0}.$$

on the meromorphic functions on $K(1)$. By the definition of ω_ν we have

$$\mathcal{L}_I(z)(\langle v', \nu_1(\cdot)(v)\rangle) = \frac{d}{d\epsilon}\langle v', \nu_1(P(z)_1\infty_0 A(\epsilon))(v)\rangle|_{\epsilon=0}$$
$$= \frac{d}{d\epsilon}\langle v', \nu_2(P(z))_1 \circ_0 \nu_0(A(\epsilon))(v)\rangle|_{\epsilon=0} = \langle v', \nu_2(P(z))_1 \circ_0 (\frac{d}{d\epsilon}\nu_0(A(\epsilon))|_{\epsilon=0})(v)\rangle$$
$$= -\langle v', \nu_2(P(z))(\omega_\nu \otimes v)\rangle = -\langle v', Y_\nu(\omega_\nu, x)v\rangle|_{x=z}.$$

On the other hand, we have the following

Proposition 3.3.5.

$$\mathcal{L}_I(z) = (z^{-2}\frac{\partial}{\partial a_0^{(1)}} + \sum_{j\in\mathbb{Z}_+} z^{j-2}\frac{\partial}{\partial a_j^{(0)}} + \sum_{j\in\mathbb{Z}_+} z^{-j-2}\frac{\partial}{\partial a_j^{(1)}})|_{K_1}.$$

where $K(1) \ni \mathcal{Q}(1;(a_j^{(0)})_{j\in\mathbb{Z}_+},(a_0^{(1)},(a_j^{(1)})_{j\in\mathbb{Z}_+}))$.

The proof is not very complicated and can be found in [Hu]. Using the proposition above, we obtain

$$\mathcal{L}_I(z)(\langle v', \nu_1(\cdot)(v)\rangle) = (z^{-2}\frac{\partial}{\partial a_0^{(1)}} + \sum_{j\in\mathbb{Z}_+} z^{j-2}\frac{\partial}{\partial a_j^{(0)}} + \sum_{j\in\mathbb{Z}_+} z^{-j-2}\frac{\partial}{\partial a_j^{(1)}})|_{K_1}(\langle v', \nu_1(\cdot)(v)\rangle)$$
$$= -\langle v', z^{-2}L_0 v\rangle - \sum_{j\in\mathbb{Z}_+}\langle v', z^{j-2}L_{-j}v\rangle - \sum_{j\in\mathbb{Z}_+}\langle v', z^{-j-2}L_j v\rangle = -\langle v', \sum_{n\in\mathbb{Z}} L_n x^{-n-2}v\rangle|_{x=z}.$$

Using these equalities we get the desired equality $Y_\nu(\omega_\nu, x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}$;

6. Derivative property: Let $B(z_0) \in K(1)$ be a sphere with punctures at ∞ and $-z_0$. We will show first that $B(z_0)$ is conformally equivalent to the sphere $\mathcal{Q}(1;(z_0,0,...,0,...),(1,\mathbf{0}))$. Obviously $B(z_0)$ is conformally equivalent (by the equivalence $z \mapsto z + z_0$) to a sphere with tubes $(\mathbb{C};\infty,0;(z \mapsto z - z_0),(z \mapsto z))$. It suffices to show that the local coordinates at ∞ agree, i.e.

$$\frac{1}{1/z - z_0} = \exp(z_0 z^2 \frac{d}{dz})z.$$

We have

$$\frac{1}{1/z - z_0} = 1/(\frac{1 - zz_0}{z}) = \frac{z}{1 - zz_0} = z(\sum_{k=0}^{\infty}(zz_0)^k) = z + z_0 z^2 + z_0^2 z^3 + ...$$

On the other hand, we have

$$\exp(z_0z^2\frac{d}{dz})z = (\sum_{k=0}^{\infty}\frac{z_0z^2}{k!}\frac{d}{dz})^kz = z+z_0z^2+z_0^2z^3+... = z(\sum_{k=0}^{\infty}(zz_0)^k) = \frac{1}{1/z-z_0}.$$

The next step is to show that $P(z-z_0) = P(z)_1\infty_0B(z_0)$. Using the solutions (3.2) of the sewing equation we see that $P(z)_1\infty_0B(z_0)$ is a sphere with punctures at ∞, $-z_0$ and $-z$ with parametrizations $w\mapsto w+z$, $w\mapsto w+z_0$ and $w\mapsto w+z$ respectively. Applying the diffeomorphism $w\mapsto w+z$ to this sphere we obtain a sphere with tubes $(\mathbb{C};2;\infty,z-z_0,0;w,w-(z-z_0),w) = P(z-z_0)$.

Finally, we have

$$\begin{aligned}
\frac{d}{dx}&Y_\nu(v_1,x)v_2|_{x=z} = -\frac{d}{dz_0}(Y_\nu(v_1,x)v_2|_{x=z-z_0})|_{z_0=0}\\
&= -\frac{d}{dz_0}(\nu_2(P(z-z_0))(v_1\otimes v_2))|_{z_0=0} = -\frac{d}{dz_0}(\nu_2(P(z)_1\infty_0B(z_0))(v_1\otimes v_2))|_{z_0=0}\\
&= -\frac{d}{dz_0}(\nu_2(P(z))_1\circ_0\nu_1(B(z_0))(v_1\otimes v_2))|_{z_0=0} = \nu_2(P(z))_1\circ_0(-\frac{d}{dz_0}\nu_1(B(z_0))|_{z_0=0})(v_1\otimes v_2)\\
&= \nu_2(P(z))_1\circ_0(-\frac{d}{dz_0}\nu_1(\mathcal{Q}(1;(z_0,0,...,0,...),(1,\mathbf{0})))|_{z_0=0})(v_1\otimes v_2)\\
&= \nu_2(P(z))(L_{-1}v_1\otimes v_2) = Y_\nu(L_{-1}v_1,x)v_2|_{x=z}.
\end{aligned}$$

7. Rationality of products and commutativity: We have

$$\begin{aligned}
Y_\nu(v_1,x_1)Y_\nu(v_2,x_2)v_3|_{x_1=z_1,x_2=z_2} &= (\nu_2(P(z_1))_2\circ_0\nu_2(P(z_2)))(v_1\otimes v_2\otimes v_3)\\
&= \nu_3(P(z_1)_2\infty_0P(z_2))(v_1\otimes v_2\otimes v_3) = \nu_3(P(z_1,z_2))(v_1\otimes v_2\otimes v_3)
\end{aligned}$$

where $P(z_1,z_2) = \mathcal{Q}(3;\infty,z_1,z_2,0;\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}),(1,\mathbf{0}))$. In a similar way we obtain

$$Y_\nu(v_2,x_2)Y_\nu(v_1,x_1)v_3|_{x_1=z_1,x_2=z_2} = \nu_3(P(z_2,z_1))(v_2\otimes v_1\otimes v_3).$$

We see that

$$\langle v',\nu_3(P(z_1,z_2))(v_1\otimes v_2\otimes v_3)\rangle = \langle v',\nu_3(P(z_2,z_1))(v_2\otimes v_1\otimes v_3)\rangle$$

because $P(z_1,z_2) = \sigma(P(z_2,z_1))$, where $S_3\ni\sigma = (1,2)$. By the meromorphicity axiom it is therefore a rational function of the form

$$\frac{g(z_1,z_2)}{z_1^pz_2^q(z_1-z_2)^r}$$

for $p,q,r\in\mathbb{N}$, $g\in\mathbb{C}[z_1,z_2]$;

8. Rationality of iterates: We have

$$\begin{aligned}
Y_\nu(Y_\nu(v_1,x_0)v_2,x_2)v_3|_{x_0=z_0,x_2=z_2} &= (\nu_2P(z_2)_1\circ_0\nu_2(P(z_0)))(v_1\otimes v_2\otimes v_3)\\
&= \nu_3(P(z_2)_1\infty_0P(z_0))(v_1\otimes v_2\otimes v_3).
\end{aligned}$$

Applying the solutions (3.2) of the sewing equation to the sphere with tubes $P(z_2)_1\infty_0 P(z_1 - z_2)$ we obtain a sphere

$$(\mathbb{C}; 3; \infty, z_0, 0, -z_2; w + z_2, w - z_0, w, w + z_2)$$

This sphere, by the diffeomorphism $w \mapsto w + z_2$, is equivalent to a sphere with tubes

$$(\mathbb{C}; 3; \infty, z_0 + z_2, z_2, 0; w, w - (z_0 + z_2), w - z_2, w) = P(z_0 + z_2, z_2)$$

hence

$$\begin{aligned} &Y_\nu(Y_\nu(v_1, x_0)v_2, x_2)v_3|_{x_0=z_0, x_2=z_2} = \nu_3(P(z_2)_1\infty_0 P(z_0))(v_1 \otimes v_2 \otimes v_3) \\ &\quad = \nu_3(P(z_0 + z_2, z_2))(v_1 \otimes v_2 \otimes v_3) = \nu_3(P(z_0 + z_2)_2\infty_0 P(z_2))(v_1 \otimes v_2 \otimes v_3) \\ &\qquad = Y_\nu(v_1, x_0 + x_2)Y_\nu(v_2, x_2)v_3|_{x_0=z_0, x_2=z_2}. \end{aligned}$$

Because of meromorphicity we have also the following equality of rational functions

$$\langle v', \nu_3(P(z_0 + z_2, z_2))(v_1 \otimes v_2 \otimes v_3)\rangle = \frac{k(x_0, x_2)}{x_0^p x_2^q (x_0 + x_2)^r}$$

for $p, q, r \in \mathbb{N}$, $k \in \mathbb{C}[x_0, x_2]$.

9. Associativity: We see from the proofs of previous axioms that

$$\begin{aligned} &Y_\nu(v_1, x_1)Y(x_2, x_2)v_3|_{x_1=z_1, x_2=z_2} = \nu_3(P(z_1)_2\infty_0 P(z_2))(v_1 \otimes v_2 \otimes v_3) \\ &\quad = \nu_3(P(z_1, z_2))(v_1 \otimes v_2 \otimes v_3) = \nu_3(P(z_2)_1\infty_0 P(z_1 - z_2))(v_1 \otimes v_2 \otimes v_3) \\ &\qquad = Y_\nu(Y_\nu(v_1, x_0)v_2, x_2)v_3|_{x_0=z_1-z_2, x_2=z_2}. \end{aligned}$$

□

Remark 3.3.6. In most cases we did not check whether the suitable spheres can be sewn (e.g. $P(z_1)$ and $P(z_2)$). In general they can not always be sewn. However, we could always choose the parameters in a way that allows sewing. Because all parameters were in fact arbitrary, all results hold.

Remark 3.3.7. The above calculations are certainly valid in the case $c = 0$, however, it is not obvious why are they correct in the case $c \neq 0$. In fact, they are valid also in this case because the sequence $\Gamma((a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}})$, described in Proposition (3.1.12), can be written as

$$\Gamma(\alpha_0, A, B) = \sum_{j\in\mathbb{Z}_+} \frac{j^3 - j}{12}\alpha^{-j}A_j B_j + \Gamma_0(\alpha, A, B),$$

where $\alpha = a_0$, $A = \bar{E}^{-1}((a_j/a_0)_{j\in\mathbb{Z}_+})$, $B = \bar{E}^{-1}((b_j)_{j\in\mathbb{Z}_+})$ (recall that in our applications $b_0 = 1$) and Γ_0 contains only terms which are the product of at least three of A_j, B_j, $j \in \mathbb{N}$, but not all A_j's or B_j's. The proof can be found in [Hu]. In the proof above always at least one of the sequences A or B was equal to $\mathbf{0}$, so $\Gamma(\alpha_0, A, B) = 0$ and it had no influence on the final result.

Remark 3.3.8. Using the above constructions we can define functors F_c and G_c, the first from the category of vertex associative algebra of rank c to the category of vertex operator algebras of central charge c and the second one in the opposite direction. One can check that these functors establish an isomorphism of the mentioned categories. The details can be found in [Hu].

Bibliography

[Ad] J. F. Adams, *Infinite Loop Spaces*, Princeton University Press (1978)

[BF] D. Ben-Zvi, E. Frenkel, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs Vol. 88, American Mathematical Society (2001)

[FLM] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press (1988)

[Hö] L. Hörmander *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, Reprint of the 1994 Edition, Springer-Verlag Berlin Heidelberg (2007)

[Hu] Y.-Z. Huang, *Two-Dimensional Conformal Geometry and Vertex Operator Algebras*, Progress in mathematics, Birkhäuser Boston (1997)

[Seg] G. B. Segal, *The definition of conformal field theory*, preprint (1988)

[Szab] B. W. Szabat, *Wstęp do analizy zespolonej*, Panstwowe Wydawnictwo Naukowe (1974)

[Vor] A. A. Voronov, *Notes on Universal Algebra*, arXiv:math/0111009v2 [math.QA] (2001)

[Vor2] A. A. Voronov, *Topics in Mathematical Physics*, course notes (http://www.math.umn.edu/ voronov/8390/index.html), (2001)