ON A LEMMA OF KORAS-RUSSELL

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A *log surface* (X, E) consists of a normal projective surface X and an effective Weil $\mathbb{Q}$-divisor E with coefficients of irreducible components not bigger than 1. By K_X we denote the canonical divisor on X. X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier.

A log surface (X, E) can be studied using the log Minimal Model Program in terms of contractions of log extremal curves which are negative with respect to $K_X + E$. We are interested in a situation where we have to run the program in a presence of some other effective divisor D on X. More precisely, we want to analyze when is it possible to find (-1)-curves on X with controlled intersection with D and E or, more concretely, curves which are both $(K_X + E)$- and $(K_X + D)$-negative.

Definition 1.1. Let D, C be effective $\mathbb{Q}$-divisors on a normal projective surface X.
(a) C is $(K_X + D)$-*negative* if $(K_X + D) \cdot C < 0$.
(b) C is $(K_X + D)$-*neutral* if $h^0(m(K_X + D + nC)) = h^0(m(K_X + D))$ for every $n, m \geq 0$.

Note that if C is irreducible and $(K_X + D)$-negative then it is $(K_X + D)$-neutral. If $\kappa(K_X + D) = -\infty$ then C is $(K_X + D)$-neutral if and only if $\kappa(K_X + D + nC) = -\infty$ for every natural n.

The lemma below generalizes Theorem 4.2 in [KR99], which treats the case of X smooth, $E := \frac{1}{r}E'$, $r \in \mathbb{N}_+$, D and E' reduced. Our proof can be considered a limit version of the original proof. Even in the latter case 1.2 is stronger by avoiding assumptions on $\mathrm{Pic}(X)$ and by replacing the assumptions (v,vi) of 4.2 loc. cit. with a weaker assumption that $K + E$ is not nef. The theorem generalizes or simplifies proofs of various other instances of 4.2 which were successfully applied to many geometric problems concerning surfaces (see [Kor07, 2.6, 2.7], [KR07, 5.10], [CNKR09, 2.4, 2.6], [Kor11, 2.2, 2.3], [KR11, 2.3, 2.4], [PUB4, 4.2]).

Lemma 1.2. *Assume:*
(1) (X, E) is a log surface which is log canonical or $\mathbb{Q}$-factorial,
(2) D is an effective divisor such that $\kappa(K + D) = -\infty$,
(3) $(K + D) \cdot (K + E) \leq 0$.
Then either $K + E$ is nef or there exists a $(K + E)$-negative log extremal rational curve ℓ which is $(K_X + D)$-neutral. Every such curve satisfies $\ell^2 \leq 0$.

Note that if $\ell^2 = 0$ then $\rho(X) = 2$ by extremality.

Proof of 1.2. With our assumptions the log Minimal Model Program works (see [Fuj12, 3.2]). Let $\mathcal{S} = \overline{NE}(X) \subseteq \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the closure of the (convex) cone of effective 1-cycles on X. Since $K + E$ is not nef, there exists a $(K + E)$-negative log-extremal curve $A \subseteq X$. Note that by [Fuj12, 3.8] such a curve is rational. We may assume that A is not $(K + D)$-neutral, that is, $\kappa(K + D + mA) \geq 0$ for some $m \in \mathbb{N}$, otherwise we take $\ell = A$. (We postpone the proof that $\ell^2 \leq 0$ to the end). Then $K + D + mA \in \mathcal{S}$. For $\alpha \geq 0$ put

$$z_\alpha = \alpha m A + K + D$$

and let $\alpha_0 = \inf\{\alpha : z_\alpha \in \mathcal{S}\}$. Since $\kappa(K + D) = -\infty$, we have $z_0 \notin \mathcal{S}$ by [Fuj12, 5.2]. The cone $\mathcal{S}$ is closed, so $\alpha_0 > 0$. We have

$$z_{\alpha_0} \cdot (K + E) = \alpha_0 m A \cdot (K + E) + (K + D)(K + E) \leq \alpha_0 m A \cdot (K + E) < 0,$$

so by the logarithmic version of the Cone Theorem (see [Mat02, Theorem 7.2.1]) there exists a finite set L of $(K+E)$-negative log-extremal curves on X, such that

$$z_{\alpha_0} - \sum_{\ell \in L} r_\ell \ell,$$

where $r_\ell > 0$, belongs to $\mathcal{S}$ and intersects $K+E$ non-negatively. In particular, L is non-empty.

Suppose $\kappa(K+D+n\ell) \geq 0$ for some $\ell \in L$ and $n \geq 0$. Then consequently

$$\begin{aligned} \varepsilon(K+D) + \frac{r_\ell}{\varepsilon}\ell \in \mathcal{S} \quad \text{for some } \varepsilon > 0, \\ \varepsilon(K+D) + z_{\alpha_0} \in \mathcal{S}, \\ \alpha_0 mA + (1+\varepsilon)(K+D) \in \mathcal{S}, \\ \frac{\alpha_0}{1+\varepsilon} mA + K + D \in \mathcal{S}. \end{aligned}$$

The latter contradicts the definition of α_0. Thus every $\ell \in L$ is $(K+D)$-neutral.

It remains to show that $\ell^2 \leq 0$. Suppose $\ell^2 > 0$. By the extremality of ℓ, $\rho(X) = 1$. Let H be an ample divisor on X. Then $K+D \equiv aH$ and $K+E \equiv a'H$ for some $a, a \in \mathbb{R}$. By assumptions $a, a' < 0$, hence $(K+D)\cdot(K+E) = aa'H^2 > 0$; a contradiction. □

Remark 1.3.

(a) If X has only rational singularities then it is $\mathbb{Q}$-factorial ([Lip69]). The converse fails (see [Fuj12, 2.3]). A normal $\mathbb{Q}$-factorial surface is automatically quasi-projective (see for example [Fuj12, 2.2]).

(b) If $(K+E)^2 < 0$ then $K+E$ is not nef, because a nef divisor is a limit of ample divisors. It is also not nef if $\kappa(K+E) = -\infty$ (see [Fuj12, 5.1]).

(c) If ℓ is a $(K+E)$-negative log extremal curve with $\ell^2 \geq 0$ then (X,E) is a log Mori fiber space. This means that either $\ell^2 > 0$ and (X,E) is a log del Pezzo surface of rank 1 or $\ell^2 = 0$ and there exists a $\mathbb{P}^1$-fibration $X \to$ (curve) with irreducible fibers and such that $E \cdot \ell < 2$. In case $\ell^2 < 0$ the curve ℓ may be contained in D or E.

(d) Assumptions imply that $X \neq \mathbb{P}^2$, so we can choose ℓ so that $(K+E)\cdot\ell > -2$ (see [Fuj12, 3.8]). If (X,E) is log canonical then $\ell \cong \mathbb{P}^1$ (see [KK94, 2.3.5]).

The following lemma shows that induction is possible.

Lemma 1.4. *Let (X,D,E) be a triple satisfying $(1)-(3)$ of 1.2 with $K+E$ not nef and let σ be the contraction of a curve ℓ given by the lemma. If $\ell^2 \neq 0$ and ℓ is not $(K+D)$-negative then $(\sigma(X), \sigma_*D, \sigma_*E)$ satisfies (1)-(3).*

Proof. The birational morphism $\sigma\colon (X,D,E) \to (\bar{X},\bar{D},\bar{E}) = (\sigma(X),\sigma_*D,\sigma_*E)$ exists by the contraction theorem (it contracts only ℓ). Put $\bar{K} = K_{\bar{X}}$. If (X,E) is $\mathbb{Q}$-factorial or projective log canonical, so is $(\bar{X},\bar{E})$ (see [Fuj12]). Let $d, d' \in \mathbb{Q}$ be such that

$$\sigma^*(\bar{K}+\bar{D}) = K+D+d\ell \quad \text{and} \quad \sigma^*(\bar{K}+\bar{E}) = K+E-d'\ell.$$

By the projection formula $d'\ell^2 = (K+E)\cdot\ell < 0$, so $d' > 0$. We have $\kappa(\bar{K}+\bar{D}) = -\infty$. Indeed, otherwise $\kappa(K+D+d\ell) \geq 0$ and hence $\kappa(K+D+n\ell) \geq 0$ for any integer $n > d$, which contradicts the properties of ℓ. Since $(K+D)\cdot\ell \geq 0$, we get

$$\begin{aligned}(\bar{K}+\bar{D})\cdot(\bar{K}+\bar{E}) = (K+D)\cdot(K+E-d'\ell) = (K+D)\cdot(K+E) - d'(K+D)\cdot\ell \leq \\ \leq (K+D)\cdot(K+E) \leq 0.\end{aligned}$$

□

Finally we find a common extremal ray negative with respect to $K+E$ and $K+D$.

Theorem 1.5. *Assume D and E are effective $\mathbb{Q}$-divisors such that:*

(1) (X,E) is a log canonical or $\mathbb{Q}$-factorial log surface,

(2) $\kappa(K+D)=-\infty$*,*
(3) $(K+D)\cdot(K+E)\leq 0$*.*
Then there is a sequence of birational extremal contractions

$$(X,D,E)\to\ldots\to(\bar{X},\bar{D},\bar{E})$$

of log extremal curves which with respect to the push-forwards of $K+E$ *and* $K+D$ *are respectively negative and neutral and such that*

(i) either $K_{\bar{X}}+\bar{E}$ *is nef*
(ii) or there exists a log extremal curve $\bar{\ell}\subseteq\bar{X}$ *such that*

$$\bar{\ell}^2\leq 0,\quad (K_{\bar{X}}+\bar{E})\cdot\bar{\ell}<0\quad\textit{and}\quad (K_{\bar{X}}+\bar{D})\cdot\bar{\ell}<0.$$

In case there exists an irreducible $(K+E)$*-negative curve* A *which is not* $(K+D)$*-neutral then* $(\bar{X},\bar{E})$ *is not nef and* $\bar{l}^2<0$*.*

Proof. The first part follows from 1.2 and 1.4. Assume A as above exists. Then $K+E$ is not nef, so there exists a $(K+E)$-negative log extremal curve ℓ such that $\ell^2\leq 0$ and $\kappa(K+D+n\ell)=-\infty$ for all $n\geq 0$. Suppose $\ell^2=0$. Now (X,E) is a log Mori fiber space over a curve, i.e. $\rho(X)=2$, ℓ is a fiber of a $\mathbb{P}^1$-fibration of X and $\ell\cdot(K+E)<0$. Let C be a section of the $\mathbb{P}^1$-fibration. Then $\{C,\ell\}$ is a basis of the vector space $\mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. Put $a=(K+D)\cdot\ell$ and suppose $a\geq 0$. Then $K+D+n\ell\equiv aC+n\ell$ is a nef divisor for $n\gg 0$, so $\kappa(K+D+n\ell)\geq 0$ for $n\gg 0$; a contradiction.

Thus $\ell^2<0$. We may assume $(K+D)\cdot\ell\geq 0$, otherwise we are done. In particular, $A\neq\ell$. Let σ be the contraction of ℓ. By 1.4 $(\sigma(X),\sigma_*D,\sigma_*E)$ satisfies (1)-(3). Let d' be as in the proof of 1.4. We have $\sigma_*A\cdot(K_{\sigma(X)}+\sigma_*E)=A\cdot(K+E-d'\ell)\leq A\cdot(K+E)<0$. and $\kappa(K_{\sigma(X)}+\sigma_*D+n\sigma_*A)\geq\kappa(K_X+D+nA)\geq 0$ for some $n\geq 0$, so we replace (X,D,E,A) with $(\sigma(X),\sigma_*D,\sigma_*E,\sigma_*A)$ and we proceed by induction. $\square$

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