On second order intuitionistic propositional logic without a universal quantifier

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Abstract

We examine second order intuitionistic propositional logic, IPC^2. Let $\mathcal{F}_\exists$ be the set of formulas with no universal quantification. We prove Glivenko's theorem for formulas in $\mathcal{F}_\exists$ that is, for $\varphi \in \mathcal{F}_\exists$, φ is a classical tautology if and only if $\neg\neg\varphi$ is a tautology of IPC^2. We show that for each sentence $\varphi \in \mathcal{F}_\exists$ (without free variables), φ is a classical tautology if and only if φ is an intuitionistic tautology. As a corollary we obtain a semantic argument that the quantifier $\forall$ is not definable in IPC^2 from $\bot, \vee, \wedge, \rightarrow, \exists$.

1 Introduction

We consider second order intuitionistic propositional logic with connectives $\bot, \rightarrow, \wedge, \vee$. Negation $\neg\varphi$ is defined as a shorthand for $\varphi \rightarrow \bot$. It is known that in intuitionistic propositional logic (IPC) no connective is definable from the others. On the other hand, in its second order version (IPC^2) one can define $\bot, \wedge, \vee, \exists$ from $\forall$ and $\rightarrow$.

In this paper we present a simple semantic argument that $\forall$ is not definable from the remaining operators. As far as we know, ours is the first argument for this fact. In this paper we concentrate on a syntactically defined fragment of IPC^2. Let $\mathcal{F}_\exists$ be the set of formulas with no universal quantification. We prove that for each *sentence* $\varphi \in \mathcal{F}_\exists$, φ is a tautology of IPC^2 if and only if it is a classical tautology. This fact does not extend to all formulas of IPC^2. Indeed, the formula $\neg\neg p \rightarrow (\exists q(p \rightarrow (q \vee \neg q)) \rightarrow p)$ is a classical tautology but if we consider the topology induced on the set $\{0\} \cup \{1/(n+1) : n \in \omega\}$ by the natural topology of the real line then φ is not true under a valutation which sets the value of p as the set $\{1/(n+1) : n \in \omega\}$.

As was shown by Połacik in [Poł98], the formula $\exists p((r \to (p \vee \neg p)) \to r)$ is not equivalent to any IPC formula. It follows that $\exists$ is not definable from the propositional connectives: $\bot, \wedge, \vee, \to$. Thus, second order intuitionistic propositional logic without universal quantifier is strictly between IPC and IPC^2.

The well known Glivenko theorem states that for any formula φ of propositional logic, φ is a classical tautology if and only if $\neg\neg\varphi$ is a tautology of IPC. We extend this theorem to formulas in $\mathcal{F}_\exists$. This fact cannot be improved to the set of all formulas of IPC^2. Indeed, $\neg\neg\forall(p \vee \neg p)$ is not a tautology of IPC^2.

Finally, let us mention the relationship between IPC and the the lambda calculus. By the Curry–Howard isomorphism, proofs in the implicational fragment of IPC correspond to terms in the lambda calculus and formulas correspond to types of these terms. Similarly, IPC^2 is a counterpart of the polymorphic lambda calculus $\lambda 2$ which is, roughly speaking, the lambda calculus with polymorphic abstraction. Then, the provability of a formula φ in IPC^2 corresponds to the inhabitation problem of the type φ. In a recent paper [Fuj05], Fujita defines a variant of the lambda calculus $\lambda^\exists$ which corresponds to a fragment of IPC^2 with $\neg, \wedge, \exists$ as the only operators. Fujita shows a Galois embedding of $\lambda 2$ into $\lambda^\exists$ which relates the properties of $\lambda 2$, like the weak normalization, the Church–Rosser property, with that of $\lambda^\exists$. This shows that the existential fragment of IPC^2 is a nontrivial part of IPC^2. Unfortunately, it seems that the translation given by Fujita does not give, e.g., the undecidability of this fragment. The undecidability of the full IPC^2 was proven by Löb in see [Löb76], see also Arts' master thesis [Art92] for a detailed exposition of this result. Another proof of this fact may be also found in the book [SU06].

In the first version of this work, we proved all the results using semantics defined on complete Heyting algebras. This line of proof would be correct only if IPC^2 were sound and complete for this semantics. In [Geu94], Geuvers states both soundness and completeness theorems for this semantics (propositions 4.15 and 4.19), but the proof contains a serious flaw. We will comment on this flaw below, after defined algebraic semantics for IPC^2. An anonymous referee who reviewed the first version of this paper asked for a proof of the completeness of IPC^2 for complete Heyting algebras: when we tried to reconstructed Geuvers's proof, we discovered the flaw. Our solution is to prove our results using a Kripke-style semantics rather than an algebraic semantics. We thank the referee for his comments.

2 Basic definitions

For a formula of IPC^2, notions of its free and bound variables are defined as usual. A sentence is a formula without free variables. When we write $\varphi(p_1, \ldots, p_n)$ we assume that the set of free variables of φ is included in $\{p_1, \ldots, p_n\}$. We write $\mathcal{F}_\exists$ for the set of formulas with no universal quantification.

Intuitionistic propositional logic is considered as a constructive part of classical propositional logic (for a detailed treatment of IPC and its second order extension, IPC^2, we refer the reader to the book by Sørensen and Urzyczyn, [SU06]). IPC may be obtained from classical logic, e.g., by deleting Peirce's law, $((\alpha \to \beta) \to \alpha) \to \alpha$, from the usual Hilbert style axiomatization of the classical propositional calculus with modus ponens as the only inference rule. However, one should be careful not to introduce nonconstructivity with some other axioms like the law of excluded middle.

Second order intuitionistic propositional logic (IPC^2) is obtained by adding the usual axioms and rules of inference for handling the existential and universal quantifiers. We present a natural deduction style proof system for IPC^2. Here Γ is a multiset of formulas of IPC^2, ψ, φ and ρ are formulas of IPC^2 and p is a propositional variable.

The letters I and E in names of rules stand for "introduction" and "elimination", respectively.

1. Axioms:
$$\Gamma, \psi \vdash \psi.$$

2. Rules for conjunction:
$$\frac{\Gamma \vdash \psi \quad || \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi \wedge \varphi}\,(\wedge\mathrm{I}), \qquad \frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \psi}, \quad \frac{\Gamma \vdash \psi \wedge \varphi}{\Gamma \vdash \varphi}\,(\wedge\mathrm{E}).$$

3. Rules for disjunction:
$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}, \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\,(\vee\mathrm{I}), \quad \frac{\Gamma, \varphi \vdash \rho \quad || \quad \Gamma, \psi \vdash \rho \quad || \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \rho}\,(\vee\mathrm{E}).$$

4. Rules for implication:
$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\,(\to\mathrm{I}), \qquad \frac{\Gamma \vdash \varphi \to \psi \quad || \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\,(\to\mathrm{E}).$$

5. A rule for negation:
$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\,(\bot\mathrm{E}).$$

6. Rules for quantifiers:

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall p\, \varphi}\,(\forall \mathrm{I}), \qquad \frac{\Gamma \vdash \forall p\, \varphi}{\Gamma \vdash \varphi[p := \psi]}\,(\forall \mathrm{E}),$$

$$\frac{\Gamma \vdash \varphi[p := \psi]}{\Gamma \vdash \exists p\, \varphi}\,(\exists \mathrm{I}), \qquad \frac{\Gamma \vdash \exists p\, \varphi \quad || \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}\,(\exists \mathrm{E}).$$

A restriction in $(\forall \mathrm{I})$ and $(\exists \mathrm{E})$ is that the variable p may not occur as a free variable of Γ or ψ

We define $\varphi \equiv \psi$ as a shorthand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. If $\varphi \equiv \psi$ is provable in IPC^2 we say that φ and ψ are equivalent. Of course, if φ and ψ are equivalent formulas then $\rho[p := \varphi]$ and $\rho[p := \psi]$ are also equivalent. We will use the fact that for each formula γ and variable p the following formula is provable in IPC^2:

$$\forall p\, \neg\gamma \equiv \neg\exists p\, \gamma. \tag{1}$$

The above formula can be seen as a definition of the construction $\forall\,\neg$ by means of an existential quantifier and negation.

2.1 Algebraic semantics

A Heyting algebra $\mathcal{H} = (H, \leq, \cup, \cap, \Rightarrow, 0, 1)$ is a structure where H is a nonempty set; $\leq$ is a partial order on H; 0 and 1 are its least and greatest elements, respectively; and $\cup$, $\cap$ and $\Rightarrow$ are the binary operations of join, meet, and relative pseudo-complementation, respectively. For elements a, b, the relative pseudo-complement of a with respect to b is the greatest element c such that $a \cap c \leq b$. A complete Heyting algebra is a Heyting algebra in which joins and meets exist for arbitrary families of elements. We denote joins and meets for a given set X by $\bigcup_{a \in X} a$ and $\bigcap_{a \in X} a$, respectively.

A canonical example of a complete Heyting algebra is an arbitrary topology $\mathcal{T} = (T, \mathcal{O}(T))$. Then, the partial order is defined by inclusion, $0 = \emptyset$, $1 = T$, join and meet are defined by set operations of sum and intersection, respectively, and $X \Rightarrow Y$ is defined as $\mathrm{int}((T - X) \cup Y)$. The infinite join of a family of open sets $\mathcal{G}$ is of course the infinite sum of elements of $\mathcal{G}$ and the infinite meet of $\mathcal{G}$ is the greatest open set contained in the intersection of all elements of $\mathcal{G}$.

Now, let v be a function from propositionals variables into a complete Heyting algebra $\mathcal{H}$ with the universe H. We call v a valuation. For an element $a \in H$, we define $v(p \mapsto a)$ as a valuation w such that $w(q) = v(q)$, for all $q \neq p$, and $w(p) = a$.

We define the truth value of a formula φ in $\mathcal{H}$ under v, denoted as $[\varphi]_v^{\mathcal{H}}$, by induction on the construction of φ as follows:

1. $[\bot]_v^{\mathcal{H}} = 0$,
2. $[p]_v^{\mathcal{H}} = v(p)$,
3. $[\psi \wedge \gamma]_v^{\mathcal{H}} = [\psi]_v^{\mathcal{H}} \cap [\gamma]_v^{\mathcal{H}}$,
4. $[\psi \vee \gamma]_v^{\mathcal{H}} = [\psi]_v^{\mathcal{H}} \cup [\gamma]_v^{\mathcal{H}}$,
5. $[\psi \to \gamma]_v^{\mathcal{H}} = [\psi]_v^{\mathcal{H}} \Rightarrow [\gamma]_v^{\mathcal{H}}$,
6. $[\exists p\psi]_v^{\mathcal{H}} = \bigcup_{a\in H} [\psi]_{v(p\mapsto a)}^{\mathcal{H}}$,
7. $[\forall p\psi]_v^{\mathcal{H}} = \bigcap_{a\in H} [\psi]_{v(p\mapsto a)}^{\mathcal{H}}$.

We say that φ is true in $\mathcal{H}$ under v if $[\varphi]_v^{\mathcal{H}} = 1$. A formula φ is a tautology of complete Heyting algebras if for all complete Heyting algebras $\mathcal{H}$ and for all valuations in $\mathcal{H}$, $[\varphi]_v^{\mathcal{H}} = 1$.

It is easy to check that complete Heyting algebras form a sound semantics for IPC^2 It is plausible that Heyting algebras are also a complete semantics for IPC^2. As noted above, however, the proof of completeness offered in [Geu94] contains a flaw. We now comment on this flaw in some detail.

The proof of corollary 4.15 in [Geu94] relies on the fact that any Heyting algebra $\mathcal{H}$ can be extended to a complete Heyting algebra $\mathcal{H}'$ such that all suprema and infima already existing in $\mathcal{H}$ are preserved in $\mathcal{H}'$. Then, Geuvers extends in this way the Heyting algebra $\mathcal{H}$ of formulas of IPC^2 to a complete Heyting algebra $\mathcal{H}'$. Relying on the fact that existing infima and suprema are preserved Geuvers assumes that the values of formulas of IPC^2 are the same in $\mathcal{H}$ and $\mathcal{H}'$. However, even if this is true it requires an additional argument. This is so because the range of quantifiers in formulas of IPC^2 is the whole algebra. So, this range is changed while going from $\mathcal{H}$ to $\mathcal{H}'$. As an example we may consider a behavior of a formula $\forall p\varphi(p)$ which begins with an universal quantifier under a valuation v in $\mathcal{H}$. Then, the set $X = \left\{[\varphi(p)]_{v(p\mapsto a)}^{\mathcal{H}} : a \in \mathcal{H}\right\}$ may be properly contained in the set $X = \left\{[\varphi(p)]_{v(p\mapsto a)}^{\mathcal{H}'} : a \in \mathcal{H}'\right\}$. Thus, even if infima existing in $\mathcal{H}$ are preserved in $\mathcal{H}'$ we have no guarantee that in $\mathcal{H}'$ it holds that $\bigcap_{a\in X} a = \bigcap_{a\in Y} a$. This fact may result in different values in $\mathcal{H}$ and $\mathcal{H}'$ of formulas with quantifiers.

2.2 Kripke style semantics

Now, we define a sound and complete semantics for IPC^2. For a more detailed treatment and for a proof of completeness we refer the reader to the book [SU06] (see Theorem 11.1.12 on page 275).

Definition 1 *A second order Kripke model is a tuple of the form*

$$\mathcal{C} = (K, \leq, \{D_c : c \in K\}),$$

where K is a non-empty set, $\leq$ is a partial ordering of K and for each $c \in K$, D_c is a non-empty collection of upward closed subsets of K which satisfies the condition

$$\text{if } c \leq c' \text{ then } D_c \subseteq D_{c'}.$$

A valuation v in $\mathcal{C}$ assigns upward closed subsets of K to propositional variables. A valuation v is admissible for a state $c \in K$ and a formula φ if for all free variables p of φ, $v(p) \in D_c$. Of course, if v is admissible for c and φ then it is also admissible for c' and φ, where $c \leq c'$.

Now, we define the satisfaction relation $\mathcal{C}, c \models \varphi[v]$, where $\mathcal{C}$ is a second order Kripke model, $c \in K$, φ is an IPC^2 formula and v is an admissible valuation for c and φ. If v is not admissible for c and φ then the relation $\mathcal{C}, c \models \varphi[v]$ is undefined.

- $\mathcal{C}, c \models p[v]$ *if* $c \in v(p)$,
- *it is never true that* $\mathcal{C}, c \models \bot[v]$,
- $\mathcal{C}, c \models \varphi \wedge \psi[v]$ *if* $\mathcal{C}, c \models \varphi[v]$ *and* $\mathcal{C}, c \models \psi[v]$,
- $\mathcal{C}, c \models \varphi \vee \psi[v]$ *if* $\mathcal{C}, c \models \varphi[v]$ *or* $\mathcal{C}, c \models \psi[v]$,
- $\mathcal{C}, c \models \varphi \to \psi[v]$ *if for all* $c' \geq c$, *if* $\mathcal{C}, c' \models \varphi[v]$ *then* $\mathcal{C}, c' \models \psi[v]$,
- $\mathcal{C}, c \models \exists p \psi(p)[v]$ *if for some* $X \in D_c$, $\mathcal{C}, c \models \psi(p)[v(p \mapsto X)]$,
- $\mathcal{C}, c \models \forall p \psi(p)[v]$ *if for all* $c' \geq c$ *and for all* $X \in D_{c'}$, $\mathcal{C}, c' \models \psi(p)[v(p \mapsto X)]$.

We write $\mathcal{C} \models \varphi$ if for each c and for each valuation v which is admissible for c and φ it holds that $\mathcal{C}, c \models \varphi[v]$.

Let us note that if a valuation v is admissible for φ and c and if $X \in D_c$, then $v(p \mapsto X)$ is also admissible for φ and c.

In order to define a sound semantics for IPC^2 we need to put one restriction on the class of second order Kripke models defined above.

Definition 2 *Let $\mathcal{C}$ be a Kripke model. An element $X \in D_c$ represents a formula φ in a state c under a valuation v if for all $c' \geq c$,*

$$c' \in X \text{ if and only if } \mathcal{C}, c' \models \varphi[v].$$

A Kripke model $\mathcal{C}$ is full if for every formula φ, every c and every valuation v admissible for φ and c, there exists $X \in D_c$ such that X represents φ in c under v.

We have the following fact which shows why we have to restrict semantics to full Kripke models, see [SU06].

Fact 3 *A second order Kripke model $\mathcal{C}$ is full if and only if for each formula φ, $\mathcal{C} \models \exists p(p \equiv \varphi)$, where p is not a free variable of φ.*

The above fact shows that in defining sematic for IPC^2 we should restrict our attention to full second order Kripke models . Indeed, only such models satisfy all formulas provable in IPC^2. For such models we have completeness theorem. It was proven in full generality by Sobolev in [Sob77], see also [SU06].

Definition 4 *Let Γ be a set of formulas of IPC^2 and let φ be an IPC^2 formula. We write $\Gamma \models \varphi$ for stating that for every full Kripke models $\mathcal{C}$, every $c \in K$ and every valuation v admissible for c and all formulas in $\Gamma \cup \{\varphi\}$, it holds that if for all $\psi \in \Gamma$, $\mathcal{C}, c \models \psi[v]$, then $\mathcal{C}, c \models \varphi[v]$.*

Let Γ be a set of formulas of IPC^2, and let φ be an IPC^2 formula. We write $\Gamma \models \varphi$ for the following:

> *for every full Kripke model $\mathcal{C}$, every $c \in K$, every valuation v admissible for c and each formula in $\Gamma \cup \{\varphi\}$, we have $C, c \models \varphi[v]$ if for every $\psi \in \Gamma$, $\mathcal{C}, c \models \psi[v]$.*

Theorem 5 *For all Γ and φ, $\Gamma \models \varphi$ if and only if $\Gamma \vdash \varphi$.*

Now, we present an example that we will use later.

Example 6 *We compute the value of the formula $\neg\neg\forall p(p \vee \neg p)$ in the model $\mathcal{C} = (\omega, \leq, \{D_i : i \in \omega\})$, where $\leq$ is the usual ordering relation and each D_i is a set of all upward closed subsets of ω. By the choice of D_i our model is obviously a full model.*

We show that $\mathcal{C}, 0 \not\models \neg\neg\forall p(p \vee \neg p)$. Indeed,

$$\mathcal{C}, 0 \not\models \neg\neg\forall p(p \vee \neg p)[v] \text{ if and only if}$$

$$\text{there exists } i \in \omega \text{ such that } \mathcal{C}, i \models \neg\forall p(p \vee \neg p)[v].$$

We show that $\mathcal{C}, 0 \models \neg\forall p(p \vee \neg p)[v]$. *We have*

$$\mathcal{C}, 0 \models \neg\forall p(p \vee \neg p)[v] \quad \text{if and only if} \quad \text{for all } i \in \omega,\ \mathcal{C}, i \not\models \forall p(p \vee \neg p)[v].$$

If we take $X_i = \omega \setminus \{0, \ldots, i\}$, *then*

$$\mathcal{C}, i \not\models p[v(p \mapsto X_i)] \text{ and } \mathcal{C}, i \not\models \neg p[v(p \mapsto X_i)].$$

Thus, $\mathcal{C}, i \not\models p \vee \neg p[v(p \mapsto X_i)]$ *and, consequently,* $\mathcal{C}, i \not\models \forall p(p \vee \neg p)[v]$.

Let us remark, that in all finite Kripke models the formula $\neg\neg\forall p(p \vee \neg p)$ is satisfied. This is because the maximal nodes of a model satisfy all classical tautologies. In a finite Kripke model $\mathcal{C}$, any node c is under some maximal node. Thus, it cannot be the case that c satisfies a negation of a classical tautology.

3 Main results

In this section we prove the main results of this paper, theorems 10, 11 and 12.

Definition 7 *For a valuation* $\varepsilon\colon \{p_1, \ldots, p_k\} \longrightarrow \{0, 1\}$ *by* $\varepsilon(\varphi)$ *we denote the (classical) value of* φ *under* ε *(if defined). If* ε *is the empty valuation, we put by convention* $\varepsilon(\bot) = 0$.

By $(\neg)^0$ *we denote just negation and by* $(\neg)^1$ *we denote the empty string. Thus,* $\bigwedge_{1 \le i \le k} (\neg)^{\varepsilon(i)} p_i$ *is a formula which describes valuation* ε *on the set of propositions* $p_1, \ldots, p_k$.

Our main lemma is the following.

Lemma 8 *Let* $\varphi(p_1, \ldots, p_k) \in \mathcal{F}_\exists$ *and let* $\varepsilon\colon \{p_1, \ldots, p_k\} \longrightarrow \{0, 1\}$. *If* $\varepsilon(\varphi) = 1$, *then*

$$\models (\bigwedge_{1 \le i \le k} (\neg)^{\varepsilon(p_i)} p_i \rightarrow \varphi(p_1, \ldots, p_k))$$

and if $\varepsilon(\varphi) = 0$, *then*

$$\models (\bigwedge_{1 \le i \le k} (\neg)^{\varepsilon(p_i)} p_i \rightarrow \neg\varphi(p_1, \ldots, p_k)).$$

Proof. The proof proceeds by induction on the complexity of φ. Below, we write $\varphi(\circ/p_i)$, where $\circ \in \{\bot, \top\}$, for a formula φ with $\circ$ substituted for the variable p_i. We use also a common convention that the empty conjuction is $\top$ and the empty disjunction is $\bot$.

For φ being a propositional variable the thesis is obvious. For $\varphi = \bot$ and ε being the empty valuation we have $\varepsilon(\bot) = 0$ and

$$\begin{aligned} \bigwedge_{1\leq i\leq 0} (\neg)^{\varepsilon(i)} p_i \to (\bot \to \bot) &\equiv \top \to (\bot \to \bot) \\ &\equiv \top. \end{aligned}$$

The inductive steps for $\vee$ and $\wedge$ are straightforward. We consider more carefully the case for the implication and for quantifiers. The implication case includes also the case for negation.

Let $\varphi = (\gamma \to \psi)$. If $\varepsilon(\gamma) = 0$, then

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models (\gamma \to \bot)$$

so also

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models (\gamma \to \psi).$$

If $\varepsilon(\psi) = 1$, then

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models \psi$$

so also

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models (\gamma \to \psi).$$

If $\varepsilon(\varphi) = 0$, then $\varepsilon(\gamma) = 1$ and $\varepsilon(\psi) = 0$. So, by the inductive assumption,

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models \gamma$$

and

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models (\psi \to \bot).$$

Putting these two together we get

$$\bigwedge_{1\leq i\leq k} (\neg)^{\varepsilon(i)} p_i \models ((\gamma \to \psi) \to \bot).$$

Now, we consider the case of existential quantifier. For simplicity we assume that φ is of the form $\varphi(p_1, \ldots, p_{k-1}) = \exists p_k \psi(p_1, \ldots, p_k)$.

If $\varepsilon(\exists p_k \psi(p_k)) = 1$, then $\varepsilon(\psi(\circ/p_k)) = 1$, for some $\circ \in \{\bot, \top\}$. Then, by the inductive assumption,

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i \models \psi(\circ/p_k)$$

so also

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i \models \exists p_k \psi(p_k).$$

Finally, let us assume that $\varepsilon(\exists p_k \psi(p_k)) = 0$. By the inductive assumption we have,

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i, p_k \models (\psi(p_k) \to \bot) \tag{2}$$

and

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i, p_k \to \bot \models (\psi(p_k) \to \bot). \tag{3}$$

We claim that in this case

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i \models \forall p_k (\psi(p_k) \to \bot). \tag{4}$$

We prove (4) by considering Kripke models semantics. It suffices to show that

$$\bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i \models \psi(p_k) \to \bot.$$

Let $\mathcal{C}$ be an arbitrary Kripke model, let a be its element and let v be a valuation such that

$$\mathcal{C}, a \models \bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i[v].$$

For the sake of contradiction we assume that there is a world b such that

$$a \le b \text{ and } \mathcal{C}, b \models \psi(p_k)[v].$$

But then, by (3), it cannot be the case that

$$\mathcal{C}, b \models p_k \to \bot[v].$$

Thus, there is a world c, such that $b \le c$ and $\mathcal{C}, c \models p_k$. Then, by monotonicity and $\mathcal{C}, b \models \psi(p_k)[v]$ we get

$$\mathcal{C}, c \models \bigwedge_{1 \le i \le k-1} (\neg)^{\varepsilon(i)} p_i \wedge p_k[v]$$

and, by (2),

$$\mathcal{C}, c \models \psi(p_k) \to \bot[v].$$

But this is a contradiction with the fact that, by monotonicity,

$$\mathcal{C}, c \models \psi(p_k).$$

So, we have proven (4).

Now, in order to show that

$$\bigwedge_{1 \leq i \leq k-1} (\neg)^{\varepsilon(i)} p_i \models \neg \exists p_k \, \psi$$

it suffices to note that by (1), we have $\forall p_k \, \neg\psi \models \neg\exists p_k \, \psi$. Then we use (4) to infer $\neg\exists p_k \psi$ from $\bigwedge_{1 \leq 1 \leq k-1} (\neg)^{\varepsilon(i)} p_i$. This ends the step for existential quantifier. $\Box$

We want to note that lemma 8 and all the theorems below hold for a syntactically larger set of formulas. Let $\mathcal{F}_{(\forall\neg)}$ be a set of such formulas in which each occurrence of an universal quantifier is before a negation. That is, for $\varphi \in \mathcal{F}_{(\forall\neg)}$, all subformulas of φ beginning with a universal quantifier are of the form $\forall p \neg\gamma$. Formulas in $\mathcal{F}_{(\forall\neg)}$ only syntactically extend the set $\mathcal{F}_\exists$. This is due to the fact, that the construction $\forall p \neg\gamma$ is definable by means of an existential quantification, see equation (1). Thus, having a formula $\varphi \in \mathcal{F}_{(\forall\neg)}$ we can translate it to an equivalent formula $\varphi' \in \mathcal{F}_\exists$. Then, lemma 8 holds for φ' but since φ and φ' are equivalent it holds also for φ. Consequently, corollary 9 and theorems 10 and 11 below are satisfied also by formulas in $\mathcal{F}_{(\forall\neg)}$.

From lemma 8 we obtain an easy corollary.

Corollary 9 *Let $\varphi(p_1, \ldots, p_k) \in \mathcal{F}_\exists$ such that φ is a classical tautology with all free variables among $p_1, \ldots, p_k$. Then the following formula is an intuitionistic tautology*

$$\left(\bigvee_{\varepsilon\colon \{1,\ldots,k\} \longrightarrow \{0,1\}} \left(\bigwedge_{1 \leq i \leq k} (\neg)^{\varepsilon(i)} p_i \right) \right) \to \varphi(p_1, \ldots, p_k).$$

Proof. By lemma 8 for each $\varepsilon\colon \{1, \ldots, k\} \longrightarrow \{0, 1\}$, $\bigwedge_{1 \leq i \leq k} (\neg)^{\varepsilon(i)} p_i \to \varphi$ is an intuitionistic tautology. It follows easily that also a formula in the statement of the corollary has to be an intuitionistic tautology. $\Box$

Glivenko's theorem states that for any propositional formula φ, φ is a classical tautology if and only if $\neg\neg\varphi$ is an intuitionistic tautology. Now, we extend this theorem to the formulas of $\mathcal{F}_\exists$ and IPC^2.

Theorem 10 *For each formula $\varphi \in \mathcal{F}_\exists$, φ is a classical tautology if and only if $\neg\neg\varphi$ is a tautology of* IPC^2.

Proof. The implication from the right to the left is obvious. Thus, let us assume that $\varphi(p_1, \ldots, p_k)$ is a classical tautology and let γ be the following formula

$$\bigvee_{\varepsilon\colon \{1,\ldots,k\} \longrightarrow \{0,1\}} \left(\bigwedge_{1 \le i \le k} (\neg)^{\varepsilon(i)} p_i \right)$$

Then, by corollary 9, $\gamma \to \varphi$ is a tautology of IPC^2. It follows that $\neg\neg\gamma \to \neg\neg\varphi$ is also an intuitionistic tautology. But γ is a classical tautology and by Glivenko's theorem, $\neg\neg\gamma$ is an intuitionistic tautology. Thus, $\neg\neg\varphi$ is a tautology of IPC^2, too. $\square$

Let us note that the last theorem does not extend to the set of all formulas of IPC^2. As it was shown in example 6 the sentence $\neg\neg\forall p(p \vee \neg p)$ is not a tautology of IPC^2 although $\forall p(p \vee \neg p)$ is a classical tautology.[1]

If we consider sentences only, we can prove the following result stronger than theorem 10.

Theorem 11 *Let $\varphi \in \mathcal{F}_\exists$ be a sentence (without free variables). Then, φ is a classical tautology if and only if φ is an intuitionistic tautology.*

Proof. The only interesting direction is from the left to the right. Thus, let us assume that $\varphi \in \mathcal{F}_\exists$ is a sentence which is a classical tautology. By corollary 9 the following formula is an intuitionistic tautology

$$\left(\bigvee_{\varepsilon\colon \emptyset \longrightarrow \{0,1\}} \bigwedge_{1 \le i \le 0} (\neg)^{\varepsilon(i)} p_i \right) \to \varphi.$$

But there is only one function $\varepsilon\colon \emptyset \longrightarrow \{0,1\}$, the empty function. Thus, the disjunction over $\varepsilon\colon \emptyset \longrightarrow \{0,1\}$, reduces to one disjunct which is the empty conjunction. Since the empty conjunction is equivalent to $\top$, the above formula reduces to

$$\left(\bigvee_{\varepsilon\colon \emptyset \longrightarrow \{0,1\}} \top \right) \to \varphi.$$

This last formula is equivalent to $\top \to \varphi$ which is equivalent just to φ. Thus, φ is provable in IPC^2. $\square$

As a corollary we obtain the following theorem.

[1] The properties of this sentence were brought to author's attention by Paweł Urzyczyn.

Theorem 12 *The universal quantifier is not definable from $\bot, \vee, \wedge, \rightarrow, \exists$ in second order intuitionistic propositional logic.*

Proof. It suffices to show that the sentence $\neg\neg\forall p(p \vee \neg p)$ is not equivalent to any IPC^2 sentence with no universal quantifier. Let us assume, for the sake of contradiction, that there is a sentence $\varphi \in \mathcal{F}_\exists$ such that it is equivalent to $\neg\neg\forall p(p \vee \neg p)$. Let us consider a one element model

$$\mathcal{C}_0 = (\{0\}, \{(0,0)\}, \{X\}),$$

where X is the power set of $\{0\}$. It is easy to verify that an IPC^2 sentence is true in $\mathcal{C}_0$ exactly when it is a classical tautology. Thus, $\neg\neg\forall p(p \vee \neg p)$ is true in $\mathcal{C}_0$. In consequence φ is also true in $\mathcal{C}_0$ and has to be a classical tautology. Then, by theorem 11, φ is also an intuitionistic tautology. But, as we showed in example 6 the formula $\neg\neg\forall p(p \vee \neg p)$ is not an intuitionistic tautology so it cannot be equivalent to φ, a contradiction. $\square$

Let us end with a presentation of a translation $\circ$ of formulas of quantified propositional calculus such that a formula φ is classically provable if and only if $\neg\neg(\varphi)^\circ$ is provable in IPC^2.

The translation is defined by induction on the structure of formulas. The translation $\circ$ is the identity on the set of propositional variables and commutes with propositional connectives and existential quantifier. For a universal quantifier step we define

$$(\forall p\, \varphi)^\circ := \neg\exists p\, \neg(\varphi)^\circ.$$

Of course, the translation does not change the properties of formulas with respect to the classical provability. The following theorem follows straightforwardly from theorem 10.

Theorem 13 *Let $\circ$ be as defined above. For each formula φ, φ is classically provable if and only if $\neg\neg(\varphi)^\circ$ is provable in IPC^2.*

4 Final remarks

We have shown that second order intuitionistic propositional logic, IPC^2, restricted to formulas in $\mathcal{F}_\exists$ behaves in some ways like IPC rather than IPC^2. It would be nice to know whether this fragment of IPC^2 behaves in other ways more like IPC than IPC^2: for example whether the real line gives a complete semantics for this fragment of IPC^2; whether this fragment has the finite model property; or whether it is decidable.

And of course there is a natural open question whether complete Heyting algebras form an adequate semantics for IPC^2.

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