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TOWARD USEFUL TYPE-FREE THEORIES. I

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Contents of Parts I and II

PART A

ISSUES FOR SEMANTICS AND MATHEMATICS

§1. The paradoxes: a continuing challenge. There is a distinction between *semantical paradoxes* on the one hand and *logical* or *mathematical paradoxes* on the other, going back to Ramsey [1925]. Those falling under the first heading have to do with such notions as *truth*, *assertion* (or *proposition*), *definition*, etc., while those falling under the second have to do with *membership*, *class*, *relation*, *function* (and derivative notions such as *cardinal* and *ordinal number*), etc. There are a number of compelling reasons for maintaining this separation but, as we shall see, there are also many close parallels from the logical point of view.

The initial solutions to the paradoxes on each side—namely Russell's theory of types for mathematics and Tarski's hierarchy of language levels for semantics—were early recognized to be excessively restrictive. The first really workable solution to the mathematical paradoxes was provided by Zermelo's theory of sets,

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subsequently improved by Fraenkel. The informal argument that the paradoxes are blocked in ZF is that its axioms are true in the *cumulative hierarchy of sets* where (i) unlike the theory of types, a set may have members of various (ordinal) levels,[2] but (ii) as in the theory of types, the level of a set is greater than that of each of its members. Thus in ZF there is no set of all sets, nor any Russell set $\{x \mid x \notin x\}$ (which would be universal since $\forall x(x \notin x)$ holds in ZF). Nor is there a set of all ordinal numbers (and so the Burali-Forti paradox is blocked).

The step to the theory BG of sets and classes developed by Bernays and improved by Gödel is intuitively modeled in the cumulative hierarchy extended to one further top level at which we find the *proper classes*; classes of lower levels are identified with sets. In BG one proves the existence of the class V of all sets, but not of all classes; further we have a class ON of all sets which are ordinal numbers. V and ON are proper classes and $V \notin V$, ON $\notin$ ON.

The theories ZF and BG (when augmented by AC, the Axiom of Choice, also intuitively true in the cumulative hierarchy) provide a framework in which practically all of current mathematics can be systematically represented in an unforced manner. The exceptions are marginal; one which will be at the center of attention here is the informal general theory of *mathematical structures*, particularly the *theory of categories*.[3] It is natural in certain situations to consider all structures of a given kind as forming a new structure, of which they are the elements. For example, the class B consisting of all structures $(A, \circ, =_A)$ with a commutative and associative binary operation $\circ$ itself forms such a structure under Cartesian product $\times$; that is, $(B, \times, \cong)$ is naturally considered as a member of B. Category theory deals with structures of a more sophisticated kind which are useful in extensive parts of algebra and topology. It is mathematically natural to impose a category structure (CAT,...) on the class CAT of all categories and thus to consider the former as a member of the latter. However, the logical problem of dealing with such is already present in simpler structures such as $(B, \ldots)$ above (or even with the class of all classes). We are blocked from forming B in ZF or BG, though in the latter we can form the class B' of all (commutative and associative) structures $(a, \circ, =_a)$, where $a \in V$. This is the basis for a distinction between "small" and "large" structures whose domains are sets, resp. classes (cf. MacLane [1977, §I.7] for its employment in category theory, e.g. with the category of all small groups, the category of all small categories, etc.). The systematic use of this distinction serves all practical ends in category theory, though it is not without awkward turns. We shall return to its troublesome aspects in Part II-C below.

ZF and BG are both *untyped* formalisms, i.e. the levels which we have in mind in their informal interpretations do not appear explicitly in the syntax.[4] Thus we can form expressions $x \in y$ without restriction, in particular the expression $x \in x$. On the other hand, one usually refers to a formal framework (a theory or structure) as being

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[2] These levels are usually called *ranks* in current set theory.

[3] Another exception is Brouwer's theory of choice sequences. In that case, the formal reduction of theories with choice sequences to theories without, accomplished by Kleene/Vesley [1965] and Kreisel/Troelstra [1970], ends with systems interpretable in ZF.

[4] There have been axiomatizations of set theory in which the ranks (footnote 2) appear explicitly, though of course as variables; cf. Scott [1960].

type-free if it admits significant instances of *self-application.* Evidently a direct account of the informal theory of categories would be type-free in this sense. There has been extensive experience in mathematical logic with type-free structures from work in *recursion theory* and with the *λ-calculus.*[5] While the structures met there have meager mathematical content, this experience among others lends hope to the pursuit of richer type-free models and theories.

There has indeed been considerable work on type-free mathematical theories, i.e. theories of classes and/or theories of functions.[6] The motivations have been various, ranging from the *ideological* (as in Frege's and Curry's programs) to specifically *useful* (as emphasized here). For some workers the challenge has simply been "logical": to find a simple, consistent but mathematically expressive type-free theory. There have been no notable successes, at least none which speak for themselves. But there are an interesting variety of "solutions" which have a corresponding variety of merits. How to assess these and where to look for further progress seems to me to require a more explicit analysis of the problem or problems to be solved than has been provided thus far. One of the main purposes of this paper is to advance such an analysis.

On the semantical side there has been an (equally) extensive pursuit of type-free frameworks, especially by workers in philosophical logic.[7] This is partly motivated by the fact that natural language abounds with directly or indirectly self-referential yet apparently harmless expressions—all of which are excluded from the Tarskian framework. Fretting about the severe restrictions placed by that solution, philosophers have sought to liberalize semantic theory so as to accomodate such expressions while still blocking the paradoxes. Another purpose of the present paper is to show how a logical analysis of problems and solutions on the semantical side closely parallels those on the mathematical side—at least to a point. But when one indulges in this kind of comparison, several striking points of difference emerge: (i) There has been no success for semantics comparable to that achieved by ZF (or even Z) for mathematics.[8] (ii) Unlike mathematics, the need for a type-free account is immediately apparent. (iii) Solutions to the semantic paradoxes have been *local* rather than taking their place within *global* (i.e. over-all) semantical frameworks; thus relatively few constraints have been considered. In my view, (iii) points to the need for more extensive criteria to be met by type-free semantical solutions.

§2. Plan of the present paper (I and II). Most of the body of the paper consists of a review of work by the author over the last seven years and of related work by others,

[5] Self-application occurs naturally elsewhere in mathematics, e.g. whenever a set of actions (such as a group of operators) is taken to be acting on itself.

[6] The bibliography to this paper will give the reader a first quick direction toward such; cf. also the historical notes in §14 below and the bibliography to Feferman [1975c].

[7] Some of this can be found in the bibliography here and in §14; the bibliography in Martin [1970] and in its second edition [1978] is a much better source.

[8] There is no problem with the idea of a cumulative hierarchy of languages L_n (or even transfinite hierarchy L_α); indeed as ordinarily construed one has $L_n \subseteq L_{n+1}$ in the Tarski hierarchy and the truth predicate T_n for L_n makes sense when applied to statements in all earlier languages (cf. §4 below). But there is (to my knowledge) no natural theory for such in which the level distinctions make no explicit appearance.

particularly Peter Aczel, aimed at constructing useful type-free theories. The main new material will be found in Part II-D. In the remainder of this Part A, I take up an analysis of both the Liar Paradox and Russell's Paradox, emphasizing the parallel features which lead to a contradiction. The possible solution routes are also paralleled. These consist in restrictions of (1°) language, (2°) logic or (3°) basic principles. The solutions of Tarski and Russell fall under (1°); in the remainder of the paper only the routes (2°) and (3°) are considered. A common move in (2°) is to pass to some sort of 3-valued logic, while in (3°) one has to deal with restrictions of the *truth scheme* (TA) or the *comprehension scheme* (CA) in a type-free language with ordinary logic. The following points are immediately observed: (i) Only the *constructive* part of ordinary reasoning is required in deriving the contradictions, and the paradoxes pose as much of a problem for the constructivist as for the classical logician. (ii) The *extensionality principle*, which is frequently considered to be fundamental for mathematical theories of classes, plays no role in Russell's paradox.[9] Particular attention will be paid in the following to *nonextensional theories* which may be considered equally well within classical or intuitionistic logic. For simplicity, though, we take ordinary logic to be represented in CPC, the first order classical predicate calculus with equality, unless otherwise specifically indicated.

A common suggestion to get around the paradoxes is that one is somehow dealing with undefined propositions, i.e. statements which are neither true nor false in the ordinary sense. Different logics which are *restrictions* of CPC have been proposed to express this idea formally. They are here considered within the single framework of logics for structures with *partial predicates*, which is the subject of Part B. Special attention is given to different interpretations of the biconditional, since that is the critical operator in both (TA) and (CA). Arguments are presented against 3-valued and other restrictive logics due to their debilitating effects on ordinary reasoning. Theories of partial predicates formulated within ordinary logic are promoted instead. It turns out that such may be treated most elegantly within an *extension* of CPC by use of a new biconditional $\equiv$ which was introduced in the paper Aczel/Feferman [1980]. The main result there yields consistency of the truth and comprehension schemes in the form

$$(\mathrm{TA})_{\equiv} \qquad \varphi \equiv \mathrm{Tr}(\ulcorner\varphi\urcorner)$$

and

$$(\mathrm{CA})_{\equiv} \qquad \forall y[y \in \{x \mid \varphi(x)\} \equiv \varphi(y)],$$

without restriction on φ. In the logic of $\equiv$, a partial predicate $P(x)$ is treated as a pair of predicates $(P(x) \equiv t)$, $(P(x) \equiv f)$ where t and f are identically true and false statements, resp. The consistency result is established by an inductive fixed-point argument; this has been used frequently in the construction of models for partial-predicate versions of (TA) and (CA) but it is here given a new twist (due to Aczel). The new approach described here absorbs the previous treatments of partial classes in the papers Feferman [1975b], [1977].

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[9] The first of these points is not novel. The second is so obvious it hardly seems worth mentioning, except as a corrective to the wide-spread and uncritical assumption of extensionality.

It is seen from the description just given that Part B is presented in a form equally applicable to semantical and mathematical theories. However, on the mathematical side its content is shown to be rather meager. In Part II-C we take up specifically mathematical criteria for a richer useful type-free theory. Such a theory is to account not only for the notions and structures of ordinary mathematical experience but also for such objects as the category of all groups, the category of all categories and the category of all functors between two given categories. Here a certain asymmetry is brought out between the roles of functions and classes, so that the former appear in a way conceptually prior to the latter (just as in certain constructive theories). Furthermore, while functions are treated naturally as partial objects in ordinary mathematical discourse, classes do not have that character. A final topic of Part II-C is how to get along without extensionality as a matter of course.

All of the preceding motivates the passage in Part II-D to nonextensional type-free theories of partial operations and total classes. Such theories had been formulated previously for constructive purposes in Feferman [1975a], [1979]. (We interpolate their connection with another approach to a semantical theory due to Aczel [1980], for what he calls *Frege structures.*) In Part II-D we shall now study the possible utility of these systems (both in their original formulation and with special modifications) as a formal framework for the general theory of mathematical structures, particularly category theory.

The paper (Part II) concludes with a discussion which retraces some of the choices made here, raises questions about alternative approaches, and looks to directions where further progress may be possible.

§3. Analysis of the Liar paradox. We here essentially follow Tarski [1956, p. 165] (from the translation of his 1935 paper on the concept of truth). (While there is nothing novel here, the details are put down for purposes of further discussion and comparison with Russell's paradox.) The contradiction results from a combination of the following features of ordinary language usage: construction of statement names and in particular of self-referential statements, acceptance of ordinary reasoning, and acceptance of the passage from the truth of a statement to the statement itself and vice versa.[10]

These features are analyzed here formally in terms of a logical system S specified by a syntax, underlying logic and basic postulates. We use 'φ', 'ψ',... to range over the statements of L (the language of S); these are assumed to be closed under the usual propositional operators here symbolized by $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. Formulas φ with at most one free variable x are indicated by $\varphi(x)$; then for t an individual term, $\varphi(t)$ denotes the result of substituting t for x in φ.

1°. *Syntax.* (i) (*Naming*). Each statement φ of L has a name in the language, i.e. there is an associated closed term $\ulcorner\varphi\urcorner$ of L.

(ii) (*Self-reference*). For each formula $\psi(x)$ we can construct a statement φ which is equivalent in S to $\psi(\ulcorner\varphi\urcorner)$.

[10] Burge [1979] argues that the contradiction does not lie in natural language itself but in theories "promoted by people" about their language: "Natural languages *per se* do not postulate or assert anything. What engenders paradox is a certain naive theory or conception of the natural concept of truth. It is the business of those interested in natural language to improve on it." (op. cit., pp. 169–170).

2°. *Logic.* The axioms and rules of ordinary propositional calculus are accepted.

3°. *Basic principles.* The following axioms are accepted for a predicate $T(x)$ which is interpreted as expressing that x is true:

(TA) $$T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi,$$

for each statement φ of L.

For the derivation of a contradiction in S we take φ with $\varphi \leftrightarrow \neg T(\ulcorner\varphi\urcorner)$ (in S). Using (TA) and transitivity of equivalence we obtain $T(\ulcorner\varphi\urcorner) \leftrightarrow \neg T(\ulcorner\varphi\urcorner)$ in S. Let $\theta = T(\ulcorner\varphi\urcorner)$. Thus both $(\theta \to \neg\theta)$ and $(\neg\theta \to \theta)$ are provable. It is a result of ordinary logic that $(\theta \to \neg\theta) \to \neg\theta$, hence $\neg\theta$ follows by *modus ponens*. But then θ follows from the second implication. Thus S is inconsistent.

Discussion. *Ad* 1°(i). The process of statement naming in natural language is accomplished uniformly by quotation. However, there are other nonuniform means which are frequently applied, e.g. numbering of statements in a text or use of such locutions as "the statement you just made", etc. In formal languages statement naming may be accomplished by an enumeration φ_n $(n = 0, 1, 2, \ldots)$ of all the well-formed statements and assignment to φ_n of a term denoting n (e.g. the numeral $\bar{n}$).

Ad 1°(ii). Natural language is rife with implicit self-reference, e.g. in such statements as "You're not listening to me", or "I never tell a lie" or "English is easy to teach foreigners" and of course by explicit fabrication, e.g. "This statement has less than ten words." The possibility of self-reference for formal systems was realized by Gödel [1931]. From this one can show in a fragment of elementary number theory that with each formula $\psi(x)$ is associated φ such that $\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)$ is provable.

By the analysis of Jeroslow [1973], one actually has a term τ in number theory such that φ is identical with $\psi(\tau)$ and for which $\ulcorner\varphi\urcorner = \tau$ is provable. In natural language we have the stronger possibility of *literal self-reference*, i.e. construction of φ identical with $\psi(\ulcorner\varphi\urcorner)$. But this can also be accomplished in any language containing at least one closed term τ, by the *ad hoc* baptismal process which takes $\ulcorner\varphi\urcorner$ to be τ for the statement $\psi(\tau)$ as φ. The conclusion from all this is that the hypothesis 1° (ii) is not by itself pernicious.

Ad 2°. It can be argued that the laws of logic implicit in ordinary reasoning are just those of the *classical 2-valued propositional* (and predicate) *calculus*. However, inspection of the argument to contradiction above shows that the law of excluded middle is not used, so the argument is already one in *intuitionistic logic*. It is usual in the latter to identify $\neg\theta$ with $\theta \to f$, where f is an identically false statement. Then the principle $(\theta \to \neg\theta) \to (\neg\theta)$ translates into $(\theta \to (\theta \to f)) \to (\theta \to f)$ which is a special case of $(\theta \to (\theta \to \chi)) \to (\theta \to \chi)$. All the other laws of $\to$ and $\leftrightarrow$ applied in the derivation of the contradiction are clearly valid intuitionistically.[11] {In fact the argument already takes place in what is called *minimal logic* (cf. Prawitz [1965]), where we have the usual axioms and rules for $\to$. The conclusion of the argument is a

[11] As already mentioned, this observation is not novel, though hardly emphasized in the literature. It was first brought to my attention some years ago by Harvey Friedman. Several readers of the ms of this paper have pointed out that this observation is already contained in Curry's derivation of a contradiction in certain logics based on combinatory systems ("Curry's paradox"); cf. Curry [1942] or Curry [1980, pp. 94–95]. The difference is that in Curry's ("illative") systems, the combinatory and logical aspects are intertwined. In deriving the paradox, the former is used to construct a "paradoxical" combinator **Y** which then leads to a variety of forms of self-reference. Still it is fair to say that when the logical part of the argument is disengaged, it can be seen to proceed entirely within intuitionistic (indeed, minimal) logic.

specific contradiction θ and $\neg\theta$, i.e. θ and $(\theta \to f)$, hence f; intuitionistically we can go on to infer any statement, but not in minimal logic. However, we could replace f by an arbitrary χ to begin with, if one wants to derive full inconsistency using only minimal logic.}

Ad 3°. Truth in natural language is used both as a *predicate* and as an *operator*. An example of the former is "What you just said is true", and of the latter is "It is true that inflation has not abated". Treated formally, truth as an operator would assign to each statement φ a new statement $T^*\varphi$ where T^* corresponds to the phrase "It is true that". Self-reference provides no way to construct φ with $\varphi \leftrightarrow \neg T^*\varphi$. The basic axioms for T^* would be:

$$(T^*A) \qquad\qquad \varphi \leftrightarrow T^*\varphi$$

for each statement φ. No contradiction results from taking these axioms in place of (TA), since we can interpret T^* as the identity operator.

§4. Solution routes for the Liar paradox. By a "solution" to the paradox is meant the production of a *consistent system S* which has more or less the properties 1°–3° of §3. Since the "more or less" is vague, there cannot be a unique definite solution to the problem posed by the paradox. Usually further criteria are brought to bear (implicitly or explicitly) to test proposed solutions. The most demanding would be to situate the solution within a coherent global (overall) semantics for natural language. We shall return to the question of such criteria in Part II-C. Here the matter is considered only locally; the following is just a preliminary survey of possible solution routes.

To obtain some generality we assume that a consistent theory S_0 with language $L_0 = L(S_0)$ is given whose syntax and semantics are regarded as unproblematic, and that S is to be found as an extension of S_0 (with $L = L(S)$ an extension of L_0). S_0 might correspond to a fragment of natural language, or it could be an axiomatic system for a part of science or mathematics.

Corresponding to 1°–3°, there are three kinds of restrictions which might be made.

1*. *Restriction of syntax.*
2*. *Restriction of logic.*
3*. *Restriction of basic principles.*

The escape route 1* is that taken in a Tarskian approach. A truth predicate T for L_0 is available only in a secondary ("higher" or "meta") language L. Statement-naming $\ulcorner\varphi\urcorner$ is provided only for φ in L_0; thus $T(\ulcorner\varphi\urcorner)$ is a sentence of L only for φ in L_0. This restriction blocks full self-reference, i.e. there is no φ of L for which $\varphi \leftrightarrow \neg T(\ulcorner\varphi\urcorner)$ holds. We may use full classical logic in L and all available instances of (TA), i.e. $T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$ just for φ in L_0 (in that sense the solution is also a restriction of 3°). Tarski's set-theoretical definition of truth for L_0 provides a consistency-proof of this system S. Evidently, the solution can be iterated to give a hierarchy of languages $L_0, L_1, L_2, L_3, \ldots$ for each of which the truth predicate is available only in the following language.[12]

[12] In his 1935 paper, Tarski stated: "In my opinion the considerations of §1 prove emphatically that the concept of truth (as well as other semantical concepts) when applied to colloquial language in conjunction with the normal laws of logic leads inevitably to confusions and contradictions." He saw the only possible way out in the "reform" of this language which would amount to a split "into a series of

For the reasons given in §1, only solution routes 2* and/or 3* will be pursued in the present paper. Actually, the Tarskian solution for L_0 can be recast alternatively as falling under 3*. Simply take a system S in which one meets the conditions 1° (i), (ii) and 2° without restriction, but where (TA) is restricted to $T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$, for φ a sentence of L_0. This is shown consistent by incorporating enough arithmetic in S to carry out Gödel numbering of $L(S)$ within S and thence all of 1° followed by Tarski's consistency proof in which T is interpreted as the truth predicate for L_0. However, this solution is rather weak: though we have sentences $T(\ulcorner\varphi\urcorner)$ in L for all φ in L, and in particular for φ such that $\varphi \leftrightarrow \neg T(\ulcorner\varphi\urcorner)$, nothing interesting can be said about these statements. Further, there is no evident means of iterating this procedure i.e. to find acceptable instances of $T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$ for φ not in L_0.

REMARKS. (i) It is common in the literature on the semantical paradoxes to present a *model* $\mathfrak{M}$ for a solution S without explicitly describing S. (Actually, the model itself is often not even described precisely; rather, an account is given which would lead to a model.) Obviously, once $\mathfrak{M}$ is constructed, one can take S to be the set of statements true in $\mathfrak{M}$. We have formulated the search for solutions as the construction of suitable consistent S for several reasons. First, there may be other means than provision of a model to establish consistency of such S, e.g. the use of proof-theoretic methods. Secondly, a criterion which might be applied is that the passage of S_0 to S is only to be a *matter of convenience* in the sense that S is supposed to be a *conservative extension of* S_0, i.e. for φ in L_0 we are to have $S \vdash \varphi$ only if $S_0 \vdash \varphi$.[13] Finally, I believe that mere presentation of a model $\mathfrak{M}$ does not give full appreciation of the merits or faults of a proposed solution when it is not said for what system S this is a model.

(ii) Solutions sometimes involve several kinds of restrictions simultaneously, e.g. as Tarski's restriction of 1° automatically required a restriction of 3°. But there is another possibility to consider: one may make an *extension* of ordinary logic provided there is a compensating restriction elsewhere. This is the approach taken in Part B, §§11–12.

§5. Analysis of Russell's paradox. This analysis parallels that of the Liar in many, but not all respects. Classes are regarded as being somehow the objectification of properties, and a theory S of classes counts such among the objects of its universe of discourse. However, there is no presumption that all objects are classes. In the formal framework of S, properties are expressed by formulas φ with a distinguished free variable, say "x". We write $\varphi(x)$ for such, or $\varphi(x, \ldots)$ when we want to stress that φ may contain other free variables ("parameters"). For a term τ, $\varphi(\tau)$ denotes the result of substituting τ for x at all its free occurrences in φ (assuming τ is free for x in φ, i.e. there are no collisions of variables).

languages ... each of which stands in the same relation to the next in which a formalized language stands to its metalanguage." But Tarski doubted that such could be done while preserving the naturalness of everyday language. Cf. the translation in Tarski [1956, p. 267].

[13] It is true that there is a necessary and sufficient model-theoretic condition for S to be a conservative extension of S_0 when these are theories formulated in first order CPC, namely: for every model $\mathfrak{M}_0$ of S_0 there exists an expansion of $\mathfrak{M}_0$ (by additional relations, etc.) to a model $\mathfrak{M}$ of S. However, this does not necessarily carry over to theories in other logics.

The following is assumed of S and $L(= L(S))$.

1°. *Syntax* (*class-naming*). With each formula $\varphi(x)$ of L is associated a term $\{x \mid \varphi(x)\}$ (in which "x" is bound).

2°. *Logic*. The axioms and rules of ordinary propositional and predicate calculus are accepted in S.

3°. *Basic principles*. There is a binary relation $y \in z$ in L which is taken to express that y is a member of the class z. The following comprehension axiom scheme is accepted in S for each $\varphi(x)$ in L:

$$\text{(CA)} \qquad \forall y[y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)]$$

The possibility of "self-reference" arises in L by consideration of formulas of the form $t \in t$. Take $r = \{x \mid \neg(x \in x)\}$ so that $\forall y[y \in r \leftrightarrow \neg(y \in y)]$ holds by (CA). Then by the rules of the predicate calculus, we have $[r \in r \leftrightarrow \neg(r \in r)]$ in S. Thus for $\theta = (r \in r)$ we have $S \vdash (\theta \leftrightarrow \neg\theta)$, so the inconsistency of S follows just as in §3.

It is common to incorporate Frege's idea of *classes as extensions of concepts* in a theory of classes, by assumption of an *axiom of extensionality*. The following is its direct expression in the present framework:

$$\text{(Ext)} \qquad \{x \mid \varphi(x)\} = \{x \mid \psi(x)\} \leftrightarrow \forall x[\varphi(x) \leftrightarrow \psi(x)].$$

Note that the inconsistency of S depends in no way on this further assumption. Thus S can be considered equally well as a theory of properties (conceived of as objects), in which $\{x \mid \varphi(x)\}$ is read as "the property of x, that $\varphi(x)$" and $y \in z$ is taken to express that y has the property z. Since the terminology of class and membership is established in mathematics, and it is the mathematical uses of such a theory that interest us, we shall not follow this reading in terms of properties. In other words, *we countenance nonextensional theories of classes.*

DISCUSSION. (i) *Ad* 1°. The hypothesis that classes may be named in the language, by the formation of abstracts $\{x \mid \varphi(x)\}$ as terms, is inessential if we replace 3° by $\exists z \forall y[y \in z \leftrightarrow \varphi(y)]$.

(ii) *Ad* 2°. On the other hand, using 1° as it stands, we do not need the full predicate calculus to derive the contradiction—its quantifier-free part ("free-variable logic") suffices if we drop the quantifier $\forall y$ in 3°. Note in any case, just as for the Liar paradox, that the inconsistency is established using only intuitionistic logic (in fact, minimal logic).

(iii) Under the given assumptions on S, we cannot define which objects in the universe are classes. It will prove convenient later to consider theories with an additional predicate $\mathrm{Cl}(z)$ expressing that z is a class. In this case (CA) is written as

$$\mathrm{Cl}(\{x \mid \varphi(x)\}) \wedge \forall y[y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)],$$

or as

$$\exists z([\{x \mid \varphi(x)\} = z \wedge \mathrm{Cl}(z) \wedge \forall y[y \in z \leftrightarrow \varphi(y)]);$$

then (Ext) would be strengthened to

$$\mathrm{Cl}(a) \wedge \mathrm{Cl}(b) \wedge \forall x[x \in a \leftrightarrow x \in b] \rightarrow a = b.$$

(iv) Such a predicate Cl is clearly unnecessary when the universe is conceived as consisting entirely of classes. In that case, extensionality is simply formulated as $\forall x[x \in a \leftrightarrow x \in b] \to a = b$.

§6. Solution routes for Russell's paradox. Here we parallel §4, assuming to begin with an initial consistent S_0 in a language $L_0 = L(S_0)$. The problem is to find a consistent extension S satisfying more or less the conditions 1°–3° of §5. As before, there are three kinds of restrictions one can make.

1*. *Restriction of syntax.*

2*. *Restriction of logic.*

3*. *Restriction of basic principles.*

The escape route 1* is that taken in Russell's theory of types, as simplified by Ramsey [1925] (i.e. the "simple" or "unramified" theory of types). Instead of variables ranging unrestrictedly over a single universe of discourse, one has, for each n, variables $x^{(n)}, y^{(n)}, \ldots$ of type n. The variables of the initial language L_0 are now taken to be of type 0. Each term τ will also be of a definite type, and $(\sigma \in \tau)$ is a formula only for σ of type n and τ of type $(n+1)$ for some n. Then formulas are built up by the operations of the predicate calculus, and if $\varphi(x^{(n)})$ is a formula, $\{x^{(n)} \mid \varphi(x^{(n)})\}$ is a term of type $(n+1)$. Then (CA) is necessarily restricted to

$$\forall y^{(n)}[y^{(n)} \in \{x^{(n)} \mid \varphi(x^{(n)})\} \leftrightarrow \varphi(y^{(n)})].$$

If extensionality is also to be included, one adds, for each n,

$$\forall x^{(n)}[x^{(n)} \in a^{(n+1)} \leftrightarrow x^{(n)} \in b^{(n+1)}] \to a^{(n+1)} = b^{(n+1)},$$

since the objects of each type $n+1$ are conceived to be classes. A set-theoretical model of the resulting system is obtained by starting with a model $\mathfrak{M}_0 = (M_0, \ldots)$ of S_0 and interpreting the variabless of type n as ranging over M_n where, for each n, M_{n+1} is taken to be the set of all subsets of M_n.

In the following we shall only pursue solutions based on restrictions of type 2* or 3*. The theory of Zermelo (or ZF) with (possible) urelements gives a solution based on a type 3* restriction. This has just one sort of variable, and the relation $\sigma \in \tau$ is formed without restriction. There is a predicate $\mathrm{Cl}(x)$, read here as "x is a set". The urelements are those x such that $\neg\mathrm{Cl}(x)$, and the axioms S_0 are taken relativized to the urelements. Instead of $\{x \mid \varphi(x)\}$ in general one has only special cases, corresponding to *separation*, *pairing*, *union*, *power set*, etc. But the *axiom of foundation* prevents any instance of self-membership $x \in x$.

In Part II-C we shall discuss mathematical criteria for a type-free theory of classes, in which there are significant instances of self-membership (and of self-application in a broader sense).

PART B

TYPE-FREE THEORIES OF PARTIAL PREDICATES (TRUTH AND MEMBERSHIP)

§7. Partial predicates and structures. At the outset we consider the route of restricting logic without restricting language. Though this direction will eventually be abandoned, it turns out that useful results can be garnered from its pursuit. It is a common move to try to pin the difficulty in the paradoxes on reasoning with *meaningless statements*, indeed with meaningless instances of basic predicates such as $T(x)$ or $(x \in y)$ (often called *truth-gaps* in the literature on the semantic paradoxes). If these are not to be banned syntactically then the logic must somehow be altered to handle reasoning with potentially meaningless statements. Various logics have been devised for this purpose; some of them will be examined in §§8–10. The notion of *partial predicates* and the associated *partial structures* provide a common framework for their comparison.

We write t and f for the two truth-values "true" and "false". Let M be an arbitrary set. By a *partial k-ary predicate* $\tilde{R}$ on M $(1 \leq k < \omega)$ is meant a partial function $\tilde{R}$ from M^k to $\{t, f\}$. Introducing the symbol "u" for "undefined", where $u \neq t$, $u \neq f$, every such predicate can be identified with a function $\tilde{R}: M^k \to \{t, f, u\}$.[14] Alternatively, we may identify $\tilde{R}$ with a *disjoint pair* $(R, \bar{R})$ of ordinary k-ary relations in M, i.e. where $R \subseteq M^k$, $\bar{R} \subseteq M^k$ and $R \cap \bar{R} = \varnothing$. In the following we shall move according to convenience from one form to another of regarding partial predicates. $\tilde{R}$ is said to be *total* if it is a total function from M^k to $\{t, f\}$, i.e. if u is not in the range of $\tilde{R}$ as a map into $\{t, f, u\}$. Viewed as a disjoint pair $(R, \bar{R})$, the predicate is total if $R \cup \bar{R} = M^k$; then $\bar{R}$ is just the complement $M^k - R$.

The set of truth-values $\{t, f, u\}$ is partially ordered by $u \leq t, u \leq f$ in addition to $u \leq u$, $t \leq t$, $f \leq f$; diagramatically:

$$\begin{array}{ccc} t & & f \\ & \vee & \\ & u & \end{array}$$

Given two k-ary partial predicates $\tilde{R}$ and $\tilde{R}'$, we put $\tilde{R} \leq \tilde{R}'$ if for all $m_1, \ldots, m_k \in M$, $\tilde{R}(m_1, \ldots, m_k) \leq \tilde{R}'(m_1, \ldots, m_k)$. This means that in their guise as partial functions from M^k to $\{t, f\}$, $\tilde{R}$ is a subfunction of $\tilde{R}'$. Equivalently, in their guise as disjoint pairs $(R, \bar{R})$ and $(R', \bar{R}')$, the relation holds when $R \subseteq R'$ and $\bar{R} \subseteq \bar{R}'$. Note that if $\tilde{R} \leq \tilde{R}'$ and $\tilde{R}' \leq \tilde{R}$ then $\tilde{R} = \tilde{R}'$.

In the following we shall consider *partial structures* $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R}_1, \ldots, \tilde{R}_n, \ldots)$ constituted from some fixed structure $\mathfrak{M}_0 = (M, \ldots)$ in the ordinary ("total") sense of the word for a language L_0, together with one or more partial predicates $\tilde{R}_n$ on M. For simplicity, much of our work with such will be illustrated by $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R})$, where $\tilde{R}$ is binary. For example, given such $\mathfrak{M}$ and $\mathfrak{M}' = (\mathfrak{M}_0, \tilde{R}')$, we put $\mathfrak{M} \leq \mathfrak{M}'$ if $\tilde{R} \leq \tilde{R}'$.

Inductive constructions of partial structures are ubiquitous in the present subject. They may be subsumed under the following general approach. Let $\mathscr{K}$ be the class of structures $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R})$, with fixed $\mathfrak{M}_0$ ordered by $\leq$. An operator Γ on $\mathscr{K}$ associates with each $\mathfrak{M}$ a new structure $\Gamma(\mathfrak{M}) = (\mathfrak{M}_0, \tilde{R}')$; we also write $\Gamma(\tilde{R})$ for $\tilde{R}'$. Γ is called a *monotonic operator* if $\mathfrak{M} \leq \mathfrak{M}' \Rightarrow \Gamma(\mathfrak{M}) \leq \Gamma(\mathfrak{M}')$.

FIXED-POINT THEOREM. *For any monotonic operator Γ and any $\mathfrak{M} \leq \Gamma(\mathfrak{M})$ there is a least $\mathfrak{M}^*$ with $\mathfrak{M} \leq \mathfrak{M}^*$ and $\Gamma(\mathfrak{M}^*) = \mathfrak{M}^*$.*

[14] This is just a special case of a general notion of *Ω-valued predicate* $R: M^k \to \Omega$ (and thence of Ω-valued structures), where Ω is any set of "truth-values". Only the special cases $\Omega = \{t, f, u\}$ and $\Omega = \{t, f\}$ are used here.

PROOF (by the usual inductive fixed-point argument). We here treat the $\tilde{R}$ as partial functions. Given $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R})$, define $\tilde{R}^{(\alpha)}$ for ordinals α by $\tilde{R}^{(0)} = \tilde{R}, \tilde{R}^{(\alpha+1)} = \Gamma(\tilde{R}^{(\alpha)})$ and $\tilde{R}^{(\lambda)} = \bigcup_{\alpha<\lambda} \tilde{R}^{(\alpha)}$ for limit λ. It is proved by induction on α that $\tilde{R}^{(\alpha)} \leq \tilde{R}^{(\alpha+1)}$. This holds for $\alpha = 0$ by hypothesis; if true for α it is true for $\alpha + 1$ by monotonicity. Finally,

$$\tilde{R}^{(\lambda+1)} = \Gamma\left(\bigcup_{\alpha<\lambda} \tilde{R}^{(\alpha)}\right) \geq \bigcup_{\alpha<\lambda} \Gamma(\tilde{R}^{(\alpha)}) = \bigcup_{\alpha<\lambda} \tilde{R}^{(\alpha+1)} \geq \bigcup_{\alpha<\lambda} \tilde{R}^{(\alpha)} = \tilde{R}^{(\lambda)},$$

so we have passage to the limits. It follows that $\alpha < \beta \Rightarrow \tilde{R}^{(\alpha)} \leq \tilde{R}^{(\beta)}$; hence each $\tilde{R}^{(\alpha)}$ is indeed a partial function. Finally, there exists a least ν with $\tilde{R}^{(\nu)} = \tilde{R}^{(\nu+1)} = \Gamma(\tilde{R}^{(\nu)})$. Take this to be $\tilde{R}^*$. Suppose $\tilde{R} \leq \tilde{R}' = \Gamma(\tilde{R}')$; then it is proved by induction on α that $\tilde{R}^{(\alpha)} \leq \tilde{R}'$; hence $\tilde{R}^* \leq \tilde{R}'$, i.e. $\tilde{R}^*$ is the least fixed-point of Γ extending $\tilde{R}$.

The hypothesis of this theorem is of course met if we start with $\mathfrak{M} = (\mathfrak{M}_0, \tilde{U})$ where $\tilde{U}$ is the completely undefined function.

Whatever the plausibility of the role of partial predicates in connection with the paradoxes (which is to be examined below), they appear naturally in other contexts. For example, let $M = \omega$ and identify t, f with 1, 0 respectively. In this case, partial k-ary predicates are partial functions from ω^k into $\{0, 1\}$. They arise naturally in recursion theory, where each partial recursive function into $\{t, f\}$, with index e say, determines a *partial recursive predicate* $(R, \bar{R})$ by

$$(x_1, \ldots, x_k) \in R \Leftrightarrow \{e\}(x_1, \ldots, x_k) \simeq t, \qquad (x_1, \ldots, x_k) \in \bar{R} \Leftrightarrow \{e\}(x_1, \ldots, x_k) \simeq f,$$

or simply $\tilde{R}(x_1, \ldots, x_k) \simeq \{e\}(x_1, \ldots, x_k)$. Of course, every partial function is extendible to a total function, say by taking $F_e(x_1, \ldots, x_k) = t$ if $\{e\}(x_1, \ldots, x_k) \simeq t$, otherwise f, but in general there is no such recursive extension. Thus u is to be read here as *undefined* relative to the specified computation procedures, and not as something inherently undefinable.

A related interpretation of partial predicates $\tilde{R}$ is as "what is *known* by a given stage σ in a computation process for a $\{t, f\}$-valued function". If by stage σ we have evaluated $\{e\}(x_1, \ldots, x_k)$ and found $\{e\}(x_1, \ldots, x_k) = t$ then $(x_1, \ldots, x_k)$ is known to be in R, while if $\{e\}(x_1, \ldots, x_n) = f$ then $(x_1, \ldots, x_k)$ is known definitely to be not in R, i.e. $(x_1, \ldots, x_k) \in \bar{R}$. But if neither is yet known, we ascribe the value u to $\tilde{R}$, with the meaning *unknown thus far* (at stage σ). This interpretation gives natural significance to the relation $\tilde{R} \leq \tilde{R}'$, which holds when $\tilde{R}'$ corresponds to what is known at a later stage σ' in the computational process.

More generally, consider an *investigation* by specified means to determine which elements of a set M have a certain property. This investigation is assumed to proceed in stages. The state of knowledge at any given stage is represented by a partial predicate $\tilde{R}$ on M, and the predicate $\tilde{R}'$ for what is known at a later stage is an extension of $\tilde{R}$. An investigation is said to be *systematic* if we have a prescribed procedure Γ for moving from any given stage to a succeeding stage. It is reasonable to prescribe that Γ is a monotone operator. Suppose we start the investigation with certain information $\tilde{R}$ handed to us, and which will not be altered by Γ. Then the least Γ-fixed point $\tilde{R}^*$ extending $\tilde{R}$ represents *all possible information that may be garnered by the given means of investigation* Γ. Of course, $\tilde{R}^*$ need not be total (unless Γ is the kind of investigation which leaves no stone unturned).

§8. Three-valued partial truth operations and semantics. Let I be an index set; an *I-ary 3-valued propositional operation* F is a map $F: \{t, f, u\}^I \to \{t, f, u\}$. We write $F(\langle p_i \rangle_{i \in I})$ in general and $F(p)$ or $F(p, q)$ when F is unary, resp. binary (with $p, q, p_i \in \{t, f, u\}$). F is said to be *monotonic* if whenever $p_i \leq q_i$ for each $i \in I$ then $F(\langle p_i \rangle_{i \in I}) \leq F(\langle q_i \rangle_{i \in I})$.

Various extensions of the familiar 2-valued operations to 3 values have been considered. We recall those proposed by Łukasiewicz (cf. papers 1–3 in McCall [1967]) and Kleene [1952, §64]. These agree for negation ($\neg$), conjunction ($\wedge$) and disjunction ($\vee$), but not for the conditional and biconditional. We symbolize Łukasiewicz' operations for the latter by $\supset$ and $\equiv$, and Kleene's by $\rightarrow$ and $\leftrightarrow$, resp. The following tables tell how these operations are computed (reading down for p arguments and across for q arguments).

	$\neg p$
t	f
f	t
u	u

$p \wedge q$	t	f	u
t	t	f	u
f	f	f	f
u	u	f	u

$p \vee q$	t	f	u
t	t	t	t
f	t	f	u
u	t	u	u

$(p \rightarrow q)$	t	f	u
t	t	f	u
f	t	t	t
u	t	u	u

$(p \leftrightarrow q)$	t	f	u
t	t	f	u
f	f	t	u
u	u	u	u

$(p \supset q)$	t	f	u
t	t	f	u
f	t	t	t
u	t	u	t

$(p \equiv q)$	t	f	u
t	t	f	u
f	f	t	u
u	u	u	t

Note that the tables for $(p \rightarrow q)$ and $(p \supset q)$ differ only in the values $(u \rightarrow u) = u$, $(u \supset u) = t$; similarly $(u \leftrightarrow u) = u$ while $(u \equiv u) = t$. We have further: $(p \wedge q) = \neg(\neg p \wedge \neg q)$, $(p \rightarrow q) = (\neg p \vee q)$, $(p \leftrightarrow q) = (p \rightarrow q) \wedge (q \rightarrow p)$, and $(p \equiv q) = (p \supset q) \wedge (q \supset p)$.

Kleene's operations are monotonic (or *regular*, in his terminology) while those of Łukasiewicz for $\supset$ and $\equiv$ are not. Kleene also introduced *weak* extensions of the 2-valued operations, obtained simply by assigning u as value if any of the arguments is u. All such are trivially monotonic. The above operations are called *strong* by Kleene; only these will be considered here.

Kleene's operations are the appropriate ones to consider for the recursion-theoretic (or more general investigative) interpretations of partial predicates discussed in the preceding section. As emphasized by Kleene (and subsequently by others), u is not to be thought of in this respect as a definite truth-value on a par with t, f (contrary to Łukasiewicz' approach), but rather as a lack of such. Each of the operations with the tables above can be composed with partial predicates $\tilde{R}$, $\tilde{S}$ considered as operations on M^k to $\{t, f, u\}$. We use the same symbols for these compositions, e.g. $\neg\tilde{R}$, $\tilde{R} \vee \tilde{S}$, $\tilde{R} \wedge \tilde{S}$, etc. It is easily seen that if $\tilde{R}$ and $\tilde{S}$ are partial recursive then so are their combinations by any of Kleene's (weak or) strong propositional operations. (For this, it is best to think of the computation procedures for $\tilde{R}$ and $\tilde{S}$ as operating in *parallel* so as to permit an answer to be provided even when one of these is undefined. The weak operations are appropriate for combinations of predicates to be computed *sequentially*.) On the other hand, we do *not* have closure of the partial recursive predicates under the operations $\supset$ and $\equiv$.

There is a direct extension of $\wedge$ and $\vee$ to infinitary conjunctions and disjunctions defined by:

$$\bigwedge_{i \in I} p_i = \begin{cases} t & \text{if each } p_i = t, \\ f & \text{if some } p_i = f, \\ u & \text{otherwise.} \end{cases}$$

$\bigvee_{i \in I} p_i$ is defined dually or as $\neg \bigwedge_{i \in I} \neg p_i$. Thus both $\bigwedge_{i \in I}$ and $\bigvee_{i \in I}$ are monotonic I-ary operations.

We now pass to a first-order language L for partial structures. For simplicity, this is described just for $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R})$, where $\tilde{R}$ is binary. The symbols of L are those of L_0 together with a single binary relation symbol R. We also consider an expanded language $L^{(M)}$ in which a constant symbol $\bar{m}$ is adjoined for each $m \in M$. The atomic formulas of L are those of L_0 together with all formulas of the form $R(\tau_1, \tau_2)$ where τ_1, τ_2 are terms of L_0. The formulas $\varphi, \psi, \ldots$ of L are generated by closing under the following operations: $\neg\varphi$, $\varphi \wedge \psi$, and $\forall x\varphi$. Then $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ and $\exists x\varphi$ are defined from these as usual. We write $\varphi(x_1, \ldots, x_n)$ for a formula with at most $x_1, \ldots, x_n$ free.

For each sentence (closed formula) φ of $L^{(M)}$ we define $\|\varphi\|_{\mathfrak{M}}$ recursively as follows (where the subscript $\mathfrak{M}$ is omitted for simplicity):

(i) If φ is an atomic sentence of $L_0^{(M)}$ then

$$\|\varphi\| = \begin{cases} t & \text{if } \mathfrak{M}_0 \models \varphi, \\ f & \text{otherwise.} \end{cases}$$

(ii) For $\varphi = R(\tau_1, \tau_2)$ with τ_1, τ_2 closed terms of $L_0^{(M)}$,

$$\|\varphi\| = \tilde{R}(\mathrm{Val}(\tau_1), \mathrm{Val}(\tau_2)).$$

(iii) $\|\neg\varphi\| = \neg\|\varphi\|$.

(iv) $\|(\varphi \wedge \psi)\| = (\|\varphi\| \wedge \|\psi\|)$.

(v) $\|\forall x\varphi(x)\| = \bigwedge_{m \in M} \|\varphi(\bar{m})\|$.

In (ii), $\mathrm{Val}(\tau)$ denotes the value in M of a closed $L_0^{(M)}$-term τ; this is completely determined by $\mathfrak{M}_0$. The following is immediate.

LEMMA 1. *For each sentence φ of $L_0^{(M)}$ we have $\|\varphi\|_{\mathfrak{M}} = t$ or f, and $\|\varphi\|_{\mathfrak{M}} = t \Leftrightarrow \mathfrak{M}_0 \models \varphi$.*

We shall also consider extensions of the language L obtained by closing under the operation $\varphi \supset \psi$ as well as by just closing under the operation $\varphi \equiv \psi$. The resulting languages are denoted by $L(\supset)$ and $L(\equiv)$ resp., and when all constants $\bar{m}\,(m \in M)$ are included, by $L^{(M)}(\supset)$ and $L^{(M)}(\equiv)$ resp.; $L(\equiv)$ may be treated as a sublanguage of $L(\supset)$ (and similarly for $L^{(M)}(\equiv)$). The semantics for these languages is obtained by extending (i)–(v) in the obvious way:

(vi) $\|\varphi \supset \psi\| = (\|\varphi\| \supset \|\psi\|)$.

(vii) $\|\varphi \equiv \psi\| = (\|\varphi\| \equiv \|\psi\|)$.

The following is immediate from the table for $\equiv$.

LEMMA 2. *For φ, ψ sentences of* $L^{(M)}(\supset)$, *and any* $\mathfrak{M}$,

$$\|\varphi \equiv \psi\|_{\mathfrak{M}} = t \Leftrightarrow \|\varphi\|_{\mathfrak{M}} = \|\psi\|_{\mathfrak{M}}.$$

LEMMA 3. *L-semantics is monotonic, i.e. for each sentence φ of $L^{(M)}$*,

$$\mathfrak{M} \leq \mathfrak{M}' \Rightarrow \|\varphi\|_{\mathfrak{M}} \leq \|\varphi\|_{\mathfrak{M}'}.$$

PROOF. By a straightforward induction on φ, using the monotonicity of the operations $\neg$, $\vee$ and $\bigwedge_{m \in M}$.

N.B. Monotonicity does *not* hold for φ in $L^{(M)}(\supset)$ or even φ in $L^{(M)}(\equiv)$. For example, let $l, m \in M$, $l \neq m$. Take $\theta = R(\bar{l}, \bar{l})$, $\psi = R(\bar{m}, \bar{m})$, $\varphi = \|\theta \equiv \psi\|$ and choose $\tilde{R} \leq \tilde{R}'$ with $\|\theta\|_{\tilde{R}} = \|\psi\|_{\tilde{R}} = u$ and $\|\theta\|_{\tilde{R}'} = t$, $\|\psi\|_{\tilde{R}'} = f$. Then $\|\varphi\|_{\tilde{R}} = t$ while $\|\varphi\|_{\tilde{R}'} = f$.

The strong Kleene operators and the infinitary $\bigwedge_{i \in I}$ and $\bigvee_{i \in I}$ are by no means the only natural ones yielding a monotonic semantics for partial structures. A wide class of such are provided by *generalized quantifiers* as handled in *generalized recursion theory* (cf. e.g. Kechris/Moschovakis [1977, pp. 694–696]). One defines a *quantifier Q on M* to be any collection of subsets of M satisfying

$$X \subseteq Y \subseteq M \,\&\, X \in Q \Rightarrow Y \in Q.$$

The *dual quantifier* $\breve{Q}$ is defined by $\breve{Q} = \{X \subseteq M : (M - X) \notin Q\}$, so $X \subseteq Y$ and $X \in \breve{Q}$ implies $Y \in \breve{Q}$. Then $\breve{\breve{Q}} = Q$.

For any quantifier Q we introduced a corresponding formal operator Qx; the formulas φ of $L(Q)$ are generated by closing under the additional operation $Qx\varphi$. Again, $L^{(M)}(Q)$ is the same, with a constant $\bar{m}$ for each $m \in M$. The semantics is extended to closed $Qx\varphi(x)$ in $L^{(M)}(Q)$ by

$$\text{(viii)} \qquad \|Qx\varphi(x)\| = \begin{cases} t & \text{if } \{m \in M : \|\varphi(\bar{m})\| = t\} \in Q, \\ f & \text{if } \{m \in M : \|\varphi(\bar{m})\| = f\} \in \breve{Q}, \\ u & \text{otherwise.} \end{cases}$$

It is readily seen that *$L(Q)$ semantics is monotonic*, i.e. Lemma 3 also holds for sentences in $L^{(M)}(Q)$.

Special quantifiers to consider are the following:

$$\forall = \{M\}, \qquad \exists = \{X \subseteq M : X \neq \varnothing\},$$
$$\forall_A = \{X \subseteq M : A \subseteq X\}, \qquad \exists_A = \{X \subseteq M : X \cap A \neq \varnothing\} \quad (A \text{ any subset of } M),$$
$$\exists_{\geq \kappa} = \{X \subseteq M : \text{card}(X) \geq \kappa\} \quad (\kappa \text{ any cardinal}).$$

Then $\breve{\forall} = \exists$, and the semantics of $\|\forall x\varphi(x)\|$ according to (viii) is the same as by (v). Further $\breve{\forall}_A = \exists_A$ and

$$\|\forall_A x\varphi(x)\| = \begin{cases} t & \text{if } A \subseteq \{m \in M : \|\varphi(\bar{m})\| = t\}, \\ f & \text{if } A \cap \{m \in M : \|\varphi(\bar{m})\| = f\} \neq \varnothing, \\ u & \text{otherwise.} \end{cases}$$

Finally

$$\|\exists_{\geq\kappa} x\varphi(x)\| = \begin{cases} t & \text{if } \operatorname{card}\{m \in M : \|\varphi(\bar{m})\| = t\} \geq \kappa, \\ f & \text{if } \operatorname{card}[M - \{m \in M : \|\varphi(\bar{m})\| = f\} < \kappa, \\ u & \text{otherwise.} \end{cases}$$

Note that the set A is explicitly definable in $L(\forall_A)$ or equivalently $L(\exists_A)$ by the formula $\chi_A(x) = \exists y_A(x = y)$. This yields $\|\chi_A(\bar{m})\| = t$ for $m \in A$ and $\|\chi_A(\bar{m})\| = f$ for $m \notin A$.

It is also possible to consider *relativized quantifiers* Q_a which associate a quantifier in the above sense with each $a \in M$. Then we have a corresponding formal operation $Q_z x \cdot \varphi$ which only binds x (distinct from z), and $\|Q_{\bar{a}} x \cdot \varphi(x)\|$ is defined as above for each assignment to z of $a \in M$. Again the semantics is monotonic. Finally we may obtain a monotonic semantics by extending the language by any combination of quantifiers and relativized quantifiers. We denote any such language by $L^+ = L(Q, \ldots)$. Note that Lemma 2 holds for $L^+(\supset)$.

§9. Three-valued models for type-free principles. To formulate principles like (TA) and (CA) we need some statement-naming and/or abstraction devices. For simplicity, this is achieved here as follows. Assume that the language L_0 contains a constant symbol $\bar{0}$ (also written 0), a binary operation symbol P and two unary operation symbols P_1, P_2. We write (τ_1, τ_2) for $P(\tau_1, \tau_2)$. Assume further that the following formulas are provable in S_0 (of which $\mathfrak{M}_0$ is assumed to be a model):

(i) $(x, y) \neq \bar{0}$;

(ii) $P_1(x, y) = x \wedge P_2(x, y) = y$.

Thus $(\cdot, \cdot)$ acts as a *pairing operation* from M^2 into $M - \{0\}$, for which P_1 and P_2 are the corresponding *projection operations*. The natural number structure may be represented by defining $x' = (x, \bar{0})$. Then P_1 also acts as the predecessor operation and we derive

(iii) $x' \neq \bar{0}$ and

(iv) $x' = y' \to x = y$

from (i) and (ii). (If preferred, one can take $'$ to be an additional basic symbol satisfying the axioms (iii) and (iv).). Any $\mathfrak{M}_0 = (M, \ldots)$ satisfying these axioms is infinite, and the natural numbers can be identified with the subset N of M generated from 0 by the $'$ operation. (Then N is definable in $L(\exists_N)$.) For $n \in N$ we write both $\bar{n}$ and n for the corresponding constant symbol in L_0.

Tuples $(\tau_1, \ldots, \tau_k)$ are introduced recursively by $(\tau_1) = \tau$ and $(\tau_1, \ldots, \tau_{k+1}) = ((\tau_1, \ldots, \tau_k), \tau_{k+1})$. There are corresponding projection operations P_i^k $(1 \leq i \leq k)$ satisfying $P_i^k(x_1, \ldots, x_k) = x_i$ in S_0. Suppose φ is any formula of L, or even of any effectively specified extension language L^+ of the kind considered in §8. We then write $\ulcorner\varphi\urcorner$ for the Gödel-number of φ or its corresponding numeral in L_0. If φ has free variables among $x_1, \ldots, x_k, y_1, \ldots, y_n$ then $(\ulcorner\varphi\urcorner, y_1, \ldots, y_n)$ serves as an operation in L_0 which "abstracts" $x_1, \ldots, x_k$, treating the $y_1, \ldots, y_n$ as paramaters. We thus define

$$\Delta_k \qquad \varphi[\hat{x}_1, \ldots, \hat{x}_k, y_1, \ldots, y_n] = (\ulcorner\varphi\urcorner, y_1, \ldots, y_n).$$

In particular, for $k = 1$ we write

$$\Delta_1 \qquad \{x \mid \varphi(x, y_1, \ldots, y_n)\} = \varphi[\hat{x}, y_1, \ldots, y_n],$$

but for $k = 0$ we write again

$$\Delta_0 \qquad \varphi[y_1, \ldots, y_n] = (\ulcorner\varphi\urcorner, y_1, \ldots, y_n),$$

which is identified with $\ulcorner\varphi\urcorner$ when $n = 0$. In Δ_k the x_i's are considered bound and may be renamed by other bound variables. (To be more precise in each case, $\ulcorner\varphi\urcorner$ is to be the Gödel-number of φ together with a specified pair of lists of variables $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_n)$ of length $k \geq 0$, $n \geq 0$, which together contain all the free variables of φ.)

The purpose of the denotational devices Δ_k is in connection with a common generalization of the truth-axioms and comprehension axioms (TA) and (CA). This is achieved by introducing for each k a $(k + 1)$-placed predicate symbol T_k where $T_k(x_1, \ldots, x_k, z)$ is read "$(x_1, \ldots, x_k)$ *satisfies* z". The appropriate axiom scheme is

$$(\mathrm{T}_k\mathrm{A}) \qquad T_k(x_1, \ldots, x_k, \varphi[\hat{u}_1, \ldots \hat{u}_k, y_1, \ldots, y_n]) \leftrightarrow \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n),$$

for each formula φ with the indicated free variables. For $k = 0$ this reduces to

$$(\mathrm{T}_0\mathrm{A}) \qquad T_0(\varphi[y_1, \ldots, y_n]) \leftrightarrow \varphi(y_1, \ldots, y_n),$$

and in particular for $n = 0$ to $T_0(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$. We may thus identify T_0 with the *truth-predicate* T. For $k = 1$ the scheme appears as

$$(\mathrm{T}_1\mathrm{A}) \qquad T_1(x, \{u \mid \varphi(u, y_1, \ldots, y_n)\}) \leftrightarrow \varphi(x, y_1, \ldots, y_n).$$

We may thus identify T_1 with the *membership relation* E or $\in$.

In 3-valued logic, there is an alternative formulation to consider of the axioms $(\mathrm{T}_k\mathrm{A})$, namely

$$(\mathrm{T}_k\mathrm{A})_\equiv \qquad T_k(x_1, \ldots, x_k, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]) \equiv \varphi(x_1, \ldots, x_k, y_1, \ldots, y_n).$$

We shall now show how to define a partial model $\mathfrak{M}^*$ for these axioms *provided* that the scheme is restricted to φ built up by monotonic operators.

FIXED-MODEL THEOREM. *Let L^+ be L or any extension $L(Q, \ldots)$ which has monotonic semantics, containing one or more of the predicate symbols T_k $(k \geq 0)$. Then for any model $\mathfrak{M}_0$ of S_0 we can find a (least) partial structure $\mathfrak{M}^* = (\mathfrak{M}_0, \ldots, \tilde{T}_k, \ldots)$ such that for each T_k in L^+ and formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$ of L^+ we have*

$$\begin{aligned} &\| T_k(\bar{m}_1, \ldots, \bar{m}_k, \varphi[\hat{u}_1, \ldots, \hat{u}_k, \bar{m}_{k+1}, \ldots, \bar{m}_{k+n}] \|_{\mathfrak{M}^*} \\ &= \| \varphi(\bar{m}_1, \ldots, \bar{m}_k, \bar{m}_{k+1}, \ldots, \bar{m}_{k+n}) \|_{\mathfrak{M}^*}. \end{aligned}$$

PROOF. For simplicity we just consider the binary relation symbol T_1 for which we write E, and look at interpretations $\mathfrak{M} = (\mathfrak{M}_0, \tilde{E})$ of this language. Let Form be the subset of M consisting of all $(\ulcorner\varphi\urcorner, m_1, \ldots, m_n)$, where $\varphi = \varphi(x, y_1, \ldots, y_n)$ is a formula of L^+ with at most x, $y_1, \ldots, y_n$ free. Then define an operator Γ on such by $\Gamma(\mathfrak{M}) = (\mathfrak{M}_0, \Gamma(\tilde{E}))$, where

$$\Gamma(\tilde{E})(m, l) = \begin{cases} \tilde{E}(m, l) & \text{for all } l \notin \text{Form}, \\ \| \varphi(\bar{m}, \bar{m}_1, \ldots, \bar{m}_n) \|_{\mathfrak{M}} & \text{for } l = (\ulcorner\varphi\urcorner, m_1, \ldots, m_n), \quad l \in \text{Form}. \end{cases}$$

Γ is monotonic. For if $\mathfrak{M} = (\mathfrak{M}_0, \tilde{E}) \le \mathfrak{M}' = (\mathfrak{M}_0, \tilde{E}')$ and $m, l \in M$ then for $l \notin \text{Form}$ we have $\tilde{E}(m,l) \le \tilde{E}'(m,l)$, while for $l = (\ulcorner\varphi\urcorner, m_1, \ldots, m_n)$ we have $\|\varphi(\bar{m}, \bar{m}_1, \ldots, \bar{m}_n)\|_{\mathfrak{M}} \le \|\varphi(\bar{m}, \bar{m}_1, \ldots, \bar{m}_n)\|_{\mathfrak{M}'}$; thus $\Gamma(\tilde{E}) \le \Gamma(\tilde{E}')$. Now the conclusion follows from the fixed-point theorem of §7, starting with $\mathfrak{M} = (\mathfrak{M}_0, \tilde{U})$ where $\tilde{U}$ is completely undefined.[15] The same argument works with any number of basic symbols T_k.

As in §7 we can formulate a more general statement, obtaining a fixed point $\mathfrak{M}^*$ for the Γ constructed in the proof extending any given $\mathfrak{M}$ which happens to satisfy $\mathfrak{M} \le \Gamma(\mathfrak{M})$.

Actually one can obtain a much more general Fixed-Model Theorem (Aczel/Feferman [1980]). Taking any basic symbols R_k, assume given any sequence of $L^{+(M)}$ sentences $\theta_{k,m_1,\ldots,m_k}$. Then we can construct $\mathfrak{M}^*$ satisfying

$$\|R_k(\bar{m}_1, \ldots, \bar{m}_k)\|_{\mathfrak{M}^*} = \|\theta_{k,m_1,\ldots,m_k}\|_{\mathfrak{M}^*}.$$

The theorem stated is the special case obtained by taking $R_{k+1} = T_k$ and $\theta_{k+1,m_1\ldots,m_k,l} = \varphi(\bar{m}_1, \ldots, \bar{m}_k)$ when $l = \varphi[\hat{u}_1, \ldots, \hat{u}_k]$.

If we specialize the Fixed-Model Theorem to L with just the predicate $T = T_0$ applied to closed φ of L we obtain a least $\mathfrak{M}^*$ satisfying

$$\|T(\ulcorner\varphi\urcorner)\|_{\mathfrak{M}^*} = \|\varphi\|_{\mathfrak{M}^*}.$$

Recasting this as a result about models of the form $(\mathfrak{M}, \tilde{T})$ where $\tilde{T} = (T, \bar{T})$ is a disjoint pair, one obtains the model-theoretic content of Kripke [1975].[16] The proof is basically the same.[17] Actually, such constructions for type-free theories of predication and classes were given much earlier by Fitch and Gilmore. The history will be picked up in the following sections and particularly in §14.

Specializing the Fixed-Model Theorem to L with just the predicate $E = T_1$, also written $\in$, we obtain a least $\mathfrak{M}^*$ such that

$$\|\forall y_1, \ldots, \forall y_n \exists a \forall x[x \in a \equiv \varphi(x, y_1, \ldots, y_n)]\|_{\mathfrak{M}^*} = t$$

for each $\varphi(x, y_1, \ldots, y_n)$ in L. This is a result due to Brady [1971] for consistency of a form of (CA) in Łukasiewicz 3-valued logic.[18] The proof is basically the same.

[15] Observe that when starting with $\tilde{U}$ for $\tilde{E}$ we have $\tilde{E}^{(\alpha)}(m,l) = u$ for all m and $l \notin$ Form. So in obtaining the least fixed point we may as well define

$$\Gamma(\tilde{E})(m, (\ulcorner\varphi\urcorner, m_1, \ldots, m_n)) = \|\varphi(\bar{m}, \bar{m}_1, \ldots, \bar{m}_n)\|_{(\mathfrak{M}_0, \tilde{E})},$$

with $\Gamma(\tilde{E})(m,l) = u$ otherwise. Translated into the language of partial predicates $\tilde{E} = (E, \bar{E})$, $\Gamma(\tilde{E}) = (E', \bar{E}')$, for $u = 0$ we have $(m, \ulcorner\varphi\urcorner)$ is in $E' \Leftrightarrow \varphi(\bar{m})$ is true in $(\mathfrak{M}_0, (E, \bar{E}))$ and $(m, \ulcorner\varphi\urcorner)$ is in $\bar{E}' \Leftrightarrow \varphi(\bar{m})$ is false in $(\mathfrak{M}_0, (E, \bar{E}))$. Thus at the fixed point $\mathfrak{M}^*$, $E(\bar{m}, \ulcorner\varphi\urcorner)$ (or $\bar{m} \in \{x \mid \varphi(x)\}$ is true (false) in $\mathfrak{M}^*$ iff $\varphi(\bar{m})$ is true (false) in $\mathfrak{M}^*$.

[16] See footnote 15 for this kind of recasting. It is also mentioned in Kripke [1975, p. 706] that his result can be extended to languages with generalized quantifiers.

[17] The inductive method is not the only way one can establish existence of fixed points. It was shown in Martin/Woodruff [1975] (independently of Kripke's work) that this can be proved by Zorn's lemma; that was also observed by Kripke. The difference is that the inductive method establish the existence of *minimal* fixed points while Zorn's lemma yields *maximal* ones.

[18] This improved Skolem [1960]; cf. the notes in §14 below. It is stated in Brady [1971] that $\mathfrak{M}^*$ is also a model for the axiom of extensionality. However, that is so only in the weak sense that $\|\forall x(x \in m_1 \equiv x \in m_2)\|_{\mathfrak{M}^*} = t \Rightarrow \|\forall y(m_1 \in y \equiv m_2 \in y)\|_{\mathfrak{M}^*} = t$.

To conclude this section we give a *counterexample* to $\exists a \forall x[x \in a \equiv \varphi(x)]$ in 3-valued models when φ is in $L(\equiv)$. This is based on the following table:

p	$\neg p \wedge \neg(p \equiv \neg p)$	$p \equiv \neg p \wedge \neg(p \equiv \neg p)$
t	f	f
f	t	f
u	f	u

Then $\exists a \forall x[x \in a \equiv \neg(x \in x) \wedge \neg(x \in x \equiv \neg(x \in x))]$ cannot receive the value t in any structure $\mathfrak{M} = (\mathfrak{M}_0, \tilde{E})$. For if it did, taking $p = \|a \in a\|$ gives a contradiction.

§10. Type-free formal systems with Łukasiewicz and Kleene logics. We now turn to the formulation of type-free systems for which the semantical constructions of §§8–9 provide a model.

Consider first the language $L(\supset)$ for Łukasiewicz predicate logic with basic operators $\neg$, $\wedge$, $\supset$ and $\forall$. A formula $\varphi(x_1, \ldots, x_k)$ is said to be *Ł-valid* if for every partial structure $\mathfrak{M}$ with domain M and any $m_1, \ldots, m_k \in M$ we have $\|\varphi(\bar{m}_1, \ldots, \bar{m}_k)\|_{\mathfrak{M}} = t$. A complete recursive Hilbert-style axiomatization of the Ł-valid formulas may be found in Rosser/Turquette [1952] (this will not be repeated here). It follows from the work of §9 that the scheme

$$(\mathrm{CA})_{\equiv} \qquad \forall y_1, \ldots, \forall y_n \exists a \forall x[x \in a \equiv \varphi(x, y_1, \ldots, y_n)]$$

is Ł-consistent *provided φ is restricted to the language L*, i.e. φ only involves $\neg$, $\wedge$ and $\forall$ in its build-up. This is the consistency result of Brady [1971]. We have similar results for the other predicates T_k, e.g. consistency of the scheme

$$(\mathrm{TA})_{\equiv} \qquad T(\ulcorner \varphi \urcorner) \equiv \varphi$$

for φ in L. These restrictions are essential, since we saw at the end of §9 that $(\mathrm{CA})_{\equiv}$ is already inconsistent in the expanded language $L(\equiv)$.[19] Assuming the means to construct self-referential statements, also $(\mathrm{TA})_{\equiv}$ leads to an inconsistency in $L(\equiv)$ by the same argument. On the other hand, Russell's paradox itself is avoided in the Ł-system with the restricted $(\mathrm{CA})_{\equiv}$, even though we have $\exists a \forall x[x \in a \equiv \neg(x \in x)]$. This is by the circumstance that $p \equiv \neg p$ has the value t when p has the value u.

We next turn to K-logic, which is obtained simply by restricting Ł-logic to the language L. Thus a formula is said to be *K-valid* if it is in L and is Ł-valid. It follows from the complete recursive axiomatization of the Ł-valid formulas that the set of K-valid formulas is recursively enumerable. However, its explicit axiomatization is another matter. The reason is that the main rule of Ł-propositional logic is *modus ponens*, in the form $\varphi, (\varphi \supset \psi)/\psi$. Since $\supset$ is not available in K-logic, we cannot use it there for the rule of modus ponens. If we take the connective $(\varphi \rightarrow \psi)$ instead (where $(p \rightarrow q) = \neg(p \wedge \neg q) = (\neg p \vee q)$) we *do* have closure under the rule φ, $(\varphi \rightarrow \psi)/\psi$ in the K-valid formulas, but we then run into trouble elsewhere; for example, the expected axioms $(\varphi \rightarrow \varphi)$, $\forall x \varphi(x) \rightarrow \varphi(\tau)$, etc. are not K-valid.

[19] The inconsistency can be demonstrated more simply in $L(\supset)$, using $\exists a(x \in a \equiv [x \in x \supset \neg(x \in x)])$ since $((p \supset \neg p) \supset p)$ is Ł-valid. This applies similarly to $(\mathrm{TA})_{\equiv}$ in $L(\supset)$.

An axiomatization of K-logic was given in Wang [1961] by use of an auxiliary symbol (here denoted) $\vdash$, which is *not iterated*. Given formulas φ, ψ in L, we say that $\varphi \vdash \psi$ is K-*valid* if for every partial structure $\mathfrak{M}$ and assignment to the free variables of φ, ψ in M we have $\|\varphi\| = t \Rightarrow \|\psi\| = t$. Wang gave a simple natural system of axioms and rules of inference for combinations $(\varphi \vdash \psi)$ which is complete for this extended notion of validity. He also axiomatized the K*-*valid* combinations $\varphi \vdash \psi$, which are defined to be those K-valid $\varphi \vdash \psi$ such that for every $\mathfrak{M}$ and assignment to the free variables, also $\|\psi\| = f \Rightarrow \|\varphi\| = f$.

Wang's system has the character of a Gentzen sequential calculus. More recently, Scott [1975] gave a complete Gentzen-style system for *sequents* $\varphi_1, \ldots, \varphi_l \vdash \psi_1, \ldots, \psi_m$ whose validity is defined like that of K*-validity, namely: for each $\mathfrak{M}$ and interpretation in M of the variables, (i) $\|\varphi_1\| = \cdots = \|\varphi_l\| = t \Rightarrow$ some $\|\psi_j\| = t$ and (ii) $\|\psi_1\| = \cdots = \|\psi_m\| = f \Rightarrow$ some $\|\varphi_i\| = f$.

The K-valid formulas are just those φ with $\vdash \varphi$ derivable in Wang's or Scott's system. Thus one obtains a very satisfactory axiomatization of K-logic in this way. The question then is how type-free principles corresponding to (TA) and (CA) are to be formulated in such K-logics. We cannot use Kleene's $\leftrightarrow$ of §8 as the principal connective, since $p \leftrightarrow \neg p$ is never t. Scott's solution is to introduce the symbol $\vdash\dashv$, where $(\varphi \vdash\dashv \psi)$ abbreviates $(\varphi \vdash \psi)$ and $(\psi \vdash \varphi)$. Of course we cannot literally form the conjunction of $(\varphi \vdash \psi)$ and $(\psi \vdash \varphi)$, but we could say that a system based on "axioms" of the form $\varphi \vdash\dashv \psi$ is consistent if there is a model which satisfies both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ for each assignment to the free variables. In this sense the scheme (for each φ in L)

$$(\mathrm{CA})_{\vdash\dashv} \qquad x \in \{u \mid \varphi(u, y_1, \ldots, y_n)\} \vdash\dashv \varphi(x, y_1, \ldots, y_n)$$

is consistent, as is the schema

$$(\mathrm{TA})_{\vdash\dashv} \qquad T(\ulcorner\varphi\urcorner) \vdash\dashv \varphi$$

for each φ in L. This is a direct consequence of the fixed-model theorem and the fact that $\|\varphi \equiv \psi\| = t \Rightarrow \|\varphi\| = \|\psi\|$. It follows by the same result that we have consistency of the schemata $(\mathrm{T}_k\mathrm{A})_{\vdash\dashv}$ in this sense for each k. Note that the use of abstracts is essential to state these consistency results, for there is no direct sense given in the formalism to such combinations as $\exists a \forall x[x \in a \vdash\dashv \varphi(x)]$.

DISCUSSION. Both the type-free systems in Ł-logic and K-logic that we have just described are superficially attractive. However, to my mind they are unsatisfactory in a number of respects which I shall now detail.

(i) The *main defect for the scheme* $(\mathrm{CA})_{\equiv}$ as restricted above is that the basic connectives $\supset$ and $\equiv$ of $L(\supset)$ may not appear in the formula to the right of the $\equiv$ sign.

(ii) Similarly, the *main defect for the scheme* $(\mathrm{CA})_{\vdash\dashv}$ is that the "operator" $\vdash\dashv$ may not be iterated.

(iii) The same criticism applies *mutatis mutandis* to the other schemes $(\mathrm{T}_k\mathrm{A})_{\equiv}$ and $(\mathrm{T}_k\mathrm{A})_{\vdash\dashv}$.

(iv) In each logic there are laws which we might expect to hold that don't. Of course $(\varphi \vee \neg \varphi)$ does not hold in either logic; but also $\neg(\varphi \wedge \neg \varphi)$ is not derivable. We have $(\varphi \supset \varphi)$ but not $\varphi \wedge (\varphi \supset \psi) \supset \psi$ in Ł-logic; nor do we derive $(\varphi \supset \neg \varphi)$

$\supset \neg \varphi$. On the other hand we don't have $(\varphi \to \varphi)$ in K-logic. In the system of Wang [1961] (or Scott [1975]) for K-logic, we don't have implication introduction (i.e. the Deduction Theorem) for the $\to$ operator. Multiplying such examples, I conclude that *nothing like sustained ordinary reasoning can be carried on in either logic.*

(v) It was stressed in the analyses of the Liar and Russell's paradoxes in §§3 and 5 that the contradiction in each case is carried out within intuitionistic logic. The conclusion was drawn that a solution to the paradoxes ought to accomodate itself equally well to a constructive as well as a classical setting. But in Ł-logic nonconstructive statements like $(\neg\neg\varphi \supset \varphi)$ are valid while in K-logic there is nothing like the constructive use of implication ($\varphi \to \psi$ is not constructively equivalent to $\neg\varphi \vee \psi$). Finally, the dual treatment of $\wedge$, $\vee$ and $\forall$, $\exists$ is completely nonconstructive.

DISCUSSION, CONTINUED. The objections so far have been based on formal, logical considerations. One may ask to what extent the proposed 3-valued solutions which have been described are satisfactory from a philosophical point of view. Some of the considerations here are, briefly, as follows.

(vi) If the basic idea is that the sentences appearing in the paradoxes are meaningless, then Ł-logic is clearly not the appropriate one. For this gives the truth-value t to compounds of entirely meaningless statements, e.g. $p \supset \neg p$ when p has the value u. Łukasiewicz' own interpretation was of "u" as *contingent*, but this has been much disputed (cf. e.g. Prior [1967]). Indeed, my impression is that no really satisfactory informal interpretation of Ł-logic has ever been given.

(vii) On the other hand Kleene's logic does seem to correspond more closely to the interpretation of u as *meaningless*. However, some argue that only his *weak connectives* are appropriate for that interpretation, since the result of combining a meaningless statement with meaningful ones should still be regarded as meaningless. The *strong connectives* are appropriate instead for the interpretation in terms of what is known in the process of an investigation, with u interpreted as *unknown.*

(viii) For the latter point of view it seems that one should ascribe a definite truth-value t or f to each statement, and that u only reflects incompleteness of our knowledge. But in that case we ought to accept statements such as $(\varphi \to \varphi)$ and $(\varphi \vee \neg\varphi)$ for each φ. It seems that considerations like this have led van Fraassen [1968] to argue for looking at partial structures within a logical framework which he calls *supervaluations.* A statement φ is said to be true in van Fraassen's sense in $\mathfrak{M} = (\mathfrak{M}_0, \tilde{R})$ if φ is true (in the usual sense) in every *total* extension $\mathfrak{M}' = (\mathfrak{M}_0, R')$ of $\mathfrak{M}$. In this picture, not all statements $R(\bar{m})$ are true or false, but all classically valid statements are automatically true.

(ix) Finally, there have been objections to "truth-gap" theories on the grounds that they too are subject to paradoxes, e.g. the Extended Liar; cf. e.g. Burge 1979. Indeed, at an informal level this criticism is valid. Namely one looks again at a φ which is supposed to be the same as $\neg T(\ulcorner\varphi\urcorner)$. According to the truth-gap argument, φ is neither true nor false. But since φ says of itself it is not true, it is true after all. So there is no gap, and the usual Liar paradox is then produced as before.

If we analyze the preceding argument in terms of the model-theoretic construction of $\mathfrak{M}^*$ in §9, the paradox disappears. In $\mathfrak{M}^*$ we have φ with $\|\varphi \equiv T(\ulcorner\varphi\urcorner)\| = t$, and by self-referential construction $\|\varphi \equiv \neg T(\ulcorner\varphi\urcorner)\| = t$. It follows that $\|\varphi\| \neq t, f$.

Thus φ is not true in $\mathfrak{M}^*$, i.e. $\|\varphi\| \neq t$, so $\|T(\ulcorner\varphi\urcorner)\| \neq t$. But that does not tell us that $\neg T(\ulcorner\varphi\urcorner)$ is true in $\mathfrak{M}^*$. The puzzle is sorted out by the inequivalence between "φ is not true" and "$\neg T(\ulcorner\varphi\urcorner)$ is true". Still, consideration of the Extended Liar does leave one with a further bit of malaise about truth-gap approaches, since the formal model-theoretic constructions don't match up with informal usage.

§11. A type-free formal system in an extension of the classical predicate calculus. In this section we will describe results concerning a new type-free system $S(\equiv)$ presented in Aczel/Feferman [1980]. $S(\equiv)$ overcomes two of the defects of 3-valued systems brought out in the preceding section, namely: (i) it is based on type-free principles like $(\mathrm{CA})_{\equiv}$ where now *any* formula of $L(\equiv)$ can stand to the right of the $\equiv$ sign, and (ii) the logic is that of full classical predicate calculus (CPC) augmented by natural laws for $\equiv$. However, $S(\equiv)$ is not fully satisfactory in other respects which will be brought out in the discussion at the end of the section and in §13.

We assume S_0 satisfies the conditions of §9, and that one or more of the predicate symbols T_k is adjoined to L_0 to provide the atomic symbols of L. We also follow the abbreviations for abstracts $\Delta_n, \Delta_1, \Delta_0$ of §9. The formulas of $L(\equiv)$ are built up using $\neg, \wedge, \equiv$ and $\forall$. Write $\varphi \not\equiv \psi$ for $\neg(\varphi \equiv \psi)$. The operators $\vee, \rightarrow, \leftrightarrow, \exists$ are defined classically as before. Let $t = (0 = 0)$, $f = \neg t$ and, for each φ,

$$D(\varphi) = [\varphi \equiv t \vee \varphi \equiv f].$$

$D(\varphi)$ is read: φ is *determinate*. The axioms for $\equiv$, denoted $\mathrm{Ax}(\equiv)$, are as follows (over and above the axioms of CPC):

(1) $\equiv$ is an equivalence relation.
(2) $\equiv$ is preserved by $\neg$, $\wedge$, $\equiv$ and $\forall$.
(3) (i) $(\varphi \equiv t) \leftrightarrow \varphi$ for φ atomic, and (ii) $(\varphi \equiv f) \leftrightarrow \neg\varphi$ for φ atomic in L_0.
(4) (i) $(\neg\varphi) \equiv t \leftrightarrow \varphi \equiv f$, and (ii) $(\neg\varphi) \equiv f \leftrightarrow \varphi \equiv t$.
(5) (i) $(\varphi \wedge \psi) \equiv t \leftrightarrow \varphi \equiv t \wedge \psi \equiv t$, and (ii) $(\varphi \wedge \psi) \equiv f \leftrightarrow \varphi \equiv f \vee \psi \equiv f$.
(6) (i) $(\forall x\varphi(x)) \equiv t \leftrightarrow \forall x[\varphi(x) \equiv t]$, and (ii) $(\forall x\varphi(x)) \equiv f \leftrightarrow \exists x(\varphi(x) \equiv f)$.
(7) (i) $(\varphi \equiv \psi) \equiv t \leftrightarrow \varphi \equiv \psi$, and (ii) $(\varphi \equiv \psi) \equiv f \leftrightarrow D(\varphi) \wedge D(\psi) \wedge \varphi \not\equiv \psi$.

The following explains (1) and (2) in more detail. (1) consists of the schemata $(\varphi \equiv \varphi)$, $(\varphi \equiv \psi) \rightarrow (\psi \equiv \varphi)$, and $(\varphi \equiv \psi) \wedge (\psi \equiv \vartheta) \rightarrow (\varphi \equiv \vartheta)$. For (2), let O be one of the n-ary operations $\neg, \wedge, \equiv$ (so $n = 1$ or 2). The statement that $\equiv$ is preserved by O is:

(2a) $$(\varphi_1 \equiv \psi_1) \wedge \cdots \wedge (\varphi_n \equiv \psi_n) \rightarrow O(\varphi_1, \ldots, \varphi_n) \equiv O(\psi_1, \ldots, \psi_n).$$

The statement that $\equiv$ is preserved by $\forall$ is given by

(2b) $$\forall x[\varphi(x) \equiv \psi(x)] \rightarrow [\forall x\varphi(x) \equiv \forall x\psi(x)].$$ [20]

We shall also consider a (seemingly) slight variant $\mathrm{Ax}'(\equiv)$ obtained by modifying (7) to (7)′, where (7)′(ii) is the same as (7)(ii) while (7)(i) is replaced by

(7)′(i) $(\varphi \equiv \psi) \equiv t \leftrightarrow D(\varphi) \wedge D(\psi) \wedge (\varphi \equiv \psi)$.

[20] The axioms for $\equiv$ given in Aczel/Feferman [1980] are a little different; the operators $\wedge$ and $\forall$ are there treated in Kleene's weak sense, so that $D(\varphi \wedge \psi) \leftrightarrow D(\varphi) \wedge D(\psi)$ and $D(\forall x\varphi(x)) \leftrightarrow \forall x D(\varphi(x))$. Here $\wedge$ and $\forall$ are treated in Kleene's strong sense. The handling of the systems is the same, otherwise.

LEMMA. *The following are consequences of both* $\mathrm{Ax}(\equiv)$ *and* $\mathrm{Ax}'(\equiv)$.

(i) $t \not\equiv f$.

(ii) $(\neg f) \equiv t$.

(iii) D *is closed under* $\neg$, $\wedge$, $\equiv$ *and* $\forall$, *i.e.*

$$D(\varphi) \to D(\neg\varphi),\ D(\varphi) \wedge D(\psi) \to D(\varphi \wedge \psi) \wedge D(\varphi \equiv \psi),$$
$$\forall x D(\varphi(x)) \to D(\forall x \varphi(x)) \quad \textit{for any } \varphi, \psi.$$

(iv) $D(\varphi)$ *for each* φ *of* L_0.

(v) $(\varphi \equiv t) \to \varphi$ *and* $(\varphi \equiv f) \to \neg\varphi$, *for each* φ.

(vi) $D(\varphi) \to [(\varphi \equiv t) \leftrightarrow \varphi] \wedge [(\varphi \equiv f) \leftrightarrow \neg\varphi]$, *for each* φ.

PROOF. (i) follows from Axiom (3)(ii) taking $\varphi = t$, using symmetry of $\equiv$. (ii)–(iv) are immediate. The statements in (v) are proved simultaneously by induction on φ. Then (vi) follows directly.

The axioms of $S(\equiv)$, resp. $S'(\equiv)$, are those of S_0 plus $\mathrm{Ax}(\equiv)$, resp. $\mathrm{Ax}'(\equiv)$, together with all formulas of the following form for T_k a symbol of L, and φ *any formula of* $L(\equiv)$:

$$(\mathrm{T}_k\mathrm{A})_\equiv \quad T_k(x_1,\dots,x_k,\varphi[\hat{u}_1,\dots,\hat{u}_k,y_1,\dots,y_n]) \equiv \varphi(x_1,\dots,x_k,y_1,\dots,y_n).$$

THEOREM. *The systems* $S(\equiv)$ *and* $S'(\equiv)$ *are both conservative extensions of* S_0.

PROOF. The first of these is the main result of Aczel/Feferman [1979]. For its proof we developed an analogue of the Church-Rosser theorem. A much simpler proof due to Aczel (in §6 of our joint paper) can be given for conservation of the system $S'(\equiv)$ over S_0. We shall follow that here; it makes use instead of the results for 3-valued models established in §9 above.[21]

The method is to expand any model $\mathfrak{M}_0$ of S_0 in L_0 to a model $\mathfrak{M} = (\mathfrak{M}_0,\dots,T_k,\dots)$ of $S'(\equiv)$ in $L(\equiv)$. $\mathfrak{M}$ is a 2-valued model. We shall use as an intermediary the 3-valued fixed-point model of §9. The crucial point is that $\|\varphi \equiv \psi\|_{\mathfrak{M}^*}$ will be evaluated according to the rules for Kleene $\leftrightarrow$. To make the different treatments of equivalence clear, let us write $\varphi^{(K)}$ for the result of replacing each operation symbol $\equiv$ in φ by the operation symbol $\leftrightarrow$. However we still write $\varphi[\hat{u}_1,\dots,y_n]$ for the term $\varphi^{(K)}[\hat{u}_1,\dots,y_n]$. The fixed-model theorem thus provides us with a partial structure $\mathfrak{M}^* = (\mathfrak{M}_0,\dots,\tilde{T}_k,\dots)$ such that

$$(1) \qquad \begin{aligned} &\|T_k(\bar{m}_1,\dots,\bar{m}_k,\varphi[\hat{u}_1,\dots,\hat{u}_k,\bar{m}_{k+1},\dots,\bar{m}_{k+n}])\|_{\mathfrak{M}^*} \\ &= \|\varphi^{(K)}(\bar{m}_1,\dots,\bar{m}_k,\bar{m}_{k+1},\dots,\bar{m}_{k+n})\|_{\mathfrak{M}^*} \end{aligned}$$

for each φ with the appropriate free variables. To simplify matters we shall illustrate the further work just with the membership relation (T_1) and drop the subscript $\mathfrak{M}^*$ in evaluations. Thus

$$(2) \qquad \|\bar{m} \in \{u \mid \varphi(u,\bar{m}_1,\dots,\bar{m}_n)\}\| = \|\varphi^{(K)}(\bar{m},\bar{m}_1,\dots,\bar{m}_n)\|$$

for all $\varphi(x, y_1,\dots, y_n)$ of $L(\equiv)$. Now define satisfaction for sentences of $L^{(M)}(\equiv)$ in

[21] The systems corresponding to $S(\equiv)$ and $S'(\equiv)$ in Aczel/Feferman [1980] were theories of classes, i.e. only involved the axioms $(\mathrm{CA})_\equiv$ for the predicate T_1. Furthermore, the abstracts $\{x \mid \varphi(x, y_1,\dots, y_n)\}$ were treated as new (iterable) term-builders. This caused (what I now view as) unnecessary complications, esp. op. cit., §4.2.

the 2-valued model $\mathfrak{M}$ as follows:

$$
(3)\quad
\begin{aligned}
&\text{(i) } \mathfrak{M} \models (\bar{m} \in \bar{l}) \Leftrightarrow \|\bar{m} \in \bar{l}\| = t;\\
&\text{(ii) } \mathfrak{M} \models \neg\varphi \Leftrightarrow \mathfrak{M} \not\models \varphi;\\
&\text{(iii) } \mathfrak{M} \models (\varphi \wedge \psi) \Leftrightarrow \mathfrak{M} \models \varphi \,\&\, \mathfrak{M} \models \psi;\\
&\text{(iv) } \mathfrak{M} \models (\varphi \equiv \psi) \Leftrightarrow \|\varphi^{(K)}\| = \|\psi^{(K)}\|;\\
&\text{(v) } \mathfrak{M} \models \forall x \varphi(x) \Leftrightarrow \text{for each } m \in M,\ \mathfrak{M} \models \varphi(\bar{m}).
\end{aligned}
$$

(In other words, the relation T_k in $\mathfrak{M}$ is the positive part of $\tilde{T}_k$ considered as a disjoint pair $(T_k, \bar{T}_k)$.) To show that $\mathfrak{M}$ is a model of $S'(\equiv)$, we first verify

$$
(4)\qquad \mathfrak{M} \models [\bar{m} \in \{u \mid \varphi(u, \bar{m}_1, \ldots, \bar{m}_n)\} \equiv \varphi(\bar{m}, \bar{m}_1, \ldots, \bar{m}_n)]
$$

for each φ. This is immediate from (2) and (3)(iv). It is straightforward to show that each of the axioms $\mathrm{Ax}'(\equiv)$ is true in $\mathfrak{M}$ under the definition (3)(iv). Here we consider only the axioms (7)$'$, leaving the others for the reader to check. Let φ, ψ be sentences of $L^{(M)}(\equiv)$. For (7)$'$, we use

$$
\mathfrak{M} \models [(\varphi \equiv \psi) \equiv t] \Leftrightarrow \|\varphi^{(K)} \leftrightarrow \psi^{(K)}\| = t
$$

and

$$
\mathfrak{M} \models [(\varphi \equiv \psi) \equiv f] \Leftrightarrow \|\varphi^{(K)} \leftrightarrow \psi^{(K)}\| = f.
$$

Now $\|\varphi^{(K)} \leftrightarrow \psi^{(K)}\|$ receives one of the values t, f only if each of $\|\varphi^{(K)}\|$ and $\|\psi^{(K)}\|$ receives the values t or f. Further, $\mathfrak{M} \models D(\varphi) \Leftrightarrow \mathfrak{M} \models (\varphi \equiv t)$ or $\mathfrak{M} \models (\varphi \equiv f)$, so $\mathfrak{M} \models D(\varphi) \Leftrightarrow \|\varphi^{(K)}\| = t$ or $\|\varphi^{(K)}\| = f$. The conclusion is that

$$
(5)\quad
\begin{aligned}
&\text{(i) } \mathfrak{M} \models [(\varphi \equiv \psi) \equiv t] \Leftrightarrow \mathfrak{M} \models D(\varphi) \,\&\, \mathfrak{M} \models D(\psi) \,\&\, \mathfrak{M} \models (\varphi \equiv \psi);\\
&\text{(ii) } \mathfrak{M} \models [(\varphi \equiv \psi) \equiv f] \Leftrightarrow \mathfrak{M} \models D(\varphi) \,\&\, \mathfrak{M} \models D(\psi) \,\&\, \mathfrak{M} \not\models (\varphi \equiv \psi).
\end{aligned}
$$

Thus (7)$'$(i), (ii) of $\mathrm{Ax}'(\equiv)$ is true in $\mathfrak{M}$. This completes the proof.

REMARKS. (i) The unexpected aspect of Aczel's proof is that it uses two different interpretations of $\equiv$, namely first as Kleene's $\leftrightarrow$ in the 3-valued model $\mathfrak{M}^*$ and then as Łukasiewicz' in passing from $\mathfrak{M}^*$ to $\mathfrak{M}$ (in (3)(iv)). The first interpretation is necessary in order to be able to apply the fixed-model theorem of §9. One also sees clearly from the last part of the proof why his method fails to give a model of $(\varphi \equiv \psi) \equiv t \leftrightarrow (\varphi \equiv \psi)$.

(ii) Consider the system $S'(\equiv)$ with axioms for the $\in$-relation (T_1),

$$
(\mathrm{CA})_{\equiv} \qquad x \in \{u \mid \varphi(u, y_1, \ldots, y_n)\} \equiv \varphi(x, y_1, \ldots, y_n).
$$

let $r = \{u \mid u \notin u\}$, so $\psi \equiv \neg\psi$ for $\psi = (r \in r)$. It follows that $\neg D(\psi)$, for otherwise we should have $t = f$. Further we can prove $\neg\psi$ in this system since we have $\psi \to (\psi \equiv t)$ (ψ being atomic) and then successively $\psi \equiv t \to (\neg\psi) \equiv t$, $(\neg\psi) \equiv t \to \psi \equiv f$, $\psi \equiv f \to (\neg\psi)$ and finally $\psi \to \neg\psi$.

(iii) By the preceding, we do *not* have closure of $S'(\equiv)$ (or $S(\equiv)$) under the rule $\varphi_1, \varphi_1 \equiv \varphi_2/\varphi_2$; otherwise the system would be inconsistent. Nor do we have closure under $\varphi_1 \equiv \varphi_2$, $\varphi_1 \leftrightarrow \varphi_1'$, $\varphi_2 \leftrightarrow \varphi_2'/\varphi_1' \equiv \varphi_2'$. In that sense, $\equiv$ *is an intensional operator.*

(iv) What informal interpretation is to be given of $\equiv$ as it is used in the systems $S'(\equiv)$ and $S(\equiv)$? In the paper Aczel/Feferman [1980] it was proposed to read $\varphi \equiv \psi$ as *φ is equivalent to ψ in consequence of basic definitions*, namely the axioms $(\mathrm{T}_k\mathrm{A})_{\equiv}$, which are taken to be definitions of application of abstracts $\varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]$, i.e. of the conditions under which they apply to elements $(x_1, \ldots, x_k)$. This is a reading for which the axioms $\mathrm{Ax}(\equiv)$ are plausible, though our intuitions are not firm in this respect. If this interpretation is accepted, our conservation results which give eliminability of the additional axioms provide one precise realization of the ideas of Behmann [1931] for a resolution of the paradoxes.[22] What is attractive about the reading is that given definitions may be "internally self-contradictory", i.e. lead to statements $\varphi \equiv \neg\varphi$ yet without leading to any inconsistency. In opposition to it, the logical points of the preceding remark may be considered as disturbing to the proposed interpretation.

(v) A clear informal interpretation of $\equiv$ would decide between the axioms (7)(i) and (7)′(i) of $\mathrm{Ax}(\equiv)$, which give some quite distinct results. Take $(\mathrm{CA})_\equiv$ for $\psi = (r \in r)$. With (7)(i) one has $(\psi \equiv \neg\psi) \equiv t$ while with (7)′(i) one has $(\psi \equiv \neg\psi) \not\equiv t$. The interpretation proposed in the preceding remark (iv) seems to me to favor (7)(i). But as we shall see in §12, the useful consequences of $S(\equiv)$ follow just as well from $S'(\equiv)$.

(vi) The systems considered provide us with the advantage of full use of ordinary reasoning, if that is understood in the sense of classical logic. But the approach of this section does not meet the overall criterion (proposed earlier) that a solution of the paradoxes ought to be equally satisfactory within the setting of constructive logic.

(vii) As a final technical point, it should be noted that there is an obvious extension of $S'(\equiv)$ to a system $S'(\supset)$ in $L(\supset)$, for which the conservation theorem still holds. In $L(\supset)$, we define $(\varphi \equiv \psi)$ as $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Then take for $\mathrm{Ax}'(\supset)$ the same (1)–(6) as before (expanding (2) to preservation of $\equiv$ by $\supset$), but with (7)′ replaced by

(7)″ (i) $(\varphi \supset \psi) \equiv t \leftrightarrow D(\varphi) \wedge D(\psi) \wedge (\varphi \equiv f \vee \psi \equiv t)$;

(ii) $(\varphi \supset \psi) \equiv f \leftrightarrow \varphi \equiv t \wedge \psi \equiv f$.

Then we take $S'(\supset)$ to consist of $\mathrm{Ax}'(\supset)$ with the $(\mathrm{T}_k A)_\equiv$ axioms. I don't know a corresponding extension $S(\supset)$ of $S(\equiv)$. For efforts in this direction cf. Bunder [1982].

§12. A type-free "modal" theory. To get a good view of the consequences of $S(\equiv)$ or $S'(\equiv)$ formulated entirely in a classical language (i.e. without the connective $\equiv$), it proves useful to pass first through a language $L(\Box)$ with laws of modal character. $L(\Box)$ is obtained from L by adjoining the unary propositional operator $\Box$, where $\neg$, $\wedge$ and $\forall$ continue to be the basic operators of L.

Take $\mathrm{Ax}(\Box)$ to consist of all formulas of the following form in $L(\Box)$:

(1) $\Box\varphi \to \varphi$.

(2) $\Box\varphi \to \Box\Box\varphi$.

(3) $\varphi \to \Box\varphi$, for any atomic φ.

(4) $\neg\varphi \to \Box\neg\varphi$, for L_0 atomic φ.

(5) $\Box\neg\neg\varphi \leftrightarrow \Box\varphi$.

(6) (i) $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$, and (ii) $\Box\neg(\varphi \wedge \psi) \leftrightarrow \Box\neg\varphi \vee \Box\neg\psi$.

(7) (i) $\Box(\forall x\varphi) \leftrightarrow \forall x\Box\varphi$, and (ii) $\Box\neg\forall x\varphi \leftrightarrow \exists x\Box\neg\varphi$.

---

[22] Cf. also Behmann [1959] and §14 below. The connection with Behmann's work was brought to our attention both by W. Craig and G. Kreisel.

REMARK. The statement that these laws are of modal character is rather loose. Though we use the $\Box$ symbol familiar from modal theory as the necessity operator, it should not be interpreted in that way here. Rather it is preferable to read $\Box\varphi$ as: φ *is established*, thinking in terms of the *investigative interpretation* discussed at the end of §7. One basic difference in the two interpretations is that $\Box(\varphi \vee \neg\varphi)$ is accepted in modal logic, but not here since by (5) and (6) $\Box(\varphi \vee \psi) \leftrightarrow \Box\varphi \vee \Box\psi$ (cf. the next lemma). There is a similar divergence with respect to $\exists$. Finally, Barcan's formula (7)(i) is not always accepted in modal logic.

In the following we define $D\varphi = \Box\varphi \vee \Box\neg\varphi$.

LEMMA. *The following are consequences of* $Ax(\Box)$:

(i) $\neg(\varphi \wedge \Box\neg\varphi)$.

(ii) $\neg(\Box\varphi \wedge \Box\neg\varphi)$.

(iii) $\Box(\varphi \vee \psi) \leftrightarrow \Box\varphi \vee \Box\psi$, and $\Box\neg(\varphi \vee \psi) \leftrightarrow \Box\neg\varphi \wedge \Box\neg\psi$.

(iv) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.

(v) $\Box(\exists x\varphi) \leftrightarrow \exists x\Box\varphi$, and $\Box\neg(\exists x\varphi) \leftrightarrow \forall x\Box\neg\varphi$.

(vi) $D\varphi \rightarrow (\varphi \leftrightarrow \Box\varphi)$.

(vii) $D\varphi$ *for each formula* φ *of* L_0.

The proofs are straightforward. We note only (iii). By definition, $(\varphi \vee \psi) = \neg(\neg\varphi \wedge \neg\psi)$, so we have the chains of equivalences

$$\Box(\varphi \vee \psi) \leftrightarrow \Box\neg(\neg\varphi \wedge \neg\psi) \leftrightarrow \Box\neg\neg\varphi \vee \Box\neg\neg\psi \leftrightarrow \Box\varphi \vee \Box\psi$$

and

$$\Box\neg(\varphi \vee \psi) \leftrightarrow \Box\neg\neg(\neg\varphi \wedge \neg\psi) \leftrightarrow \Box(\neg\varphi \wedge \neg\psi) \leftrightarrow \Box\neg\varphi \wedge \Box\neg\psi.$$

Now for each of the relation symbols $T_k(x_1, \ldots, x_k, z)$ of $L - L_0$ define

$$(^-) \qquad \bar{T}_k(x_1, \ldots, x_k, z) = \Box\neg T_k(x_1, \ldots, x_k, z).$$

It follows from (i) of the preceding lemma that $(T_k, \bar{T}_k)$ form a disjoint pair. The associated abstraction axioms are here formulated as follows, for each symbol T_k of L and each formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$ of $L(\Box)$.

$$(\mathrm{T}_k\mathrm{A})_\Box \quad \begin{cases} T_k(x_1, \ldots, x_k, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]) \leftrightarrow \Box\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n), \\ \bar{T}_k(x_1, \ldots, x_k, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]) \leftrightarrow \Box\neg\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n), \end{cases}$$

We define $S(\Box)$ to consist of S_0 plus $\mathrm{Ax}(\Box)$ plus $(\mathrm{T}_k\mathrm{A})_\Box$ for T_k in L.

THEOREM. $S(\Box)$ *is interpretable in both* $S(\equiv)$ *and* $S'(\equiv)$ *by* $\Box\varphi = (\varphi \equiv t)$. *Hence* $S(\Box)$ *is a conservative extension of* S_0.

PROOF. Note first that $\Box\neg\varphi \leftrightarrow \varphi \equiv f$ for any φ. Then it is a routine check to show that $\mathrm{Ax}(\Box)$ follows from $\mathrm{Ax}(\equiv)$; here (7)(i) is used only in the form

$$D(\psi) \rightarrow [(\varphi \equiv \psi) \equiv t \leftrightarrow \varphi \equiv \psi],$$

which follows from both forms (7)(i) and (7)′(i). Now for $(\mathrm{T}_k\mathrm{A})_\Box$, let $\theta = T_k(x_1, \ldots, x_k, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n])$, and $\bar{\theta}$ the same with $\bar{T}_k$, and write φ for $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$. The axiom $(\mathrm{T}_k\mathrm{A})_\equiv$ tells us that $\theta \equiv \varphi$. Hence $\theta \equiv t \leftrightarrow \varphi \equiv t$, so $\theta \leftrightarrow \Box\varphi$, θ being atomic. Also $\theta \equiv f \leftrightarrow \varphi \equiv f$; so $\Box\neg\theta \leftrightarrow \Box\neg\varphi$, i.e. $\bar{\theta} \leftrightarrow \Box\neg\varphi$ by the definition $(^-)$, Thus the axioms $(\mathrm{T}_k\mathrm{A})_\Box$ are verified.

As always, the two cases of $(\mathrm{T}_k\mathrm{A})_\Box$ of special interest to us are those for $k = 0, 1$:

$$(\mathrm{TA})_\Box \qquad \begin{cases} T(\ulcorner\varphi\urcorner) \leftrightarrow \Box\varphi, \\ \bar{T}(\ulcorner\varphi\urcorner) \leftrightarrow \Box\neg\varphi; \end{cases}$$

$$(\mathrm{CA})_\Box \qquad \begin{cases} x \in \{u \mid \varphi(u, y_1, \ldots, y_n)\} \leftrightarrow \Box\varphi(x, y_1, \ldots, y_n), \\ x \mathrel{\bar{\in}} \{u \mid \varphi(u, y_1, \ldots, y_n)\} \leftrightarrow \Box\neg\varphi(x, y_1, \ldots, y_n). \end{cases}$$

Consistency of a scheme $(\mathrm{CA})_\Box$ is due to Fitch [1966].

§13. Theories in the CPC for partial predicates as disjoint pairs. Finally we pass to a completely classical language. The price for doing this is to replace each basic relation symbol T_k of L (outside L_0) by a *pair of basic symbols* $(T_k, \bar{T}_k)$. The resulting language is denoted $L(+/-)$. It is useful here to take all of $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$ as basic operators. A formula of $L(+/-)$ is said to be *positive* over L_0 if it is equivalent to one built up without $\neg$ from atomic formulas and negations of L_0-atomic formulas. Every formula φ has associated with it a formula φ^+ which is positive over L_0 and which approximates φ. One way to obtain φ^+ is to put φ in prenex disjunctive normal form and replace each occurrence $\neg T_k$ or $\neg \bar{T}_k$ by $\bar{T}_k$ or T_k, resp. Let $\varphi^- = (\neg\varphi)^+$ be the positive approximant of the negation of φ. We can also define φ^+ and φ^- inductively as follows:

(i) $\varphi^+ = \varphi$ for all atomic φ.

(ii) If φ is L_0-atomic, $\varphi^- = \neg\varphi$; if $\varphi = T_k(\ldots)$ then $\varphi^- = \bar{T}_k(\ldots)$; and if $\varphi = \bar{T}_k(\ldots)$ then $\varphi^- = T_k(\ldots)$.

(iii) $(\neg\varphi)^+ = \varphi^-$ and $(\neg\varphi)^- = \varphi^+$.

(iv) $(\varphi \wedge \psi)^+ = \varphi^+ \wedge \psi^+$ and $(\varphi \wedge \psi)^- = \varphi^- \vee \psi^-$.

(v) $(\varphi \vee \psi)^+ = \varphi^+ \vee \psi^+$ and $(\varphi \vee \psi)^- = \varphi^- \wedge \psi^-$.

(vi) $(\forall x\varphi)^+ = \forall x\varphi^+$ and $(\forall x\varphi)^- = \exists x\varphi^-$.

(vii) $(\exists x\varphi)^+ = \exists x\varphi^+$ and $(\exists x\varphi)^- = \forall x\varphi^-$.

LEMMA 1. (i) *For each φ both φ^+ and φ^- are positive over L_0.*

(ii) *If φ is positive over L_0 then φ is equivalent to φ^+.*

As basic axioms in $L(+/-)$ we take

$$\mathrm{Dis}(T_k, \bar{T}_k) \qquad \neg\left(T_k(x_1, \ldots, x_k, z) \wedge \bar{T}_k(x_1, \ldots, x_k, z)\right)$$

which express that $(T_k, \bar{T}_k)$ form a disjoint pair. Dis is used to denote the collection of all these axioms for T_k a symbol of L.

LEMMA. 2. Dis *implies* $(\varphi^+ \to \varphi)$ *and* $(\varphi^- \to \neg\varphi)$ *for each* φ.

Now we take $S(+/-)$ to consist of $S_0 +$ Dis plus the following for each symbol T_k of the language and each formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$:

$$(\mathrm{T}_k\mathrm{A})_{(+/-)} \qquad \begin{cases} T_k(x_1, \ldots, x_n, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]) \leftrightarrow \varphi^+(x_1, \ldots, x_k, y_1, \ldots, y_n), \\ \bar{T}_k(x_1, \ldots, x_n, \varphi[\hat{u}_1, \ldots, \hat{u}_k, y_1, \ldots, y_n]) \leftrightarrow \varphi^-(x_1, \ldots, x_k, y_1, \ldots, y_n). \end{cases}$$

THEOREM. *$S(+/-)$ is interpretable in $S(\Box)$ by the definition $(^-)$ in §12 of $\bar{T}_k$ in $L(\Box)$. Hence $S(+/-)$ is conservative over S_0.*

PROOF. Assume $\mathrm{Ax}(\Box)$ and define $\bar{T}_k(\ldots) = \Box\neg T_k(\ldots)$ as in §12. Then one proves $\varphi^+ \leftrightarrow \Box\varphi$ and $\varphi^- \leftrightarrow \Box\neg\varphi$ by induction on φ. Note that here $\vee$ and $\exists$ are treated as basic operators while in $S(\Box)$ they were treated as defined operators. For example, to prove $(\varphi \vee \psi)^+ \leftrightarrow \Box(\varphi \vee \psi)$ and $(\varphi \vee \psi)^- \leftrightarrow \Box\neg(\varphi \vee \psi)$, assuming $\varphi^+ \leftrightarrow \Box\varphi$, $\varphi^- \leftrightarrow \Box\neg\varphi$, $\psi^+ \leftrightarrow \Box\psi$, $\psi^- \leftrightarrow \Box\neg\psi$, we use $(\varphi \vee \psi)^+ = \varphi^+ \vee \psi^+ \leftrightarrow \Box\varphi \vee \Box\psi \leftrightarrow \Box(\varphi \vee \psi)$ (by the Lemma (iii) of §12) while $(\varphi \vee \psi)^- = \varphi^- \wedge \psi^- \leftrightarrow \Box\neg\varphi \wedge \Box\neg\psi \leftrightarrow \Box\neg(\varphi \vee \psi)$ (again by Lemma (iii) of §12). $\exists$ is handled similarly.

As usual the two cases of the schemes $(\mathrm{T}_k\mathrm{A})_{(+/-)}$ of special interest to us are those for $k = 0, 1$.

$$(\mathrm{TA})_{(+/-)} \qquad \begin{cases} T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi^+, \\ \bar{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi^-, \end{cases} \quad \text{for each sentence } \varphi;$$

$$(\mathrm{CA})_{(+/-)} \qquad \begin{cases} x \in \{u \mid \varphi(u, y_1, \dots, y_n)\} \leftrightarrow \varphi^+(x, y_1, \dots, y_n), \\ x \mathrel{\bar{\in}} \{u \mid \varphi(u, y_1, \dots, y_n)\} \leftrightarrow \varphi^-(x, y_1, \dots, y_n), \end{cases}$$
$$\text{for each formula } \varphi(x, y_1, \dots, y_n).$$

The consistency of the scheme $(\mathrm{CA})_{(+/-)}$ is (essentially) due to Gilmore [1974].[23] Gilmore obtained a fixed-model result for structures of the form $\mathfrak{M} = (M, \in, \bar{\in})$ by an inductive argument, observing that positive formulas are preserved under $\leq$. Extensions of this were given in Feferman [1975b], [1977].

We now look at mathematical consequences for a theory of classes. The axioms $\mathrm{Dis}(\in, \bar{\in})$ and $(\mathrm{CA})_{(+/-)}$ are assumed in the following without further remark. The first problem is to see for which φ we have the ordinary instance of

$$(\mathrm{CA}) \qquad x \in \{u \mid \varphi(u, y_1, \dots, y_n)\} \leftrightarrow \varphi(x, y_1, \dots, y_n)$$

derivable. The second problem is to see for which φ we have

$$x \mathrel{\bar{\in}} \{u \mid \varphi(u, y_1, \dots, y_n)\} \leftrightarrow x \notin \{u \mid \varphi(u, y_1, \dots, y_n)\}.$$

There are some immediate easy answers: (CA) holds for all φ which are positive over L_0, and the above equivalence of $\bar{\in}$ with $\notin$ holds when $\varphi^- \leftrightarrow \neg\varphi$, so it holds when $\neg\varphi$ is positive over L_0.

DEFINITION. (i) $\mathrm{Cl}(a) = \forall x(x \in a \vee x \mathrel{\bar{\in}} a)$.

(ii) $\mathrm{Cl} = \{a \mid \mathrm{Cl}(a)\}$.

We are thus treating Cl both as a formula and as an object. But since the formula is positive, we have $a \in \mathrm{Cl} \leftrightarrow \mathrm{Cl}(a)$. Note that $a \in \mathrm{Cl}$ iff $\forall x[x \mathrel{\bar{\in}} a \leftrightarrow x \notin a]$, and that $a \mathrel{\bar{\in}} \mathrm{Cl}$ is false for all a.

LEMMA 3. $\mathrm{Cl} \notin \mathrm{Cl}$.

PROOF. Suppose $\mathrm{Cl} \in \mathrm{Cl}$. The Russell argument is adapted to yield a contradiction. Let $r = \{a \mid a \in \mathrm{Cl} \wedge a \mathrel{\bar{\in}} a\}$. Then $a \in r \leftrightarrow a \in \mathrm{Cl} \wedge a \mathrel{\bar{\in}} a \leftrightarrow a \in \mathrm{Cl} \wedge a \notin a$. Also by $(\mathrm{CA})_{(+/-)}$, $a \mathrel{\bar{\in}} r \leftrightarrow (a \in \mathrm{Cl} \wedge a \mathrel{\bar{\in}} a)^- \leftrightarrow a \mathrel{\bar{\in}} \mathrm{Cl} \vee a \in a$. Since $\mathrm{Cl} \in \mathrm{Cl}$, we have $a \mathrel{\bar{\in}} \mathrm{Cl} \leftrightarrow a \notin \mathrm{Cl}$. Thus $a \mathrel{\bar{\in}} r \leftrightarrow a \notin \mathrm{Cl} \vee a \in a \leftrightarrow a \notin r$. Hence $r \in \mathrm{Cl}$. It follows that $r \in r \leftrightarrow r \notin r$, which gives a contradiction.

DEFINITION. (i) $a \subseteq b \leftrightarrow \forall x(x \in a \rightarrow x \in b)$.

(ii) $a \equiv b \leftrightarrow a \subseteq b \wedge b \subseteq a$.

[23] Gilmore first publicized his work in 1967; cf. the notes in §14 below.

Since we are not assuming extensionality it is necessary to consider the relation $\equiv$ of extensional equality. Lemma 3 can be strengthened to: $\neg\exists c(c \equiv \mathrm{Cl} \wedge c \in \mathrm{Cl})$, using the same argument.

The elements of Cl are called *total classes*, thinking of a *partial class* c as one for which we need not have $x \notin c \leftrightarrow x \mathrel{\bar{\in}} c$, or as a pair of disjoint classes $(\{x \mid x \in c\}, \{x \mid x \mathrel{\bar{\in}} c\})$. By Lemma 3, Cl is an example of a partial class which is not total. We shall now derive some closure conditions on Cl which allow us to construct new total classes from given ones.

Let $\varphi(x, y_1, \ldots, y_n, a_1, \ldots, a_m)$ be a formula without any $\bar{\in}$ symbols. We say that φ is in $\in (a_1, \ldots, a_m)$*-form* if each atomic subformula $s \in t$ of φ is of the form $s \in a_i$ for some i. (There are no other restrictions, e.g. we might have $a_i \in a_i$ in φ.) Write $x_1, \ldots, x_n \in a$ for $x_1 \in a \wedge \cdots \wedge x_n \in a$.

LEMMA 4. *If* $\varphi = \varphi(x, y_1, \ldots, y_n, a_1, \ldots, a_m)$ *is in* $\in (a_1, \ldots, a_m)$ *form, then*

$$a_1, \ldots, a_m \in \mathrm{Cl} \rightarrow (\varphi^+ \leftrightarrow \varphi) \wedge (\varphi^- \leftrightarrow \neg\varphi).$$

Hence

$$a_1, \ldots, a_m \in \mathrm{Cl} \rightarrow \{x \mid \varphi(u, y_1, \ldots, y_n, a_1, \ldots, a_m)\} \in \mathrm{Cl}.$$

PROOF. We can prove this by induction on φ in $\in (a_1, \ldots, a_m)$-form, relative to given $a_1, \ldots, a_m$ in Cl. The basis step is with φ of the form $(s \in a_i)$. Here $(s \in a_i)^- = (s \mathrel{\bar{\in}} a_i) \leftrightarrow \neg(s \in a_i)$.

DEFINITION. (i) $V = \{x \mid x = x\}$ and $\Lambda = \{x \mid x \neq x\}$.

(ii) $\{y_1, y_2\} = \{x \mid x = y_1 \vee x = y_2\}$.

(iii) $a \cup b = \{x \mid x \in a \vee x \in b\}$ and $a \cap b = \{x \mid x \in a \wedge x \in b\}$.

(iv) $-a = \{x \mid x \notin a\}$.

(v) $a \times b = \{x \mid x = (P_1(x), P_2(x)) \wedge P_1(x) \in a \wedge P_2(x) \in b\}$.

(vi) $\mathscr{D}(a) = \{x \mid \exists y(x, y) \in a\}$.

(vii) $\breve{a} = \{x \mid x = (P_1(x), P_2(x)) \wedge (P_2(x), P_1(x)) \in a\}$.

(viii) $\bigcup a = \{x \mid \exists y(y \in a \wedge x \in y)\}$.

(ix) $\bigcap a = \{x \mid \forall y(y \in a \rightarrow x \in y)\}$.

(x) $\mathscr{P}a = \{x \mid \mathrm{Cl}(x) \wedge x \subseteq a\}$.

LEMMA. 5. (i) $\Lambda, V, \{y_1, y_2\} \in \mathrm{Cl}$.

(ii) $a, b \in \mathrm{Cl} \rightarrow a \cup b, a \cap b, -a, a \times b, \mathscr{D}a, \breve{a} \in \mathrm{Cl}$.

(iii) $a \in \mathrm{Cl} \wedge a \subseteq \mathrm{Cl} \rightarrow \bigcup a, \bigcap a \in \mathrm{Cl}$.

PROOF. (i) and (ii) are by Lemma 4. (iii) requires an additional argument. By $(\mathrm{CA})_{(+/-)}$ we have $x \in \bigcup a \leftrightarrow \exists y(y \in a \wedge x \in y)$ and $x \mathrel{\bar{\in}} \bigcup a \leftrightarrow \forall y(y \mathrel{\bar{\in}} a \vee x \mathrel{\bar{\in}} y)$. Then the hypothesis $a \in \mathrm{Cl} \wedge a \subseteq \mathrm{Cl}$ show $x \mathrel{\bar{\in}} \bigcup a \leftrightarrow x \notin \bigcup a$. Also $x \in \bigcap a \leftrightarrow \forall y(y \mathrel{\bar{\in}} a \vee x \in y)$ and $x \mathrel{\bar{\in}} \bigcap a \leftrightarrow \exists y(y \in a \wedge x \mathrel{\bar{\in}} y)$. Once more the hypothesis gives $x \mathrel{\bar{\in}} \bigcap a \leftrightarrow x \notin \bigcap a$ and indeed $x \in \bigcap a \leftrightarrow \forall y(y \in a \rightarrow x \in y)$.

Since $\forall x(x \in V)$, we have $V \in V$ in particular. This is our first instance of *self-application*, though by itself not one of special interest.

DISCUSSION. It appears that with Lemma 5 one is setting a course for a reasonable development of a (nonextensional) type-free theory of classes in this framework. However, the next steps usually taken run into *obstacles*, which are now taken up.

(i) First of all, $a \in \mathrm{Cl}$ does not imply $\mathscr{P}a \in \mathrm{Cl}$, since $\mathscr{P}V$ is extensionally equal to Cl.[24]

(ii) The usual way of defining ordered pairs set-theoretically, $\langle y_1, y_2 \rangle = \{\{y_1\}, \{y_1, y_2\}\}$, is also available to us here. However, that is unnecessary as ordered pairs (y_1, y_2) are already provided by the theory S_0. Thus in defining the notion of function from one class to another, we can take

$$(c:a \to b) \leftrightarrow c \subseteq a \times b \wedge \forall x \in a \exists! y \in b[(x, y) \in c].$$

But if we then define $b^a = \{c \,|\, \mathrm{Cl}(c) \wedge (c:a \to b)\}$ we are not able to show that $a, b \in \mathrm{Cl} \to b^a \in \mathrm{Cl}$. (In this case the counterexample is $\{0, 1\}^V \notin \mathrm{Cl}$.) Since the formation of *function classes* is essential to the definition of the real number system and analysis, this is a *critical defect*. (It is such even more so in view of our stated aim to deal with the analogous functor categories.)

(iii) The *natural number system* would have to be treated prior to analysis. One way to try to introduce numbers would follow the Fregean approach through the *cardinals*. Define.

$$(a \sim b) \leftrightarrow \exists c(\mathrm{Cl}(c) \wedge c:a \to b \wedge \forall y \in b \exists! x \in a[(x, y) \in c]).$$

Then $\sim$ is an equivalence relation on classes. The corresponding equivalence "classes" would be

$$[a] = \{b \,|\, \mathrm{Cl}(b) \wedge a \sim b\}.$$

However, we do not have $[a] \in \mathrm{Cl}$ when $a \in \mathrm{Cl}$. Further, without extensionality we do not have the usual property $[a] = [b] \leftrightarrow a \sim b$, only $[a] \equiv [b] \leftrightarrow a \sim b$. Thus we cannot develop the theory of cardinals as cardinal equivalence types.

(iv) An alternative approach would be to try to define the natural numbers as the smallest class a containing 0 and closed under $'$, where these are defined in S_0 by §9. Formally, this suggests taking

$$\mathbf{N} = \{x \,|\, \forall a[\mathrm{Cl}(a) \wedge 0 \in a \wedge \forall y(y \in a \to y' \in a) \to x \in a]\}.$$

While the matrix is equivalent to $\forall a[\mathrm{Cl}(a) \to 0 \mathrel{\bar{\in}} a \vee \exists y(y \mathrel{\bar{\in}} a \vee y' \in a) \vee x \in a]$, there is still the negative subformula "Cl(a)" which cannot be circumvented. Thus we cannot prove that $\mathbf{N}$ satisfies its defining condition. Even if it did, we would not obtain full induction on $\mathbf{N}$, since not every formula defines a member of Cl.

(v) One way around this problem is to build $\mathbf{N}$ in from the beginning by use of the quantifiers $\forall_{\mathbf{N}}$ and $\exists_{\mathbf{N}}$ (§8). Since these are monotonic, the fixed-model theorem and all the results of §§9–12 can be extended to include them in the language. Then $\mathbf{N}$ is definable as $\{x \,|\, \exists_{\mathbf{N}} y(x = y)\}$; we have $\mathbf{N} \in \mathrm{Cl}$, and by the semantics of $\exists_{\mathbf{N}}$ we have

$$(\mathbf{N}) \qquad \begin{cases} 0 \in \mathbf{N}, \forall x(x \in \mathbf{N} \to x' \in \mathbf{N}), \quad \text{and} \\ \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x')) \to \forall x(x \in \mathbf{N} \to \varphi(x)) \quad \text{for each } \varphi. \end{cases}$$

While this may be considered "cheating", it serves at least to show that one can

[24] One might think to define $\mathscr{P}a$ instead as $\{x \,|\, x \subseteq a\}$. Then one would have $x \in \mathscr{P}a \leftrightarrow \forall y(y \mathrel{\bar{\in}} x \vee y \in a) \leftrightarrow (-x) \cup a \equiv V$. Take $a = V - \{\mathrm{Cl}\}$ and $x = \mathrm{Cl}$. Then $x \subseteq a$ but $(-x) \equiv \Lambda$, so $(-x) \cup a \not\equiv V$. Thus it is not the case that $x \in \mathscr{P}a \leftrightarrow x \subseteq a$ under this definition even for $a \in \mathrm{Cl}$.

obtain consistent type-free theories of classes which include the (**N**) axioms for **N** as a total class.

(vi) Similarly the notion of *well-ordering relation* with the usual inductive properties can be handled in this framework by the incorporation of a suitable generalized quantifier, but cannot be managed directly without such. In any case, we cannot develop the theory of *ordinals* as isomorphism types of well-ordered relations, for the same reasons as in (iii).

(vii) To conclude, the mathematical content of the type-free theory of classes finally reached in this section is rather meager. Part II of this paper is devoted to the construction of much richer type-free mathematical theories. In addition, we should want such to be a theory of *total classes* to begin with (i.e. with the sole relation $\in$ of membership), since the idea of partial classes (in contrast to that of partial functions) is not felt to be a natural one, mathematically.

§14. Summary and historical notes. In Part B (§§7–13) we have rung a series of changes on the theme of type-free semantical and mathematical theories of partial predicates. In doing so we made a transition from theories in 3-valued logics to theories in the classical predicate calculus (CPC), passing via extensions of CPC by the additional operators $\equiv$ and $\Box$. This has in effect constituted a transition from the solution route 2° for the paradoxes ("restriction of logic") to the solution route 3° ("restriction of basic principles"). In addition, the treatment unified the semantical and mathematical theories by means of a more general theory of predicates T_k. At each step the defects and disadvantages of a given formal solution were weighed against its attractions and advantages. Only at the end (in §13) did specifically mathematical goals make their appearance. These goals will take control in Part II in order to move toward improved solutions. The work carried out so far will help there to focus both on what is to be avoided and what is to be accomplished. In addition, the criterion that a solution should be equally satisfactory from a constructive point of view, which has not so far been met, will be dealt with there.

The work described above has not been presented in the historical order in which it evolved, and the references to other sources have been rather perfunctory. We thus conclude this part with some notes on relevant work, presented in (essentially) chronological order. Even so, what follows is far from comprehensive, especially on the semantical side. In addition, the notes only attempt to indicate what I take to be the main direction or character of the contributions (insofar as they are connected with the present work).[25]

Behmann [1931] presented informal ideas for the avoidance of paradoxical abstraction, by analysis of reduction procedures such as that of $t \in \{x : \varphi(x)\}$ to $\varphi(t)$. Paradoxical abstraction leads to nonterminating reduction sequences. These ideas were spelled out more fully in Behmann [1959] but never in exact form. (The handling of the system $S(\equiv)$ in Aczel/Feferman [1980] is motivated by similar ideas.).

In 1941 and during the 1950s, Ackermann published a series of papers on type-free systems; cf. e.g. Ackerman [1950] and [1957]. Some of these were simplified and

[25] I believe a serious comparative study would be of value in this subject, since in many cases it is not easy to assess what is accomplished.

extended by Schütte; cf. Schütte [1952] and [1960, Chapter VIII]. A common feature of these systems is that they do not contain the full law of excluded middle, but are classical in other logical respects; in that sense they are based on a form of 3-valued logic. However, they are not keyed to any prior semantics. Some of the systems make use of an additional unary propositional operator **B** ("Beweisbar") which has a $\Box$-like character.

Fitch [1948] inaugurates a series of papers (continuing to Fitch [1980]) in which the inductive method is used to set up consistent combinatory systems with strong means for representing logical and mathematical notions. I trace the use of the inductive method in this subject (for the construction of partial fixed-point models) to Fitch's work. (Readers may not find the connection so clear, since his systems involve an unusual mixing of combinatory and logical syntax and since his pursuit of extensionality complicates matters.)

Halldén [1949] is an original and ambitious essay on the philosophical side setting up a "logic of nonsense" to deal with the paradoxes and other problems. This is based on a form of (Kleene) weak 3-valued logic. Halldén also suggested modal extensions. Unfortunately, this work is not easily available. (More recent treatments of some of his systems are to be found in Segerberg [1965] and Woodruff [1973].)

The study of the comprehension scheme $(\mathrm{CA})_{\equiv}$ in Łukasiewicz 3-valued logic was initiated by Skolem [1960] (cf. also Skolem [1963] and the work of Brady below).

As explained in §10, Wang [1961] gave a complete Gentzen-style axiomatization of Kleene's strong 3-valued logic.[26] Apparently a successor to that paper with applications to set theory was planned, but does not appear to have been published.

Fitch [1963] is central to his approach inaugurated in 1948, described above. Somewhat differently, in an abstract Fitch [1966], he states the consistency of the scheme $\exists a \forall x[x \in a \leftrightarrow \Box\varphi(x)]$ for arbitrary φ in an extension of ordinary predicate calculus. But in a detailed paper in this JOURNAL, vol. 32 (1967), pp. 93–103, Fitch returns to a modal extension of a combinatory system closer to his system $C\Delta$ of [1963] (though weaker in other respects); cf. also Fitch's article in ***Monist***, vol. 51 (1967), pp. 104–109.

In 1967 Gilmore wrote a report on his system of partial set theory where one works in the classical logic of a disjoint pair $(\in, \bar{\in})$ (as taken up in §13 above). This was reported to the 1967 Institute on Set Theory at UCLA (but not published until Gilmore [1974]), and is where I first saw the use of partial predicate models and the inductive method of building fixed-point models.[27]

In the discussion (viii) of §10 we have already mentioned the interesting idea of van Fraassen [1968] to use partial models so as to avoid the semantic paradoxes but at the same time retain classical logical validity through the device of supervaluations. Though the motivation to stay within ordinary reasoning is the same as here, the solution is different. As far as I know, the logic of supervaluations has not been

[26] Such systems keep being rediscovered; cf. e.g. Thomason [1969] and Scott [1975].

[27] At the time I did not pay much attention to these methods and results, and in fact my view then of work on type-free theories was rather negative. What I did not realize was the potential utility of type-free theories for fairly specific mathematical purposes. In contrast, the work I had seen was dominated (at least implicitly) by an attempt to reconstruct Frege's global program for the foundations of mathematics. It took me a while to recognize that formal work for the latter could be enlisted in the cause of the former.

pursued in any detailed systematic way, though the idea has been applied further by van Fraassen and others, e.g. Skyrms [1970]; cf. also the collection Martin [1970] of essays on the Liar paradox for a number of related discussions and suggestions of alternative approaches. In addition that volume contains a very good bibliography (further extended in the 1978 edition), particularly on the semantical side of the subject.

Brady [1971] much strengthened Skolem's result for consistency of forms of $(CA)_{\equiv}$ in Łukasiewicz 3-valued logic, using Gilmore's inductive method. He extended this in 1972 to a system containing Bernays-Gödel set theory, but the idea of the proof is the same.

The paper Nepeĭvoda [1973] presents an infinitary system for Kleene 3-valued logic in the context of number theory. Though the stated interest there is in the subject of predicativity, in effect he builds a partial model for $(CA)_{\mapsto}$ in that system. (Nepeĭvoda had several related papers in the period 1973–74.)

Kripke [1975] appears to be the first paper on the semantical side to make use of the inductive construction of fixed-point models, in this case with Kleene strong 3-valued logic. Kripke's paper contains an interesting discussion of the problems which that construction solves and considerations of some alternatives. The construction itself was carried out independently by Kindt [1976]. Also independently, Martin/Woodruff [1975] dealt with fixed points for partial truth predicates, but instead of producing minimal fixed points inductively, applied Zorn's lemma to obtain maximal fixed points.

My own work on type-free theories dates to 1974, with the first publications being in 1975.[28] The first of these, [1975a], was concerned with nonextensional theories of partial functions and total classes (called operations and classifications, resp.) for the formalization of Bishop-style constructive mathematics. While this provided examples of self-membership, the framework did not appear adequate for applications to an unrestricted theory of structures and categories. Pursuit of the latter led me to theories of partial functions and partial classes wherein, like the constructive systems, functions appear prior to classes. This "two-stage" approach was a principal new feature of the work. It can be described in terms of the presentation here as the assumption that S_0 contains a suitably strong theory of partial functions (and secondarily that abstracts $\{x \mid \varphi(x, y_1, \ldots, y_n)\}$ in S are functions of their parameters $y_1, \ldots, y_n$ in the sense of S_0). In this paper such an assumption will only make its explicit appearance and be motivated in Part II.

My notes [1975b] presented a form of $S(\Box)$ for $(CA)_{\Box}$ over a theory S_0 of partial functions, reading $\Box\varphi$ as "φ is established in the course of a (possibly transfinite) investigation" (cf. §12 above). This was followed in [1975c] by a system like $S_{(+/-)}$ for $(CA)_{(+/-)}$ over such S_0. In my notes [1976] I initiated a comparative (and parallel) study of semantical and mathematical type-free theories in various logics for partial models. The present paper grew out of those notes and incorporates most of the material from there. A Part II was planned for the paper [1975c] but never

[28] Actually my interest in the foundations of category theory goes back somewhat earlier. In Feferman [1969] I had applied the reflection principle in ZF to avoid the distinction between "small" and "large" categories.

published; however, much of the intended material was eventually used in Feferman [1977], which described the $\Box$-systems and also sketched applications to category theory.

The paper Scott [1975] combines features of (one stage of) my [1975c] paper with the system $S_{\vdash\dashv}$ for $(CA)_{\vdash\dashv}$ in Kleene 3-valued logic, described above in §10. Cantini [1979a] adapted Nepeĭvoda [1973] and Feferman [1975c] to a semantical type-free theory; the approach through a modal system like $S(\Box)$ was similarly adapted in Cantini [1979b].

The axioms for $S(\equiv)$ and the idea for a conservation proof by Church-Rosser methods were first introduced in 1977 at a symposium at Yale in honor of Professor Fitch. As a result of later improvements and additions by Aczel, particularly with the approach to $S'(\equiv)$ described in §11 above, the work appeared finally as Aczel/Feferman [1980]. No special strong assumptions were made on S_0 there, though it can also be made part of a two-stage theory as indicated above. The interesting constructive semantical theory of Aczel [1980] will be described in Part II, where its relevance will become clear.

I have concentrated here on works most directly relevant to the approaches and matters taken up in this part; admittedly even that has been far from comprehensive.[29] Nothing has been said about quite different type-free approaches e.g. via stratified theories or illative combinatory systems.[30] It is my plan to comment on those and others at the end of Part II. Finally, the issue of extensionality vs. nonextensionality, which has been largely ignored here, will be taken up there.

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[30] Good introductions to the work of the Curry school on illative combinatory logic are to be found in Bunder [1980] and Curry [1980].

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