Degrees of logics with Henkin quantifiers in poor vocabularies.

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Abstract

We investigate some logics with Henkin quantifiers. For a given logic L, we consider questions of the form: what is the degree of the set of L–tautologies in a poor vocabulary (monadic or empty)? We prove that the set of tautologies of the logic with all Henkin quantifiers in empty vocabulary $L^*_\emptyset$ is of degree $\mathbf{0}'$. We show that the same holds also for some weaker logics like $L_\emptyset(\mathsf{H}_\omega)$ and $L_\emptyset(\mathsf{E}_\omega)$.

We show that each logic of the form $L^{(k)}_\emptyset(Q)$ with the number of variables restricted to k is decidable. Nevertheless – following the argument of M. Mostowski from [Mos89] – for each reasonable set theory no concrete algorithm can provably decide $L^{(k)}(Q)$, for some Q. We improve also some results related to undecidability and expressibility for logics $L(\mathsf{H}_4)$ and $L(\mathsf{F}_2)$ of Krynicki and M. Mostowski from [KM92].

1 Introduction

This paper considers the problem of complexity of some logics with Henkin quantifiers in poor vocabularies. It follows investigations presented in [KL79], [Mos89], [Mos91], and [KM92]. For a more general framework see the survey [KM95]. In [KL79] it was shown that the logic with the simplest Henkin quantifier (called *the Henkin quantifier*) in monadic vocabulary is decidable. On the other hand the same logic with at least one function symbol or at least one binary predicate has nonarithmetical set of tautologies (see theorem 3).

It follows that only logics restricted to poor vocabularies (monadic or empty) can be decidable or have arithmetical degrees of unsolvability. In this work we study degrees of some logics with Henkin quantifiers in poor vocabularies.

In [Mos89] it was shown that the logic with Henkin quantifiers in empty vocabulary is not decidable, in this paper we prove that it is of degree $\mathbf{0}'$ (recursive with recursively enumerable oracle). The idea of the proof is essentially based on model theoretical proof from [Mos91] of the Skolem–Löwenheim property of $L^*_\emptyset$ (the logic with Henkin quantifiers in empty vocabulary). We show by a similar argument that the degree of some logics appearingly weaker than $L^*_\emptyset$ is also $\mathbf{0}'$.

In [KM92] it is shown that the word problem for semigroups is effectively reducible to the tautology problem for $L_\emptyset(\mathsf{H}_\omega)$. We show here that the class E_ω (which is semantically contained in H_ω) is sufficient for this reduction. As a corollary we obtain that the degree of $L_\emptyset(\mathsf{E}_\omega)$ is also $\mathbf{0}'$.

Then we consider logics with finitely many variables. We show that for each Henkin quantifier Q and each k the logic $L^{(k)}_\emptyset(Q)$ ($L(Q)$ restricted to formulae with variables $x_0, \ldots, x_{k-1}$) is decidable. Moreover, by an argument similar as in [Mos89] we show that for each reasonable set theory T there is Q such that for any algorithm A the statement "A decides $L^{(k)}_\emptyset(Q)$" cannot be proven in T. Moreover, such Q can be found between relatively weak quantifiers E_ω.

In the next part of this paper we consider logics with Henkin quantifiers in monadic vocabularies. By σ_n we mean the vocabulary with monadic predicates $P_1, \ldots, P_n$. We improve the results from [KM92] by showing that, for some n, logics $L_{\sigma_n}(\mathsf{H}_4)$ and $L_{\sigma_n}(\mathsf{F}^2_2)$ are not decidable. Additionally we give an argument showing that these logics are not equivalent to $L(\mathsf{F}_\omega)$ – the logic with all unary Krynicki quantifiers.

2 Basic concepts and facts

We recall here shortly basic relevant facts and definitions related to Henkin quantifiers (for details see the survey [KM95]).

Definition 1 *By a Henkin prefix we mean a triple $Q = (A_Q, E_Q, D_Q)$, where A_Q and E_Q are disjoint finite sets of variables and $D_Q \subseteq A_Q \times E_Q$ is the dependency relation of Q. If $(x, y) \in D_Q$, then we say that the existential variable y depends in Q on x.*

Usually we write down Henkin prefixes in a more readable form. Particularly the prefix $\mathsf{H}_n = (\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}, \{(x_1, y_1), \ldots, (x_n, y_n)\})$ is written as follows:

$$\begin{array}{c} \forall x_1 \ \exists y_1 \\ \vdots \\ \forall x_n \ \exists y_n \end{array}$$

Dependencies between variables are determined by the same rule as in the first order case: *each existential variable depends on all universal variables on the left of it.* When we forget about concrete variables of a prefix Q then we can think of Q as being a quantifier. We use here (as in the literature of the topic) ambiguous terminology applying the same terms for quantifiers and quantifier prefixes.

The simplest Henkin quantifier is of the form

$$\begin{array}{c} \forall x \exists y \\ \forall z \exists w \end{array}$$

and is called *the Henkin quantifier*. We denote this quantifier by H. We write $\mathcal{H}$ for the set of all Henkin quantifiers. For each vocabulary σ and $A \subseteq \mathcal{H}$, by $L_\sigma(A)$ we mean the logic being the extension of elementary logic in vocabulary σ by additional quantifiers from A.

The set of $L_\sigma(A)$–formulae is defined by the formation rules for elementary logic and an additional one: *if a prefix Q belongs to A and φ is a formula, then $Q\varphi$ is also a formula* – with the natural modification when we treat Q as a quantifier. We write L^*_σ for $L_\sigma(\mathcal{H})$.

The semantics is given by the inductive translation into second order formulae defined by the inductive step for $Q\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ as $\exists f_1, \ldots, \exists f_k \forall x_1, \ldots, \forall x_n \varphi(x_1, \ldots, x_n, f_1(\overline{x}_1), \ldots, f_k(\overline{x}_k))$, where $A_Q = \{x_1, \ldots, x_n\}$, $E_Q = \{y_1, \ldots, y_k\}$, and, for $i = 1, \ldots, k$, $\overline{x}_i$ is the list of all universal variables of Q on which y_i depends in Q.

For proving that the logic with Henkin quantifiers extends first order logic it suffices to observe that even in the logic with the simplest, nonlinear Henkin quantifier we can define the class of all infinite models. The sentence defining this class,

$$\exists t \ \begin{array}{c} \forall x \ \exists y \\ \forall z \ \exists w \end{array} \ ((x = z \equiv y = w) \wedge (t \neq y)),$$

is called the Ehrenfeucht sentence.

The following theorem was proven independently by H. B. Enderton and by W. J. Walkoe. A proof independent from the Axiom of Choice was given in [Mos91], see also [KM95].

Theorem 2 (Enderton–Walkoe, see [End70], [Wal70]) *There is an effective procedure f such that for each existential, second order formula φ, $f(\varphi) = Q\psi$, where*

- *Q is a Henkin prefix,*
- *ψ is quantifier free in the same vocabulary as φ,*
- *φ is semantically equivalent to $Q\psi$.*

As a corollary from [KL79] we have the following.

Theorem 3 *For each vocabulary σ with at least one function symbol or one binary predicate, the set of $L_\sigma(\mathsf{H})$–tautologies is not arithmetical.*

Proof. Let us observe that the expressive power of one unary function symbol is covered by any vocabulary with one binary predicate.

It was shown in [KL79] that the standard model (ω, s) can be characterized (up to isomorphism) by a single $L(\mathsf{H})$–sentence, say φ. Moreover, addition and multiplication are definable in $L(\mathsf{H})$ by means of the successor only. So, there is an effective translation of first order formulae into $L_\sigma(\mathsf{H})$–formulae $\psi \mapsto \psi^*$ such that $(\omega, +, \times) \models \psi$ is equivalent to $(\omega, s) \models \psi^*$. Therefore, the translation $\psi \mapsto (\varphi \Rightarrow \psi^*)$ reduces the first order truth in $(\omega, +, \times)$ to the tautology problem for $L_\sigma(\mathsf{H})$. **Q.E.D**

In this paper we consider the following classes of Henkin quantifiers (or prefixes):

- $\mathcal{H}$ – the class of all Henkin quantifiers,
- $\mathsf{H}_\omega = \{\mathsf{H}_n : n = 2, 3, \ldots\}$, where H_n is defined above,
- $\mathsf{E}_\omega = \{\mathsf{E}_n : n = 1, 2, 3, \ldots\}$, where E_n is the quantifier[1]

$$\begin{matrix} \forall x\ \exists y_1 \ldots \exists y_n \\ \forall z\ \exists w_1 \ldots \exists w_n \end{matrix} .$$

[1]This notation is a new one. The class E_ω has not been studied earlier.

Proposition 4 *Each quantifier* E_n *can be defined by the quantifier* H_{2n}*, so the logic* $L(\mathsf{E}_\omega)$ *is weaker or equivalent to* $L(\mathsf{H}_\omega)$*.*

Proof. For the proof it suffices to observe that

$$M \models \begin{matrix} \forall x_1 \; \exists y_1 \ldots \exists y_n \\ \forall z_1 \; \exists w_1 \ldots \exists w_n \end{matrix} \; \varphi[\nu]$$

if and only if

$$M \models \begin{matrix} \forall x_1 \; \exists y_1 \\ \vdots \\ \forall x_n \; \exists y_n \\ \forall z_1 \; \exists w_1 \\ \vdots \\ \forall z_n \; \exists w_n \end{matrix} \; ((\bigwedge_{i,j\leq n}(x_i = x_j \wedge z_i = z_j)) \Rightarrow \varphi)[\nu].$$

Q.E.D

For a first order formula φ, the formula $\mathsf{E}_n\varphi$ can be written as

$$\exists f_1 \ldots \exists f_{2n} \forall x \forall y \psi$$

and the formula $\mathsf{H}_{2n}\varphi$ can be written as

$$\exists f_1 \ldots \exists f_{2n} \forall x_1 \ldots \forall x_{2n} \psi',$$

where ψ and ψ' are first order. Thus, the logic $L(\mathsf{E}_\omega)$ seems to be essentially weaker than $L(\mathsf{H}_\omega)$ since in the latter we have no restriction on the number of considered arguments, here $x_1, \ldots, x_{2n}$. However, let us observe that there is no class A having essentially infinite number of nonequivalent nonlinear Henkin quantifiers such that $L(A)$ is known to be weaker than L^*.

Definition 5 *By a simple positive formula we mean a formula of the form* $Q\varphi$*, where* Q *is a quantifier prefix and* φ *is a quantifier free formula.* $Q\varphi$ *is a simple positive sentence if it is a simple formula and has no free variables.*

We assume here the common convention according to which $M \models \varphi$ means that for all valuations ν in M, $M \models \varphi[\nu]$. Therefore, the statement $M \models \varphi$ is meaningful also when φ has free variables.

Definition 6 *Let ν be a valuation in a structure M and σ be a finite, relational vocabulary. By a σ–type T of a valuation ν with respect to variables $x_1, \ldots, x_k$ we mean a conjunction of all atomic formulae and their negations which hold between $x_1, \ldots, x_k$ under ν in M. The type of ν is therefore first order, quantifier free formula. We call $x_1, \ldots, x_k$ the variables of the type T. If σ is empty, then the type T is a conjunction of equalities and inequalities.*

We have the following obvious lemma.

Lemma 7 *For each relational, monadic vocabulary σ.*
If two valuations, ν_1, ν_2 in the structure M have the same type T and the set of free variables of a formula $\varphi \in L^*_\sigma$ is the subset of the variables of T then $M \models \varphi[\nu_1]$ if and only if $M \models \varphi[\nu_2]$.

Of course if σ is not relational or relational but not monadic then the lemma is trivially false! The lemma says that independently of underlying logic for poor vocabularies (monadic or empty) the satisfiability relation, $M \models \varphi[\nu]$, depends only on the type of ν with respect to the free variables of φ.

Definition 8 *Let $\sigma = (P_0, \ldots, P_{n-1})$ be a monadic vocabulary and M be a model of vocabulary σ. For each $\varepsilon : n \longrightarrow \{0, 1\}$ we define a constituent of M:*

$$C_\varepsilon = \{a \in |M| : M \models ((\neg)^{\varepsilon(0)} P_0(a) \wedge \ldots \wedge (\neg)^{\varepsilon(n-1)} P_{n-1}[a])\},$$

where $(\neg)^0$ is the lack of negation and $(\neg)^1$ is just negation.

3 Henkin quantifiers in empty vocabulary

3.1 The degree of $L^*_\emptyset$

In what follows, we are going to prove

Theorem 9 *The following sets are of degree $\mathbf{0}'$:*

1. *The set of $L^*_\emptyset$–sentences true in all infinite models.*
2. *The set of $L^*_\emptyset$ tautologies (true in all models).*

We will give the proof in a few steps.

Let $P(M)$ be a set of all simple $L^*_\emptyset$–sentences true in a model M. For the empty vocabulary there is only one (up to isomorphism) countable model. We will identify this model with ω.

Lemma 10 *For all infinite models M, M' in the empty vocabulary, $P(M) = P(M')$.*

Proof. We prove the lemma by applying the Skolem–Löwenheim theorem. If $\varphi \in P(M)$ then φ is equivalent to a Σ^1_1–sentence $\exists \mathbf{R} \psi(\mathbf{R})$. By the Skolem–Löwenheim theorem $\psi(\mathbf{R})$ has witnesses for $\mathbf{R}$ in M if and only if it has witnesses for $\mathbf{R}$ in M'. **Q.E.D**

Lemma 11 *For each infinite model M there is an effective procedure using $P(M)$ as an oracle which assigns to each $L^*_\emptyset$–formula φ a quantifier free formula φ' such that $M \models (\varphi \equiv \varphi')$.*

Proof. For the proof we will describe the translating procedure, which for a given formula φ produces φ'. The translation is inductive and all steps are effective except the step for eliminating Henkin quantifiers which uses $P(M)$ as an oracle. We will describe this step which is the only non trivial part of the procedure.

We have as an input a formula $Q\varphi(\bar{x})$, where $\varphi(\bar{x})$ is quantifier free and $\bar{x}$ is the list of all free variables of $Q\varphi(\bar{x})$. We have finitely many types of valuations with respect to $\bar{x}$. Using $P(M)$ as an oracle we construct the list $T_1, \ldots, T_k$ of all types such that $\forall \bar{x} Q(T_i(\bar{x}) \Rightarrow \varphi(\bar{x}))$ belongs to $P(M)$ for $i = 1, \ldots, k$. (The prefix $\forall \bar{x} Q$ is treated here as a single Henkin prefix.) By lemma 7 the formula $T_1(\bar{x}) \vee \ldots \vee T_k(\bar{x})$ is equivalent to $Q\varphi(\bar{x})$ in M. **Q.E.D**

Now, we estimate the complexity of the oracle set, $P(M)$.

Lemma 12 *The tautology problem for simple positive sentences in empty vocabulary in infinite models is Π^0_1 – complete. Moreover, the set of simple formulae true in the class of all models is also Π^0_1–complete.*

Proof. It is known that the set of first order formulae satisfiable in infinite models is Π^0_1–complete. By the Skolem–Löwenheim theorem, for each first order formula $\psi(\mathbf{R})$, where $\mathbf{R}$ is the sequence of all non logical symbols, $\psi(\mathbf{R})$ is satisfiable in infinite models if and only if $\exists\mathbf{R}\psi$ is true in every infinite model. By theorem 2 we can find a simple sentence equivalent to $\exists\mathbf{R}\psi$. Thus, the set of simple sentences which are true in every infinite model is Π^0_1–hard, in the sense that each Π^0_1–set is effectively reducible to it.

One the other hand, it is Π^0_1 since the above translation can be reversed. This translation reduces the tautology in infinite model problem for simple positive sentences to the satisfiability in infinite models problem for first order logic.

For the second claim we observe that the set of simple formulae which are true in every finite model is Π^0_1. Therefore, the set of simple sentences true in all models is Π^0_1. Π^0_1–completeness of the problem follows from the first part. **Q.E.D**

Proof. [**of theorem 9**] By lemma 12 the set $P(\omega)$ is Π^0_1–complete. Additionally for each infinite M, we have by lemma 10 that $P(M) = P(\omega)$. Finally, by lemma 11 there is an algorithm which using $P(\omega)$ as an oracle decides $L^*_\emptyset$. This gives the first part of the theorem.

By lemma 10, for each infinite M, $P(M) = P(\omega)$. Therefore, by lemma 12, the set $P(\omega)$ is Π^0_1–complete. Finally, by lemma 11, there is an algorithm which decides $L^*_\emptyset$ using $P(\omega)$ as an oracle. This gives the first part of the theorem.

To prove the second part, it suffices to observe that the set of tautologies of finite models is Π^0_1 and it can be separated from the set of $L^*_\emptyset$–tautologies in infinite models by the Ehrenfeucht sentence (see page 4). **Q.E.D**

Lemmas 10 and 11 give as a corollary the following theorem from [Mos91].

Theorem 13 *For every sentence $\varphi \in L^*_\emptyset$, φ is a tautology in infinite models or φ is a contrtautology in infinite models. In the other words, the theory of infinite models in $L^*_\emptyset$ is complete.*

3.2 The degrees of $L_\emptyset(\mathsf{H}_\omega)$ and $L_\emptyset(\mathsf{E}_\omega)$

It has been proven in [KM92] that $L_\emptyset(\mathsf{H}_\omega)$ is undecidable. The proof has been done by an interpretation of the word problem for semigroups in the set of $L_\emptyset(\mathsf{H}_\omega)$–tautologies. In this section we improve this result by showing that the quantifiers E_ω are sufficient for this interpretation.

Theorem 14 *The logic $L_\emptyset(\mathsf{E}_\omega)$ is undecidable.*

Proof. It is known that the word problem for semigroups with two generators is undecidable (see e. g. [Dav77]). We will show a reduction of the word problem to the tautology problem for $L(\mathsf{E}_\omega)$. In this proof we skip the details which are the same as in the proof of the undecidability theorem for $L_\emptyset(\mathsf{H}_\omega)$ in [KM92].

If $A = \{a, b\}$ is an alphabet and $X \in A^*$, then for each variable x we can define a translation μ_x of X to the terms with two unary function symbols f and g: $\mu_x(\varepsilon) = x$, $\mu_x(aX) = f(\mu_x(X))$, $\mu_x(bX) = g(\mu_x(X))$. Let $X_i, Y_i \in A^*, i \leq k$, then the following statements are equivalent:

- $((\bigwedge_{i<k} X_i = Y_i) \Rightarrow X_k = Y_k)$ is a consequence of the theory of semigroups,
- $((\forall x \bigwedge_{i<k} \mu_x(X_i) = \mu_x(Y_i)) \Rightarrow \forall z(\mu_z(X_k) = \mu_z(Y_k)))$ is a first order tautology,
- $\forall f, g((\forall x \bigwedge_{i<k} \mu_x(X_i) = \mu_x(Y_i)) \Rightarrow \forall z(\mu_z(X_k) = \mu_z(Y_k)))$ is a second order tautology,
- $\exists z \exists f, g \forall x(\bigwedge_{i<k} \mu_x(X_i) = \mu_x(Y_i)) \wedge \mu_z(X_k) \neq \mu_z(Y_k))$ is a second order contrtautology.

In the next step we eliminate all complex terms in the last formula by adding new function symbols. We proceed in the following manner. A formula of the form $\exists z \exists h_0, \ldots, \exists h_n \forall x \forall y(\varphi(f(t(w))))$, where t is a complex term and $w \in \{x, y, z\}$, is replaced by the formula $\exists z \exists h_0, \ldots, \exists h_n \exists h_{n+1} \forall x \forall y[(y = t(x) \Rightarrow h_{n+1}(x) = f(y)) \wedge (\varphi(h_{n+1}(w)))]$. The essential property of this replacement is that we use only two universal variables and we increase only the number of existential, second order variables. In the last step we replace the sequence of second order variables by a proper E_n. If ψ is a so obtained formula, then $((\bigwedge_{i<k} X_i = Y_i) \Rightarrow X_k = Y_k)$ is a consequence of the theory of semigroups if and only if $\neg\psi$ is a tautology. It follows, that a deciding method for $L(\mathsf{E}_\omega)$ gives a deciding method for the word problem for semigroups. However, it is known that the latter problem is undecidable. **Q.E.D**

Later, the second author has shown that the above theorem remains true even for logics $L_\emptyset(\mathsf{H}_{10})$ and $L_\emptyset(\mathsf{E}_{10})$, see [Zda02].

Let us observe that the translation described in the above proof is a reduction of the Σ^0_1–complete problem (the word problem for semigroups, see e. g. [Dav77]), to the problem of truth in all infinite models for negations of simple formulae of $L(\mathsf{E}_\omega)$. Thus, by lemma 12, the tautology problems for simple positive sentences with all Henkin quantifiers and for that with quantifiers from E_ω only are equivalent. Since, by theorem 13, any sentence of $L^*_\emptyset$ is a tautology or contrtautology in infinite models then we obtain the following.

Theorem 15 *There is an effective procedure f such that for every simple positive sentence $Q\varphi$ in empty vocabulary, $f(Q\varphi) = \ulcorner \mathsf{E}_n\psi \urcorner$, where $\mathsf{E}_n\varphi$ is also a simple sentence in empty vocabulary such that $Q\varphi$ and $\mathsf{E}_n\psi$ are equivalent in infinite models.*

Theorem 16 *The sets of tautologies of $L_\emptyset(\mathsf{H}_\omega)$ and of $L_\emptyset(\mathsf{E}_\omega)$ have degree $\mathbf{0}'$.*

Proof. By theorem 9 the considered sets have degree at most $\mathbf{0}'$. By the reduction of the Σ^0_1–complete problem from the proof of theorem 14 these sets cannot be of any lower degree. **Q.E.D**

Now let us compare the complexity of the consistency problem for first order sentences with the tautology problem for simple formulae with Henkin quantifiers. We have the following.

Theorem 17 ([Mos89]) *There is an effective procedure translating first order sentences into simple sentences of $L^*_\emptyset$, $\delta \longmapsto Cons_\delta$ such that for every δ, δ is consistent if and only if $Cons_\delta$ is a tautology.*

Applying the result from theorem 15 we can strengthen this result. We know that there is an effective procedure translating first order sentences into sentences of $L(\mathsf{E}_\omega)$ in empty vocabulary, $\varphi \longmapsto Cons_\varphi$, such that for every φ, φ is consistent if and only if $Cons_\varphi$ is a tautology. Moreover, since the consistency problem for first order logic is Π^0_1–complete, by lemma 12 there is also an effective translating procedure in the opposite direction. Then the sets of consistent first order formulae and of simple formulae being $L(\mathsf{E}_\omega)$–tautologies in empty vocabulary are Π^0_1–complete. Therefore we have the following.

Theorem 18 *The sets of consistent first order formulae and simple formulae being $L(\mathsf{E}_\omega)$–tautologies in empty vocabulary are effectively equivalent.*

Let us observe that, by theorem 3, allowing non monadic vocabularies the complexity of the set of tautologies with the simplest Henkin quantifier falls beyond the arithmetical hierarchy.

3.3 Some decidable but not determined sets

In this section we show that the set of tautologies of logics with Henkin quantifiers essentially depends on some independent set-theoretic statements. In particular, consistency of a first order sentence can be expressed by means of a simple formula with a Henkin quantifier (see theorem 17). Then, it was observed in [Mos89] that for any reasonable set theory T there is a Henkin quantifier Q such that no fixed algorithm can decide $L^*_\emptyset(Q)$ provably in T. By theorem 15 we can improve this obtaining the following.

Theorem 19 *For every T – finitely axiomatizable, self-consistent ($T \nvdash \neg Cons_T$, where $Cons_T$ is the standard arithmetical sentence expressing consistency of T) extension of ZFC there is n such that no algorithm A can be proven in T to decide the tautology problem for $L(\mathsf{E}_n)$ ($T \nvdash$ "A is an algorithm deciding $L(\mathsf{E}_n)$").*

Nevertheless it might still hold that

$$T \vdash \exists x(\text{“}x \text{ is an algorithm deciding } L(\mathsf{E}_n)\text{”}).$$

Now, we are going to give an example of a logic with this property.

By $L^{(k)}(Q)$ we denote the logic obtained from the first order logic by adding the quantifier Q and restricting the number of variables to k.

Theorem 20 *For every $Q \in \mathcal{H}$ and every $k \in \omega$ the set of tautologies of $L^{(k)}(Q)$ in infinite models is decidable.*

Proof. For obtaining an algorithm which decides $L^{(k)}(Q)$ we apply the elimination of quantifiers from lemma 11. It works with the oracle for simple formulae in $L^{(k)}(Q)$. However, in this logic we have only finitely many (up to logical equivalence) simple formulae with variables $x_0, \ldots, x_{k-1}$. Since the oracle is finite the considered set is decidable. **Q.E.D**

Combining the results from theorem 19 and the last theorem we can state:

Corollary 21 *Let T satisfy conditions from theorem 19. Then for each $Q \in \mathcal{H}$ and each $k \in \omega$ there is $n \in \omega$ such that*

$$T \vdash \exists x < \overline{n} \text{“} x \textit{ is an algorithm deciding } L^{(k)}(Q) \textit{ in infinite models.”}.^{2}$$

Moreover, for sufficiently large $k \in \omega$ and $Q \in \mathcal{H}$, for all $n \in \omega$

$$T \nvdash \text{“} n \textit{ is an algorithm deciding } L^{(k)}(Q) \textit{ in infinite models.”}.$$

Proof. The first part of theorem follows from the fact that the proof of the previous theorem can be done within the theory T. The proof of the second part is exactly the same as the proof of the corresponding theorem in [Mos89]. **Q.E.D**

In fact we cannot even prove in T the completeness of any given axiomatization of $L^{(k)}(Q)$ since any such complete axiomatization would provide an algorithm deciding $L^{(k)}(Q)$. This is so, because for every sentence φ of $L^{(k)}(Q)$ either φ or $\neg\varphi$ is a tautology in infinite models.

4 Vocabularies with unary predicates

In the remaining part of this paper we consider logics restricted to vocabularies with unary predicates.

4.1 Undecidability of $L(\mathsf{H}_4)$ and $L(\mathsf{F}_2^2)$

In [KL79] it has been proven that the logic $L(\mathsf{H})$ in the language with unary predicates is decidable. On the other hand in [KM92] some appearingly only slightly stronger languages has been proven to be undecidable. Here we are going to improve these results.

Definition 22 *A set $R \subseteq \omega$ is Diophantine if for some quantifier free, arithmetical formula $\varphi(x, y_1, \ldots, y_n)$ we have:*

$$R = \{x \in \omega : \exists y_1, \ldots, y_n \in \omega \;\; \varphi(x, y_1, \ldots, y_n)\}.$$

[2] By $\overline{n}$ we denote a proper standard term naming the number n.

By the result of Matijasevič, see [Mat70], every recursively enumerable set is Diophantine.

Now we are going to strengthen theorems from [KM92]. By the quantifier F_n we mean the Krynicki quantifier defined as

$$M \models \mathsf{F}_n x_1, \ldots, x_n, , y_1, \ldots, y_n \ \varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$$
if and only if
there is a unary operation f defined on the universe of M such that
$$(M, f) \models \forall x_1 \ldots \forall x_n \ \varphi(x_1, \ldots, x_n, f(x_1), \ldots, f(x_n)).$$

We will write F_ω for the class of all quantifiers F_n.

Similarly, by the quantifier F_2^2 we mean the quantifier defined as

$$M \models \mathsf{F}_2^2 x_1^1, x_2^1, x_1^2, x_2^2, y_1, y_2 \ \varphi(x_1^1, x_2^1, x_1^2, x_2^2, y_1, y_2)$$
if and only if
there is a binary operation f defined on the universe of M such that
$$(M, f) \models \forall x_1^1 \forall x_2^1 \forall x_1^2 \forall x_2^2 \varphi(x_1^1, x_2^1, x_1^2, x_2^2, f(x_1^1, x_2^1), f(x_1^2, x_2^2)).$$

In [KM92] it has been proven that the logics $L(\mathsf{H}_4)$ and $L(\mathsf{F}_2^2)$ in the vocabulary with infinitely many unary predicates are undecidable. We can improve these theorems by the following.

Theorem 23 *The logics $L(\mathsf{H}_4)$ and $L(\mathsf{F}_2^2)$ in the vocabulary with finitely many unary predicates are undecidable.*

Proof. The proof of the relevant theorem in [KM92] relies on the ability of interpreting the truth of an existential arithmetical sentences in the problem of satisfiability for $L(\mathsf{H}_4)$–formulae (or $L(\mathsf{F}_2^2)$–formulae) in the language with infinitely many unary predicates P_i. The cardinalities of predicates interpret numerical variables. However, we do not need infinitely many variables for obtaining undecidability.

Let K be a Diophantine set which is not recursive. Let $\varphi(x, y_1, \ldots, y_n)$ be a quantifier free, arithmetical formula such that

$$K = \{x \in \omega : \exists y_1, \ldots, \exists y_n \in \omega \ \varphi(x, y_1, \ldots, y_2)\}.$$

With each $k \in \omega$ we construct the sentence $\varphi'_k \in L(\mathsf{H}_4)$ (resp. $\varphi'_k \in L(\mathsf{F}_2^2)$) having the following property: $N \models \exists y_1, \ldots, \exists y_n \varphi(k, y_1, \ldots, y_n)$ if and only if φ'_k is satisfied.

φ'_k is obtained from φ as follows. Firstly, we eliminate all nested terms in φ by adding some existential quantifiers. Then, we fix the list of all variables $y_1, \ldots, y_r$ occurring in this new formula and we choose m such, that $r < 2^m$. Each variable y_i will be replaced by i-th constituent

$$(\neg)^{\varepsilon_i(1)} P_1(x) \wedge \ldots \wedge (\neg)^{\varepsilon_i(m)} P_m(x),$$

where $\varepsilon_i : \{0, \ldots, m\} \longrightarrow \{0, 1\}$, $\varepsilon_i(j) = 1$ when the integral part of $i/2^j$ is odd, and $\varepsilon_i(j) = 0$ otherwise. As before, $(\neg)^0$ means lack of the negation, and $(\neg)^1$ is just the negation. As in [KM92] we have $L(\mathsf{H}_4)$ (or $L(\mathsf{F}^2_2)$) – formulae $I(S_1, S_2)$, $\Psi_+(S_1, S_2, S_3)$ and $\Psi_\times(S_1, S_2, S_3)$ expressing the following properties:

- $I(S_1, S_2)$ says that cardinalities of S_1 and S_2 are same;
- $\Psi_+(S_1, S_2, S_3)$ says that the result of addition of cardinalities of S_1 and S_2 gives the cardinality of S_3;
- $\Psi_\times(S_1, S_2, S_3)$ says that the result of multiplication of cardinalities of S_1 and S_2 gives the cardinality of S_3.

Finally we replace all atomic formulae $y_i = y_j$ by $I(P_{\varepsilon_i}, P_{\varepsilon_j})$, formulae $y_i + y_j = y_k$ by $\Psi_+(P_{\varepsilon_i}, P_{\varepsilon_j}, P_{\varepsilon_k})$, $y_i \times y_j = y_k$ by $\Psi_\times(P_{\varepsilon_i}, P_{\varepsilon_j}, P_{\varepsilon_k})$ and we skip added existential quantifiers.
Let φ_0 be so obtained formula, then φ'_k is $\exists^{=k} x(P_{\varepsilon_0}(x) \wedge \varphi_0) \wedge \neg\gamma$, where γ is the Ehrenfeucht sentence. So, having any decision method for formulae φ'_k, $k \in \omega$ we would obtain a decision method for K. **Q.E.D**

A finite model M for vocabulary with m unary predicates determines the vector $v_M \in \omega^{2^m}$ such that the j-th coordinate of v_M is the cardinality of j-th constituent of M. Isomorphic models determine the same vector. Therefore closed on isomorphisms classes of finite models can be identified with subsets of ω^{2^m}.

Theorem 24 *For each recursively enumerable $K \subseteq \omega$ there is a sentence φ of $L(\mathsf{H}_4)$ (or $L(\mathsf{F}^2_2)$) with unary predicates such that if $R \subseteq \omega^{2^m}$ describes the class of finite models for φ then*

$$K = \{n \in \omega : \exists k_1, \ldots, \exists k_{2^m-1} (n, k_1, \ldots, k_{2^m-1}) \in R\}.$$

The last theorem is a consequence of the proof of theorem 23. As a formula defining R we can take φ_0.

It can be contrasted with the classes of models definable by means of unary Krynicki quantifiers. These are just semilinear subsets in the vector space ω^{2^m}, or equivalently, relations definable in Presburger arithmetic (see [MW03]).

We have the following.

Theorem 25 *The logic $L(\mathsf{F}^2_2)$ and $L(\mathsf{H}_4)$ are not equivalent to the logic $L(\mathsf{F}_\omega)$ in a vocabulary containing at least one monadic predicate.*

It is relatively easy to construct an explicit example of a sentence φ of $L(\mathsf{F}^2_2)$ or $L(\mathsf{H}_4)$ with one monadic predicate such that φ defines a class of models which is not definable in $L(\mathsf{F}_\omega)$. E.g. the class of finite models (U, P) such that $card(U)^2 = card(M - U)$ is definable in both: $L(\mathsf{F}^2_2)$ and $L(\mathsf{H}_4)$.

5 Final remarks

In this paper we have solved some open problems. However, fortunately not all of them. The majority of still open problems is related to logics with Henkin quantifiers in monadic vocabularies. Here is a list of some of them:

1. **Exact borderline of decidable logics.**

   We know from [KL79] that for a monadic vocabulary σ, the logic $L_\sigma(\mathsf{H})$ is decidable. Moreover, we know that there is n, such that $L_{\sigma_n}(\mathsf{H_4})$ is undecidable. However, we do not know for which values of n $L_{\sigma_n}(\mathsf{H_4})$ is undecidable. The same question is open for logics $L(\mathsf{E_n})$ in monadic vocabularies.

   The question of decidability of $L(\mathsf{H_3})$ in monadic vocabularies remains open.

2. **Classes of models definable in $L(\mathsf{H}_3)$.**

   Classes of models which are definable in monadic $L(\mathsf{H})$ or in monadic $L(\mathsf{F}_\omega)$ can be described as semilinear sets (see [KL79] and [MW03], in case of $L(\mathsf{H})$ the other direction is wrong). The analogue for $L(\mathsf{H}_4)$ does not hold (see theorem 25). Which classes of models are definable by $L(\mathsf{H}_3)$–formulae in the monadic vocabulary? Are they also semilinear?

3. **The relation between quantifiers $\mathsf{H_n}$ and $\mathsf{E_n}$.**

   We know that $\mathsf{E_n} \leq \mathsf{H_{2n}}$ for each n. However, we do not know the smallest k such that $\mathsf{E_n} \leq \mathsf{H_k}$. Additionally, the problem whether the class E_ω is weaker than H_ω is also open.

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