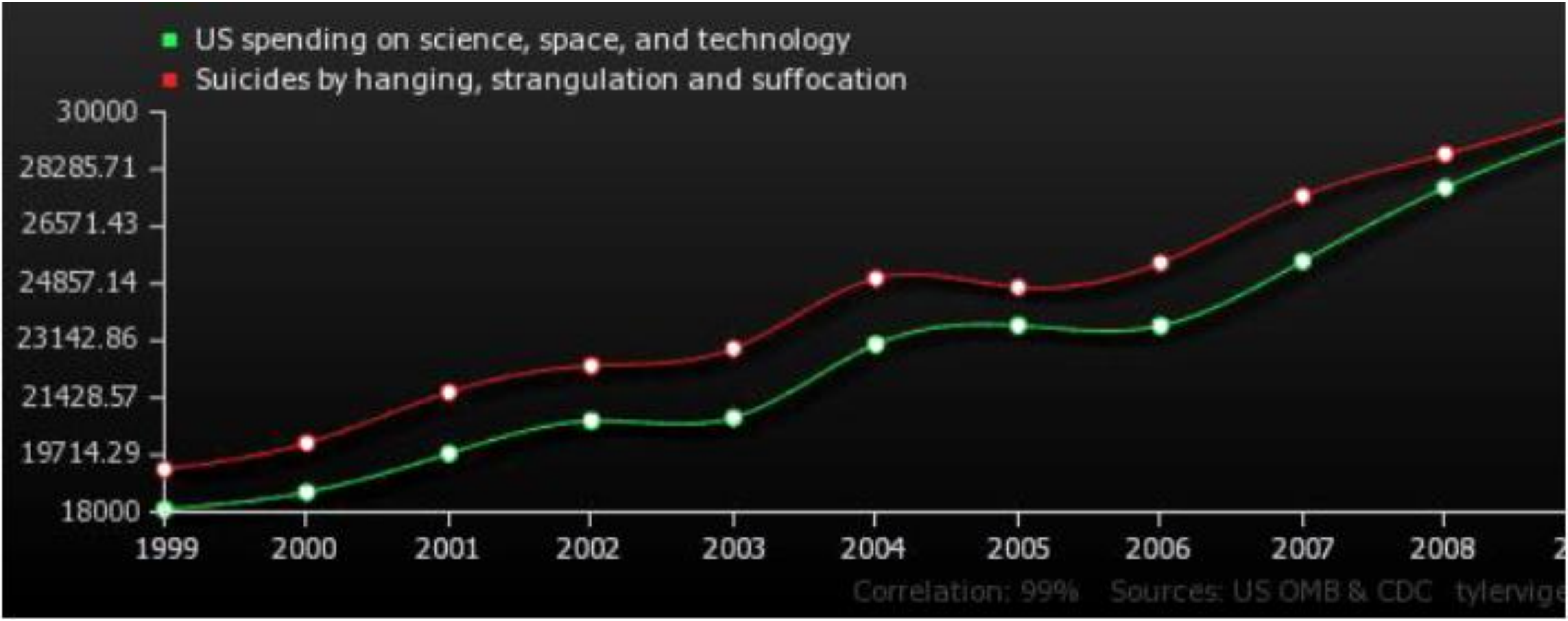
Nonlinear eigenvalue problems and *PT*-symmetric quantum mechanics



Nonlinear eigenvalue problems and *PT*-symmetric quantum mechanics

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Washington University in St. Louis

Perspectives of Modern Complex Analysis

Będlewo, 21-25 July 2014

PT-symmetric quantum theory is an extension of QM into the complex plane

$$H = H^{\dagger}$$ ($\dagger$ means transpose + complex conjugate)

- guarantees real energy and probability-conserving time evolution
- but … is a **mathematical** axiom and not a **physical** axiom of quantum mechanics

Dirac Hermiticity can be generalized...

The idea: Replace Dirac Hermiticity by the *physical* and *weaker* condition of *PT* symmetry

P = parity

T = time reversal

(physical because *P* and *T* are elements of the Lorentz group)

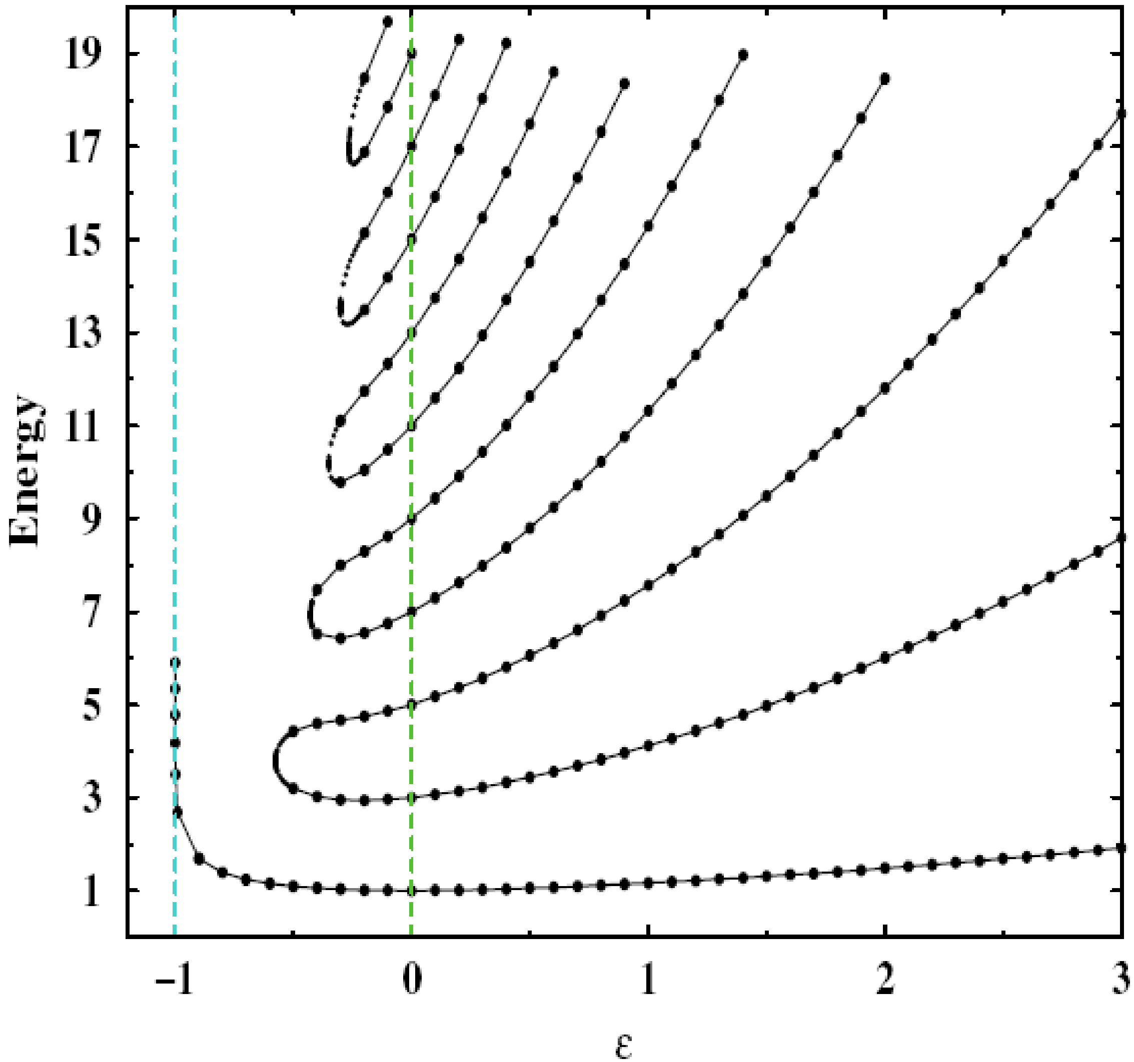
Example:

$$H = p^2 + ix^3$$

This Hamiltonian has ***PT* symmetry!**

Class of ***PT***-symmetric Hamiltonians discovered in 1998:

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$



Transition

at ε = 0

CMB and S. Boettcher
Physical Review Letters **80**, 5243 (1998)

Some of my work on *PT* symmetry

- CMB and S. Boettcher, *Physical Review Letters* **80**, 5243 (1998)
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PT papers (2008-2010)

- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, *Phyical Review Letters* **100**, 103904 (2008)
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, *Physical Review Letters* **100**, 030402 (2008)
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- P. Mannheim and J. O'Brien, *Physical Review Letters* **106**, 121101 (2011)
- L. Feng, M. Ayache, J. Huang, Y. Xu, M. Lu, Y. Chen, Y. Fainman, A. Scherer, *Science* **333**, 729 (2011)

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- L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, A. Scherer, *Nature Materials* **12**, 108-113 (2013)
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- Y. Sun, W. Tan, H.-Q. Li, J. Li, H. Chen, *Physical Review Letters* **112**, 143903 (2014)
- B. Peng, Ş. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, CMB, and L. Yang, *Nature Physics* **10,** 394 (2014)
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- J. M. Lee, S. Factor, Z. Lin, I. Vitebskiy, F. Ellis, and T. Kottos, *Physical Review Letters* **112**, 253902 (2014)
- M. Brandstetter, M. Liertzer, C. Deutsch, P. Klang, J. Schöberl, H. E. Türeci, G. Strasser, K. Unterrainer, and S. Rotter, *Nature Communications* **5**, 4034 (2014)
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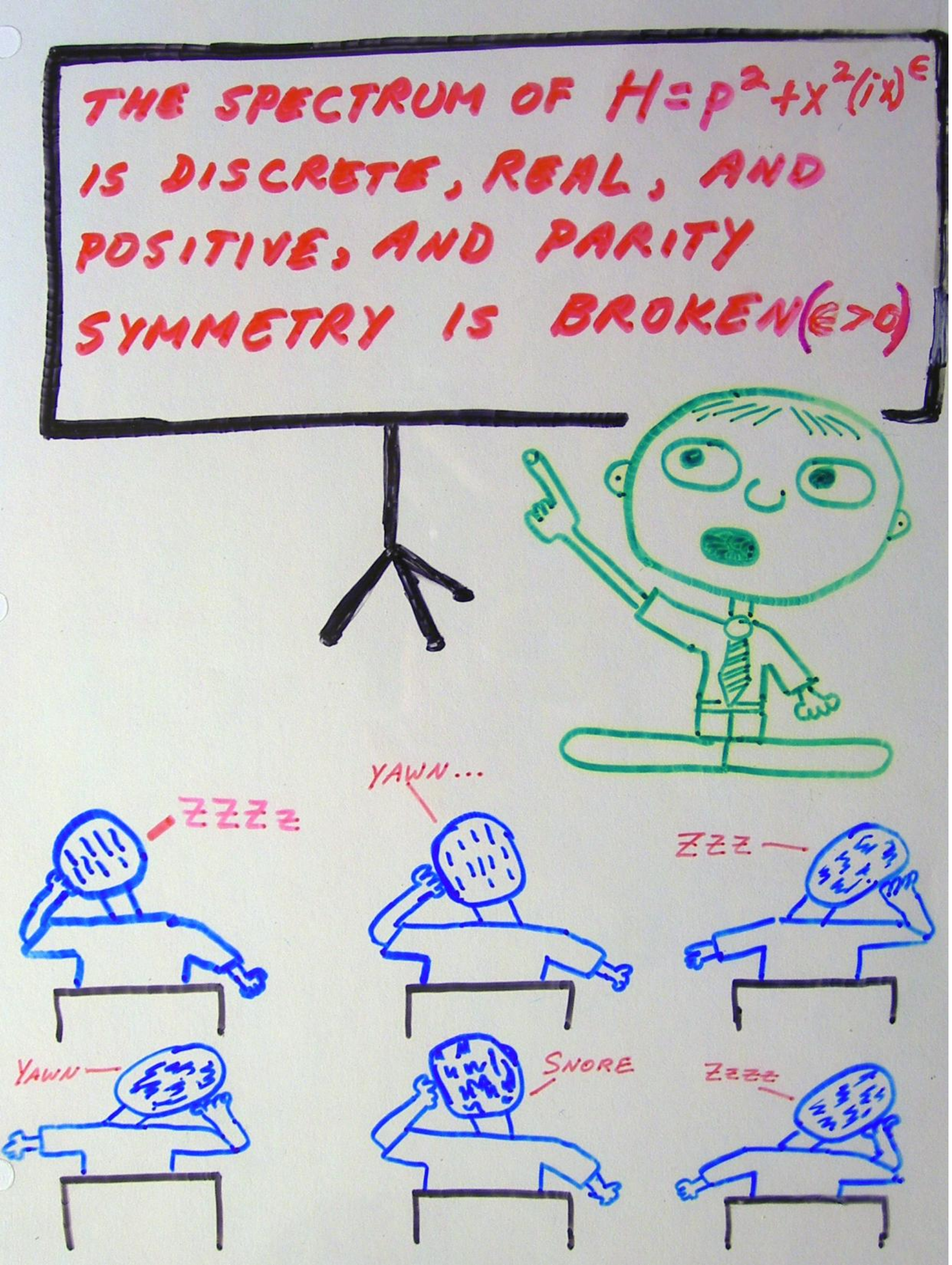
- L. Chang, X. Jiang, S. Hua, C. Yang, J. Wen, L. Jiang, G. Li, G. Wang, and M. Xiao, *Nature Photonics* **8**, 524 (2014)

Developments in *PT* Quantum Mechanics

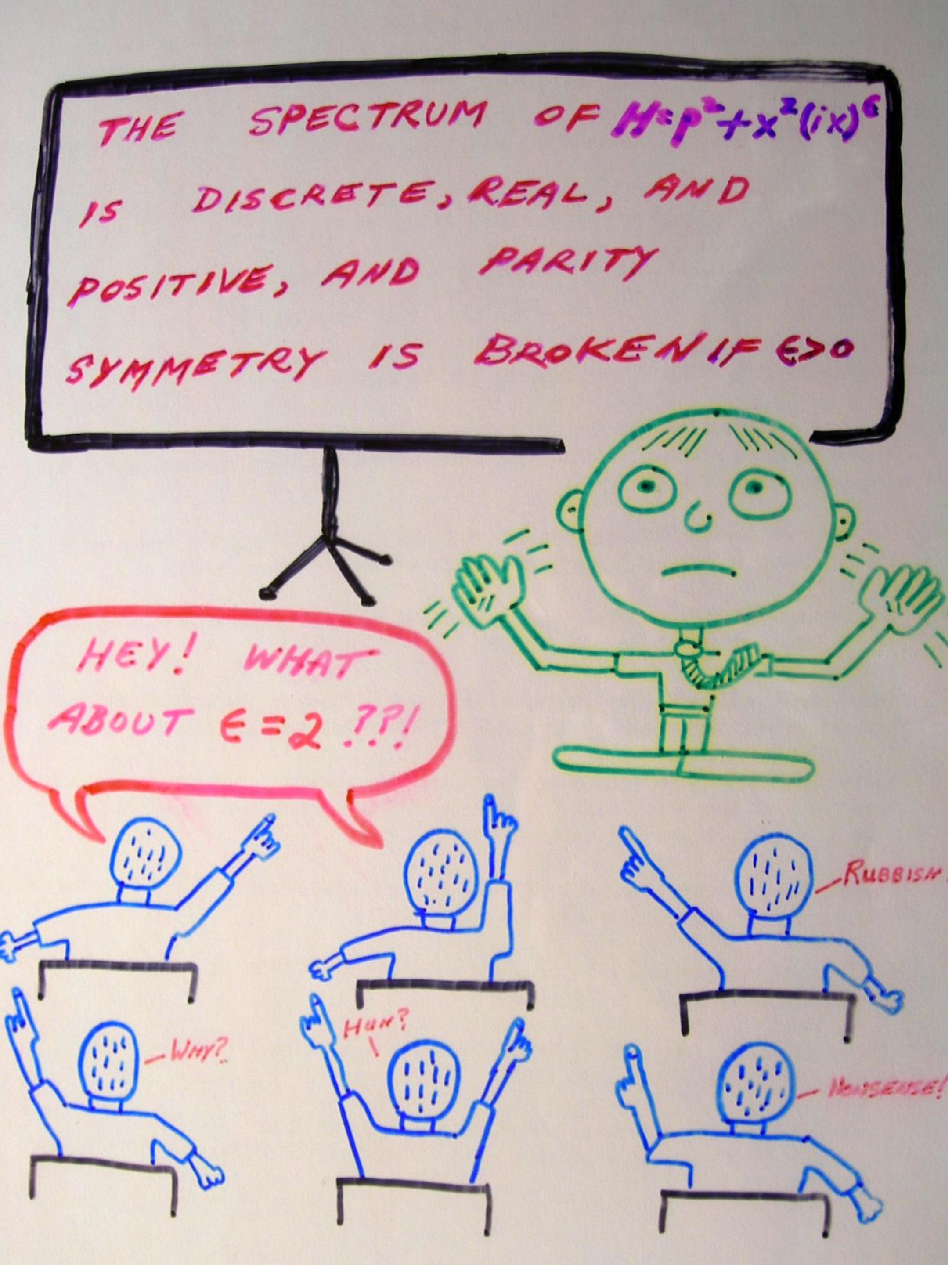
(Since its 'official' beginning in 1998)

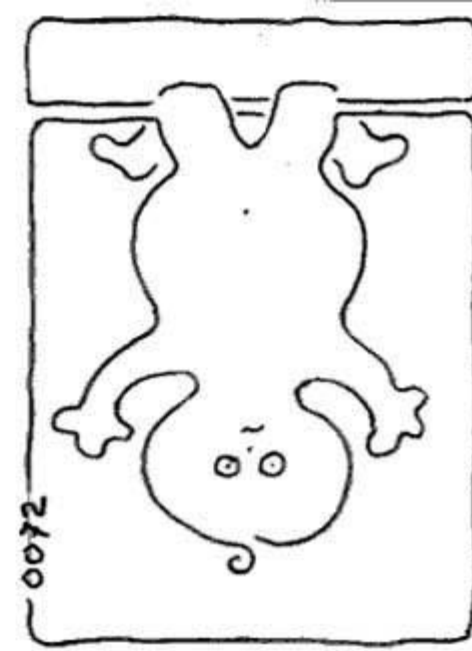
- Nearly 20 international conferences – *FOUR this summer!*
- Nearly 2000 published papers
- Website: "***PT* symmeter**" <http://ptsymmetry.net>
- Many many ***many*** experimental results in last four years!



Rigorous proof of real eigenvalues:

"ODE/IM Correspondence"
P. Dorey, C. Dunning, and R. Tateo,
J. Phys. A **40**, R205 (2007)





Upside-down potential with real positive eigenvalues?!

$$V(x) = -x^4$$

Z. Ahmed, CMB, and M. V. Berry,
J. Phys. A: Math. Gen. **38**, L627 (2005)
[arXiv: quant-ph/0508117]

CMB, D. C. Brody, J.-H. Chen, H. F. Jones,
K. A. Milton, and M. C. Ogilvie,
Phys. Rev. D **74**, 025016 (2006)
[arXiv: hep-th/0605066]

Hermitian Hamiltonians: BORING!

Eigenvalues are always real – nothing interesting happens



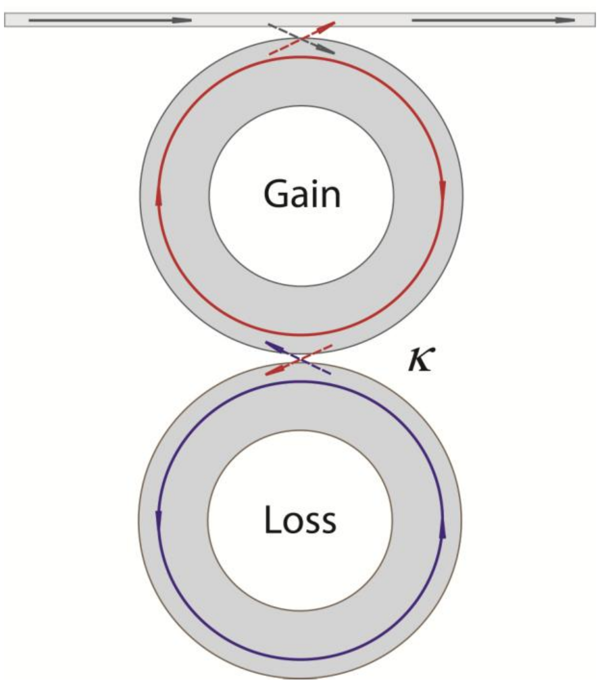
PT-symmetric Hamiltonians:
ASTONISHING!

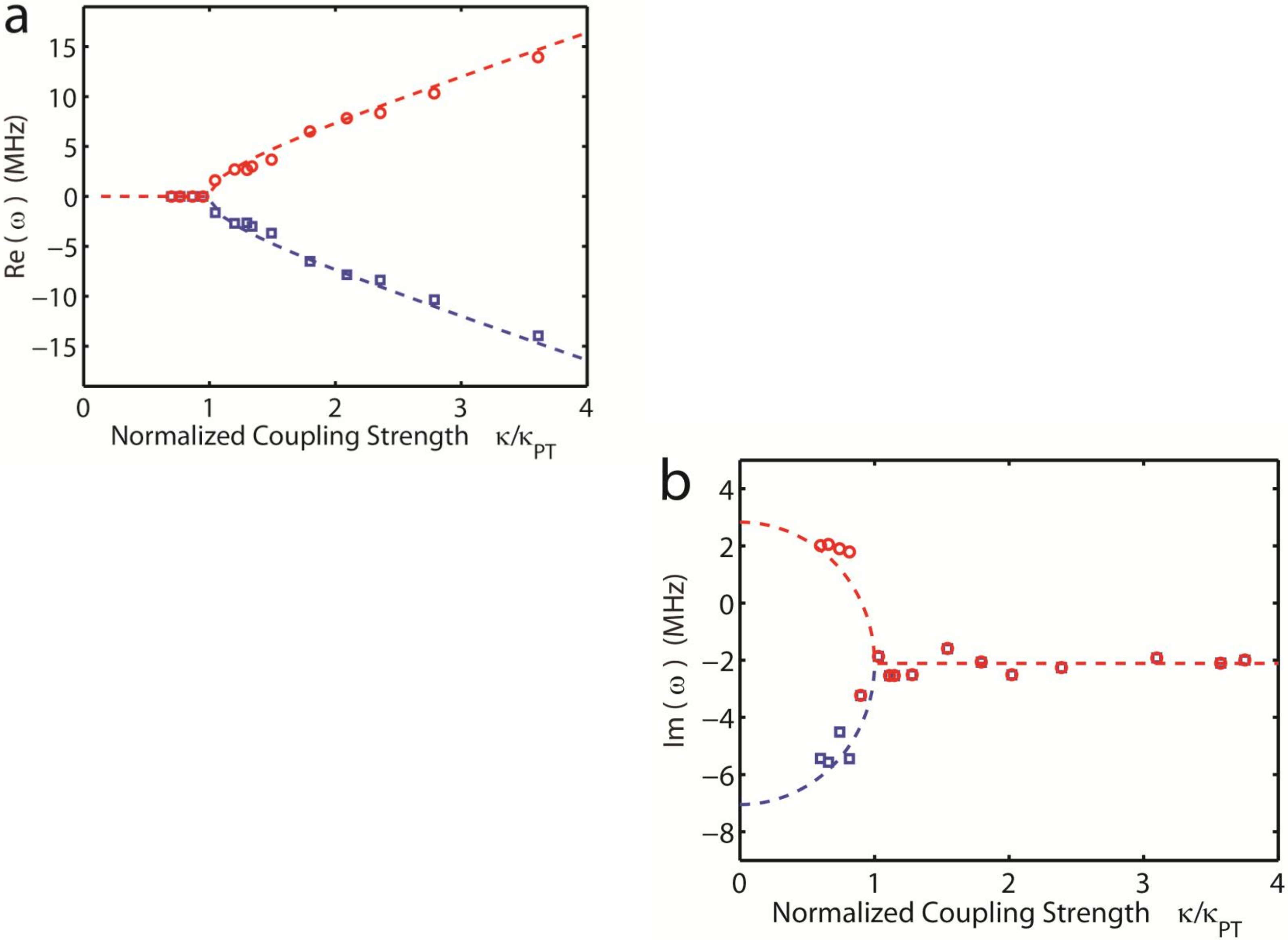
Transition between parametric regions of broken and unbroken ***PT*** symmetry...
Can be observed experimentally!

A recent experiment…

"Nonreciprocal light transmission in parity-time-symmetric whispering-gallery microcavities," B. Peng, S. K. Ozdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, CMB, L. Yang, *Nature Physics* **10**, 394 (2014) [arXiv: 1308.4564]

"Twofold Transition in **PT**-Symmetric Coupled Oscillators," CMB, M. Gianfreda, B. Peng, S. K. Ozdemir, and L. Yang *Physical Review A* **88**, 062111 (2013) [arXiv:hep-th/1305.7107]




a
Re (ω) (MHz)
15
10
5
0
−5
−10
−15
0
1
2
3
4
Normalized Coupling Strength κ/κPT
b
Im (ω) (MHz)
4
2
0
−2
−4
−6
−8
0
1
2
3
4
Normalized Coupling Strength κ/κPT

At a physical level, *PT*-symmetric quantum systems are intermediate between closed and open systems.

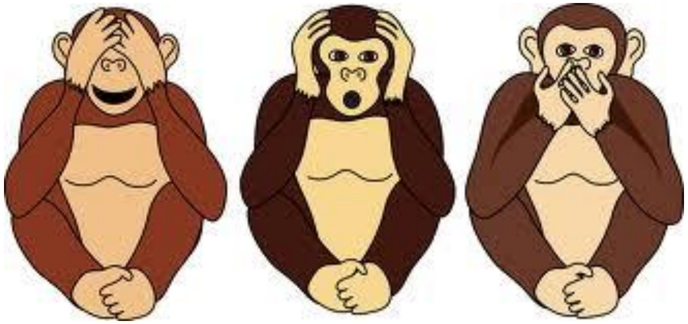
Hermitian H

PT-symmetric H

Non-Hermitian H

_PT_ quantum mechanics is fun! You can re-visit things you already know about traditional Hermitian quantum mechanics.

Three examples:



1. "Ghost Busting: *PT*-Symmetric Interpretation of the Lee Model,"
CMB, S. Brandt, J.-H. Chen, and Q. Wang, *Phys. Rev. D* 71, 025014 (2005) **[arXiv: hep-th/0411064]**

2. "No-ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model,"
CMB and P. Mannheim, *Phys. Rev. Lett.* 100, 110402 (2008) **[arXiv: hep-th/0706.0207]**

3. "*PT*-Symmetric Interpretation of Double-Scaling"
CMB, M. Moshe, and S. Sarkar , *J. Phys. A: Math. Theor.* 46, 102002 (2013) **[arXiv: hep-th/1206.4943]**
and
"Double-Scaling Limit of the O(N)-Symmetric Anharmonic Oscillator"
CMB and S. Sarkar, *J. Phys. A: Math. Theor.* 46, 442001 (2013) **[arXiv: hep-th/1307.4348]**

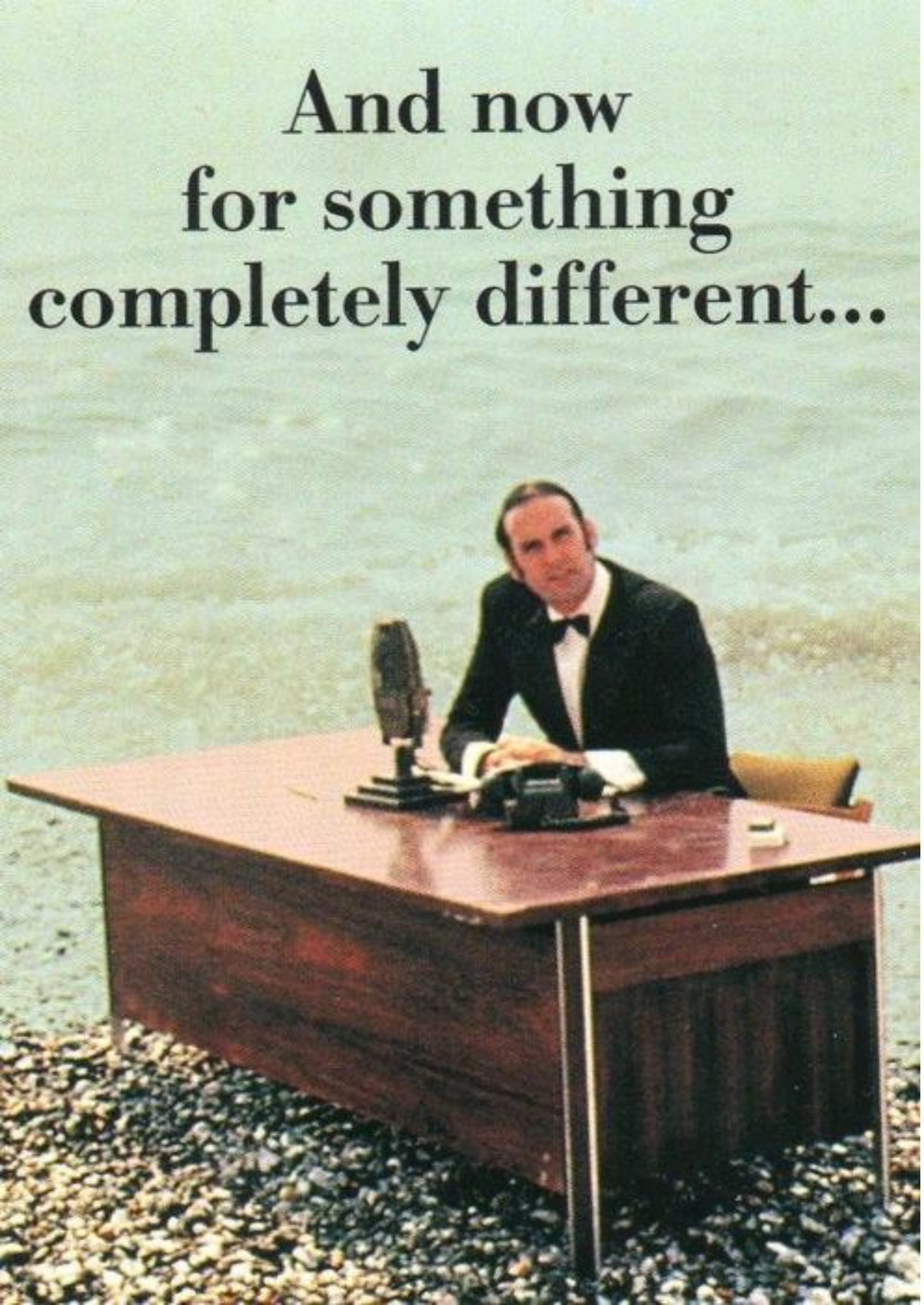
Three current *PT* research problems:

(1) Conformal Liouville quantum field theory

Interaction term: $\mathbf{exp}(i\phi)$, ***S*** duality

(2) Electromagnetic back-reaction force

(3) Nonlinear systems and nucleotide (DNA) chemical simulations



Nonlinear eigenvalue problems...

Outline of talk

(1) Beginning

(2) Middle

(3) End

Linear eigenvalue problems...

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \qquad \psi(\pm\infty) = 0$$

Difficult because this is a *global* (not a local) problem with widely separated boundary conditions!

Example of a difficult global problem...

Difficult problem with widely separated boundary conditions

Problem even with not so distant boundary conditions

For linear problems WKB gives a good approximation for large eigenvalues

$$\int_{x_1}^{x_2} dx \sqrt{E_n - V(x)} \sim (n + 1/2)\pi \quad (n \to \infty)$$

Example 1: harmonic oscillator

$$V(x) = x^2$$

$$E_n \sim n \quad (n \to \infty)$$

Example 2: anharmonic oscillator

$$V(x) = x^4$$

$$E_n \sim B n^{4/3} \quad (n \to \infty) \qquad B = \left[\frac{3\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}\right]^{4/3}$$

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$

WKB approximation works for *PT* as well:

$$E_n \sim \left[\frac{\Gamma\left(\frac{3}{2}+\frac{1}{\varepsilon+2}\right)\sqrt{\pi}\,n}{\sin\left(\frac{\pi}{\varepsilon+2}\right)\Gamma\left(1+\frac{1}{\varepsilon+2}\right)}\right]^{\frac{2\varepsilon+4}{\varepsilon+4}} \qquad (n\to\infty)$$

Hyperasymptotics

Leading asymptotic behavior for large positive *x*

$$\psi(x) \sim C[V(x)-E]^{-1/4} \exp\left[\int^x ds \sqrt{V(s)-E}\right] \quad (x \to \infty)$$

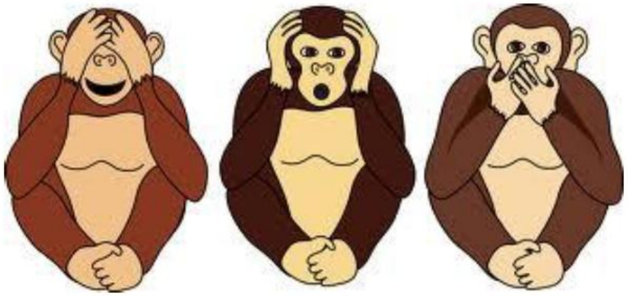
NOTE: Only ONE arbitrary constant!

Second arbitrary constant is invisible because it is contained in the *subdominant* solution:

$$\psi(x) \sim D[V(x)-E]^{-1/4} \exp\left[-\int^x ds \sqrt{V(s)-E}\right] \quad (x \to \infty)$$

This is the *physical* solution. *Unstable* under small changes in *E*.

Three characteristic properties of solutions



(1) **Oscillatory** in *classically allowed* region (*n*th eigenfunction has *n* nodes)

(2) **Monotone decay** in *classically forbidden* region

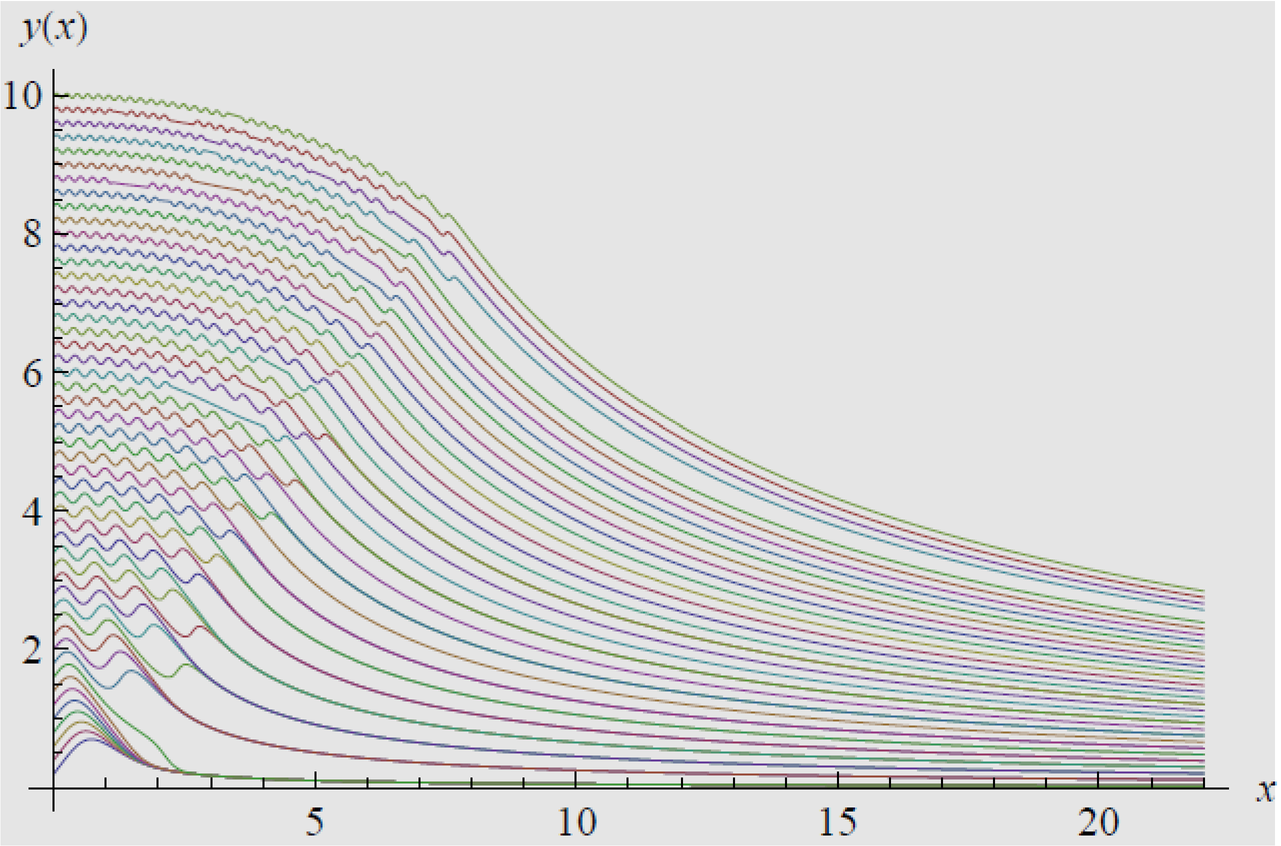
(3) **Transition** at the boundary (*turning point*)

Nonlinear toy eigenvalue problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a$$

Some references:

[1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978), chap. 4.
[2] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Phys. Rev. Lett. **104**, 061601 (2010).
[3] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Ann. Phys. **325**, 2332-2362 (2010).
[4] J. Gair, N. Yunes, and C. M. Bender, J. Math. Phys. **53**, 032503 (2012).



Solutions for 50 initial conditions

Note: (1) oscillation (2) monotone decay (3) transition

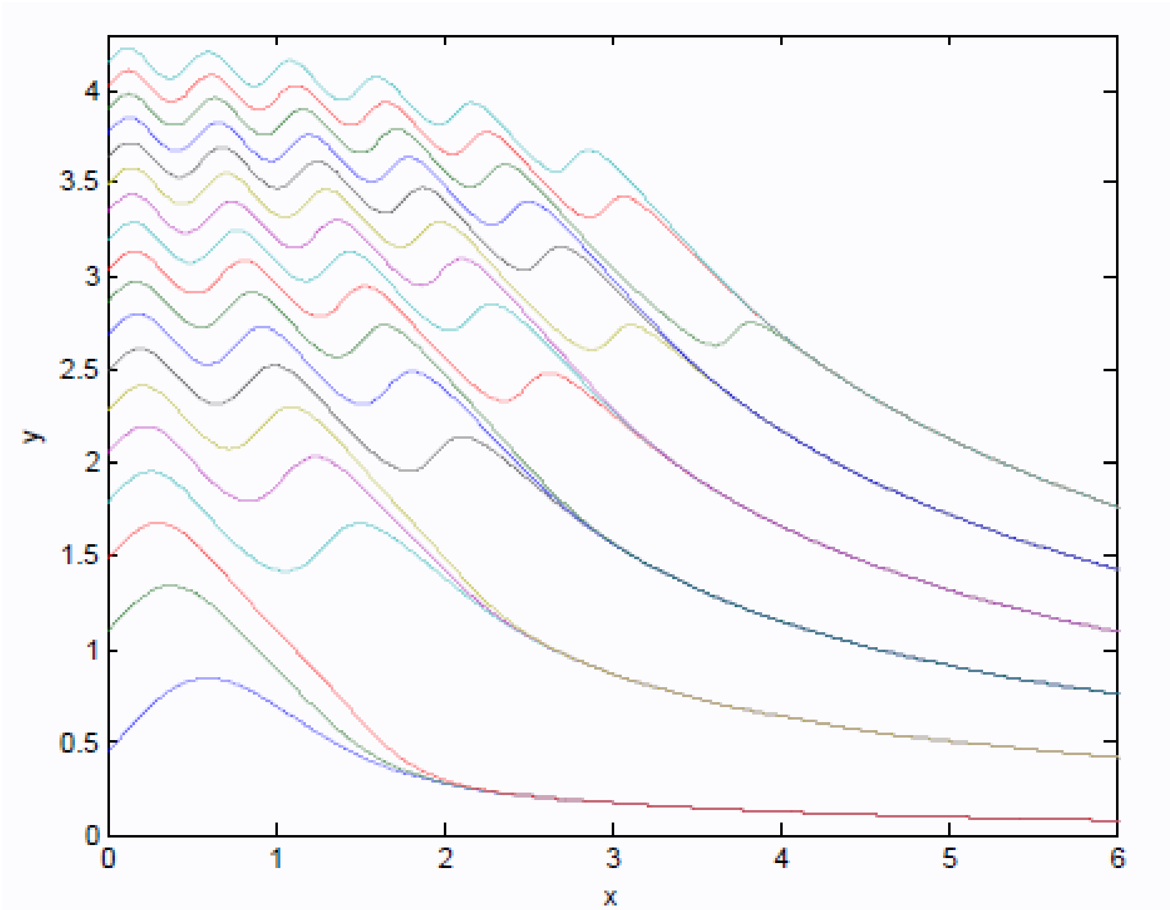
Asymptotic behavior for large *x*

Solution behaves like:

$$y(x) \sim \frac{m + 1/2}{x}$$

where *m* = 0, 1, 2, 3, ... is an integer

There's a ***big*** problem here...



m **= 10**
m **= 8**
m **= 6**
m **= 4**
m **= 2**
m **= 0**

Where are the odd-*m* solutions?!?

Furthermore, no arbitrary constant appears in the asymptotic behavior!!

Where is the arbitrary constant?!?

Higher-order asymptotic behavior for large *x* still contains no arbitrary constant!

$$y(x) \sim \frac{m+1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty)$$

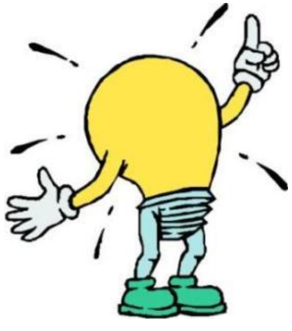
$$\begin{aligned}
c_1 &= \frac{(-1)^m}{\pi}(m+1/2), \\
c_2 &= \frac{3}{\pi^2}(m+1/2), \\
c_3 &= (-1)^m \left[\frac{(m+1/2)^3}{6\pi} + \frac{15(m+1/2)}{\pi^3}\right], \\
c_4 &= \frac{8(m+1/2)^3}{3\pi^2} + \frac{105(m+1/2)}{\pi^4}, \\
c_5 &= (-1)^m \left[\frac{3(m+1/2)^5}{40\pi} + \frac{36(m+1/2)^3}{\pi^3} + \frac{945(m+1/2)}{\pi^5}\right], \\
c_6 &= \frac{38(m+1/2)^5}{15\pi^2} + \frac{498(m+1/2)^3}{\pi^4} + \frac{10395(m+1/2)}{\pi^6}.
\end{aligned}$$

Hyperasymptotic analysis

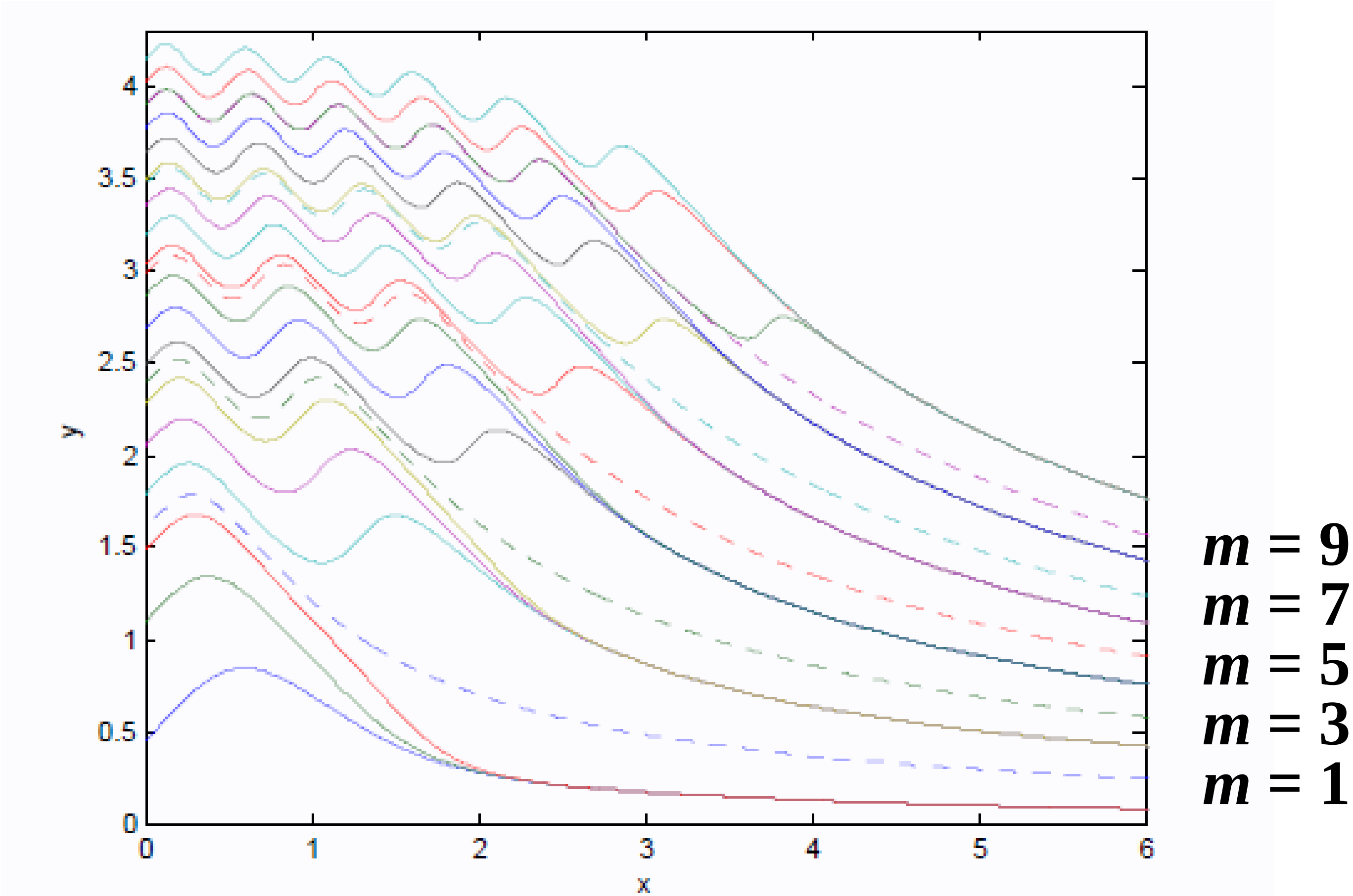
$$Y(x) \equiv y_1(x) - y_2(x)$$

$$\begin{aligned} Y'(x) &= \cos[\pi x y_1(x)] - \cos[\pi x y_2(x)] \\ &= -2\sin\left[\tfrac{1}{2}\pi x y_1(x) + \tfrac{1}{2}\pi x y_2(x)\right]\sin\left[\tfrac{1}{2}\pi x y_1(x) - \tfrac{1}{2}\pi x y_2(x)\right] \\ &\sim -2\sin\left[\pi\left(m+\tfrac{1}{2}\right)\right]\sin\left[\tfrac{1}{2}\pi x Y(x)\right] \quad (x\to\infty) \\ &\sim -(-1)^m \pi x Y(x) \quad (x\to\infty). \end{aligned}$$

$$Y(x) \sim K\exp\left[-(-1)^m \pi x^2\right] \quad (x\to\infty)$$



Aha! *K* is the arbitrary constant!
Odd *m* unstable, even *m* stable



$$y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots\}$$

Eigenvalues correspond to odd *m* ...

Separatrices **(*unstable*) begin at eigenvalues**

We calculated up to *m*=500,001

Let $m = 2n - 1$

We determined that for large *n* the *n*th eigenvalue grows like the *square root* of *n* times a constant *A*, and we used Richardson extrapolation to show that

A= 1.7817974363...

and then we <u>guessed</u> *A*!!!

A surprising result:

$$a_n \sim A\sqrt{n} \quad (n \to \infty)$$

$$A = 2^{5/6}$$

This is a nontrivial problem...

Another *nontrivial* problem

...and we found the analytic solution!

Some scaling changes of variable:

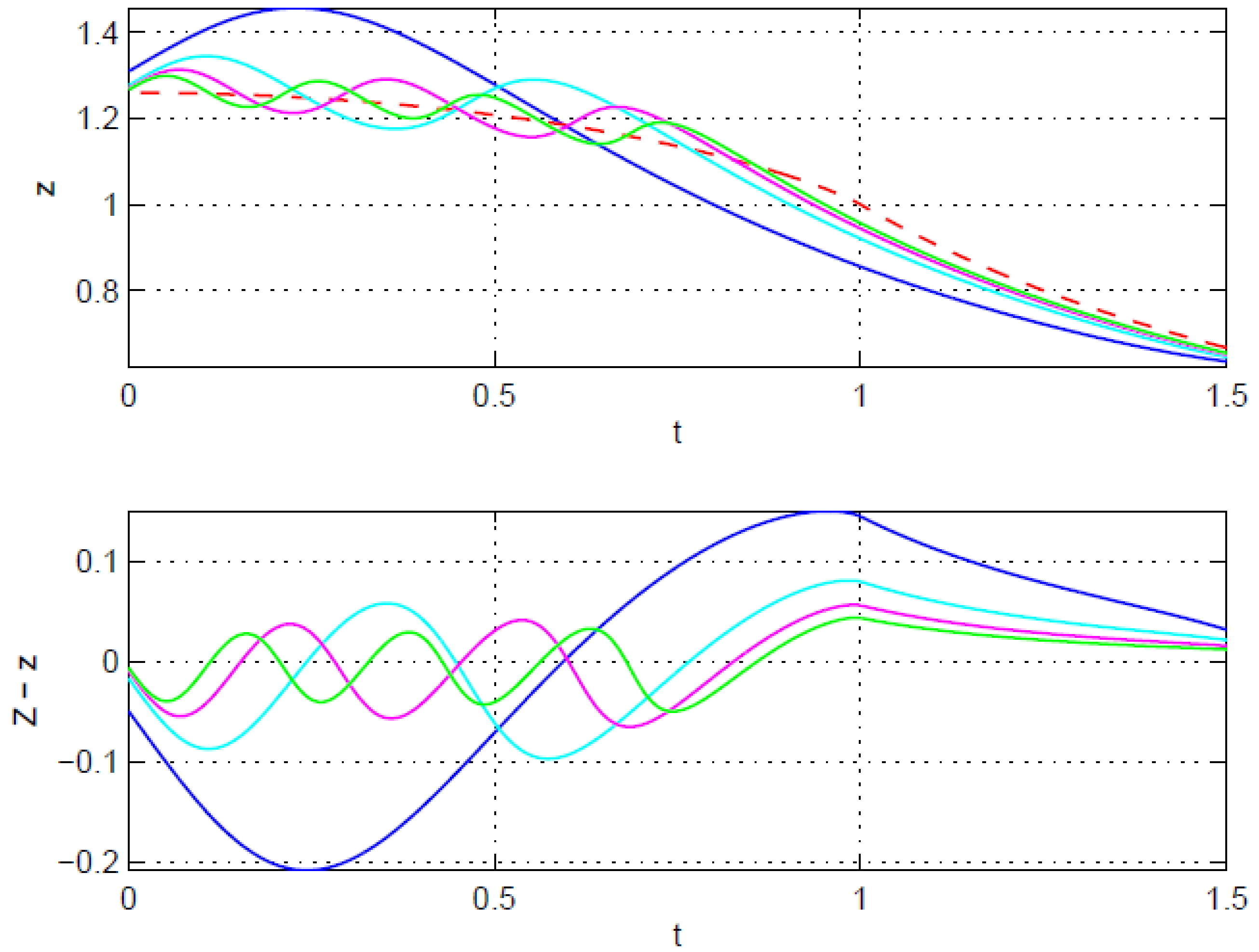
$$m = 2n - 1$$

$$x = \sqrt{2n - 1/2}\,t, \quad y(x) = \sqrt{2n - 1/2}\,z(t)$$

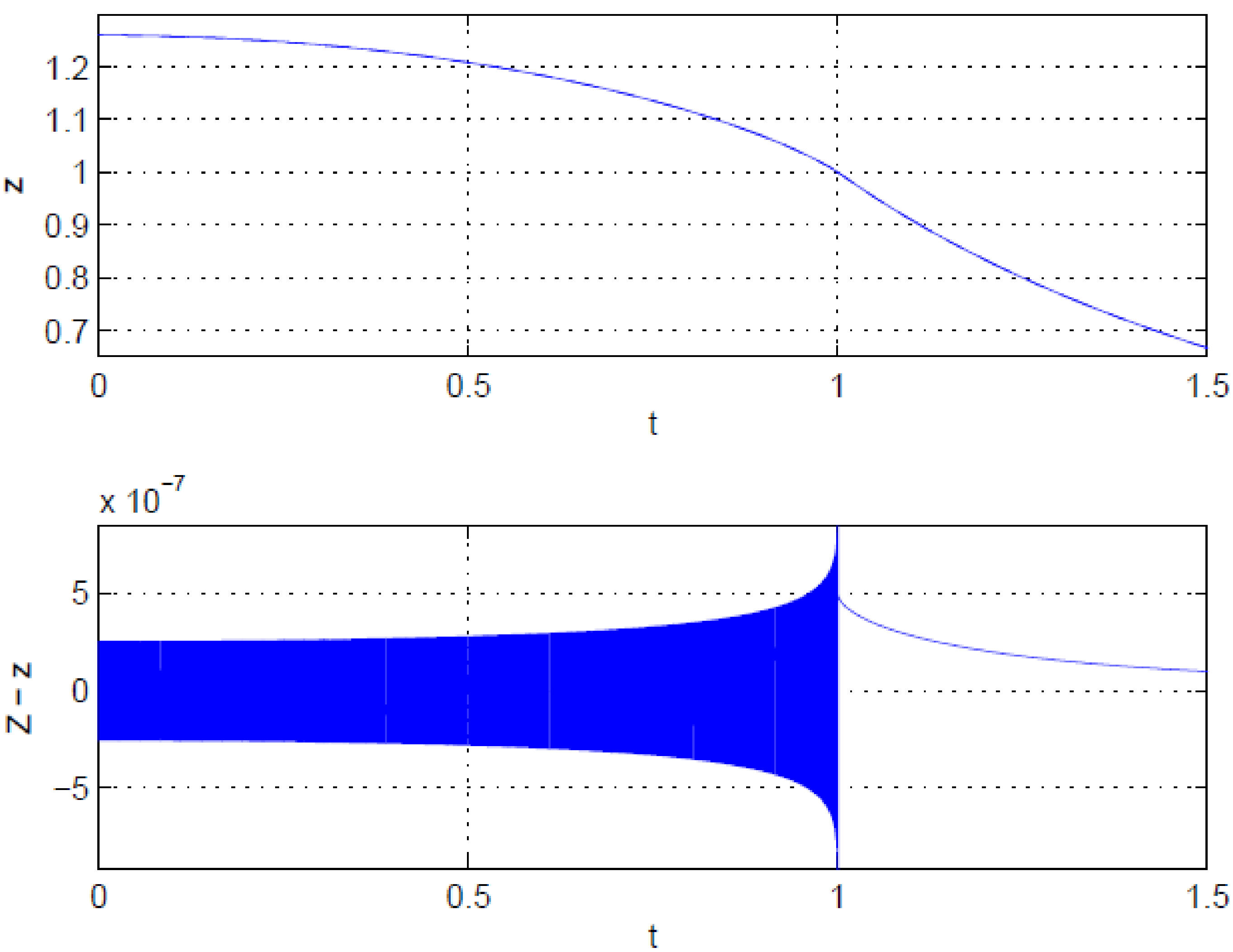
$$\lambda = (2n - 1/2)\pi$$

$$z'(t) = \cos[\lambda t z(t)]$$

For large λ, the eigenfunctions (*separatrix curves*) approach a *limiting curve*, which we call *Z*(*t*)...



First four separatrix curves

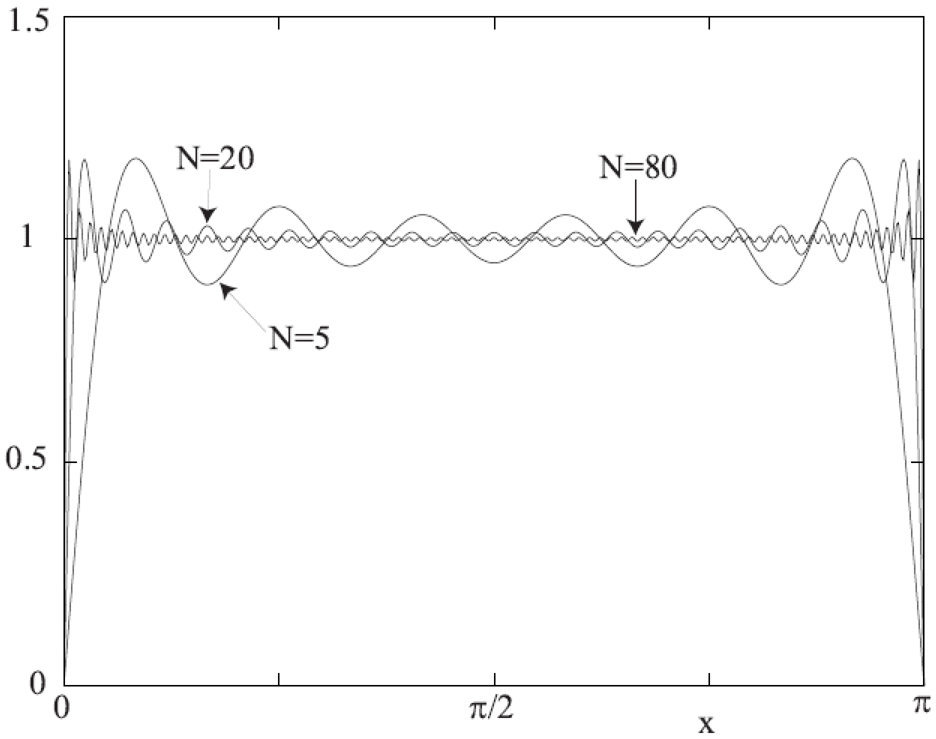


***m* = 500,001 separatrix curve**

Convergence to *Z* is like convergence of Fourier series

$$f(x) = 1$$

$$S_{2N+1}(x) = \frac{4}{\pi}\sum_{n=0}^{N}\frac{\sin[(2n+1)x]}{2n+1}$$



Analytic calculation of the constant *A*

Multiply $z'(t) = \cos[\lambda t z(t)]$ by $z(t) + t z'(t)$

Integrate from 0 to t and use double-angle formula for cosines:

$$[z(t)]^2 - [z(0)]^2 + t^2/2 + \eta(t) = \mathrm{O}(1/\lambda) \quad (\lambda \to \infty),$$

$$\eta(t) = \int_0^t ds\, s \cos[2\lambda s z(s)]$$

Problem is to calculate η(*t*)

η(*t*) is just one of a doubly-infinite set of moments defined as:

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Note that $\eta(t) = A_{2,0}(t)$

For large λ these moments satisfy the linear difference equation

$$A_{n,k}(t) = -\tfrac{1}{2}A_{n-1,k+1}(t) - \tfrac{1}{2}A_{n+1,k+1}(t) \quad (n \geq 2)$$

To get this result we multiply the integrand in η by 1:

$$\frac{z(s) + sz'(s)}{z(s)} - \frac{sz'(s)}{z(s)}$$

The moments are associated with a semi-infinite linear one-dimensional random-walk in which random walkers become static as they reach n=1

The random-walk analysis goes as follows: We let $\alpha_{n,k}$ be the numerical coefficient of the integrals in $A_{n,k}$. The initial condition is $\alpha_{n,0} = 0$ if $n \neq 2$ and $\alpha_{2,0} = 1$. Integration by parts gives the relations between the coefficients:

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0,$$

$$2\alpha_{2,k} + \alpha_{3,k-1} = 0,$$

$$2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).$$

$$\alpha_{n,k} = \frac{(-1)^n (n-1)k!}{2^k (k/2 + n/2)!(k/2 - n/2 + 1)!}.$$

$$\eta(t) = -\int_0^t ds\, z(s) z'(s) - \frac{1}{2\sqrt{\pi}} \sum_{p=1}^{\infty} \frac{\Gamma(p+1/2)}{(p+1)!} \int_0^t ds\, z'(s) \frac{s^{2p+2}}{[z(s)]^{2p+1}}$$

No explicit reference to λ, so we pass to limit of large λ.
In this limit the $z(t)$ oscillates rapidly and
approaches the smooth and non-oscillatory function $Z(t)$.

We get an integral equation satisfied by $Z(t)$:

$$[Z(t)]^2 - [Z(0)]^2 + \tfrac{1}{2}t^2 - \int_0^t ds\, Z(s) Z'(s) + \int_0^t ds\, Z(s) Z'(s) \sqrt{1 - s^2/[Z(s)]^2} = 0.$$

Differentiate integral equation with respect to *t* :

$$Z(t)Z'(t) + t + Z'(t)\sqrt{[Z(t)]^2 - t^2} = 0$$

Let $Z(t) = t\,G(t)$

$$\frac{K}{t^3} = \left(1 + 3[G(t)]^2\right)\left(G(t) + \sqrt{[G(t)]^2 - 1}\right)\frac{\sqrt{[G(t)]^2 - 1} - 2G(t)}{\sqrt{[G(t)]^2 - 1} + 2G(t)}$$

$G(1) = 1$ gives $K = -4$

We thus get $Z(0) = 2^{1/3}$

and from this we get $A = 2^{5/6}$

CMB, A. Fring, and J. Komijani
J. Phys. A: Math. Theor. **47**, 235204 (2014)
[arXiv: math-ph/1401.6161]

Possible connection with the *power series constant P*???

W. K. Hayman, *Research Problems in Function theory* [Athlone Press (University of London), London, 1967]

J. Clunie and P. Erdös, Proc. Roy. Irish Acad. **65**, 113 (1967).
J. D. Buckholtz, Michigan Math. J. **15**, 481 (1968).

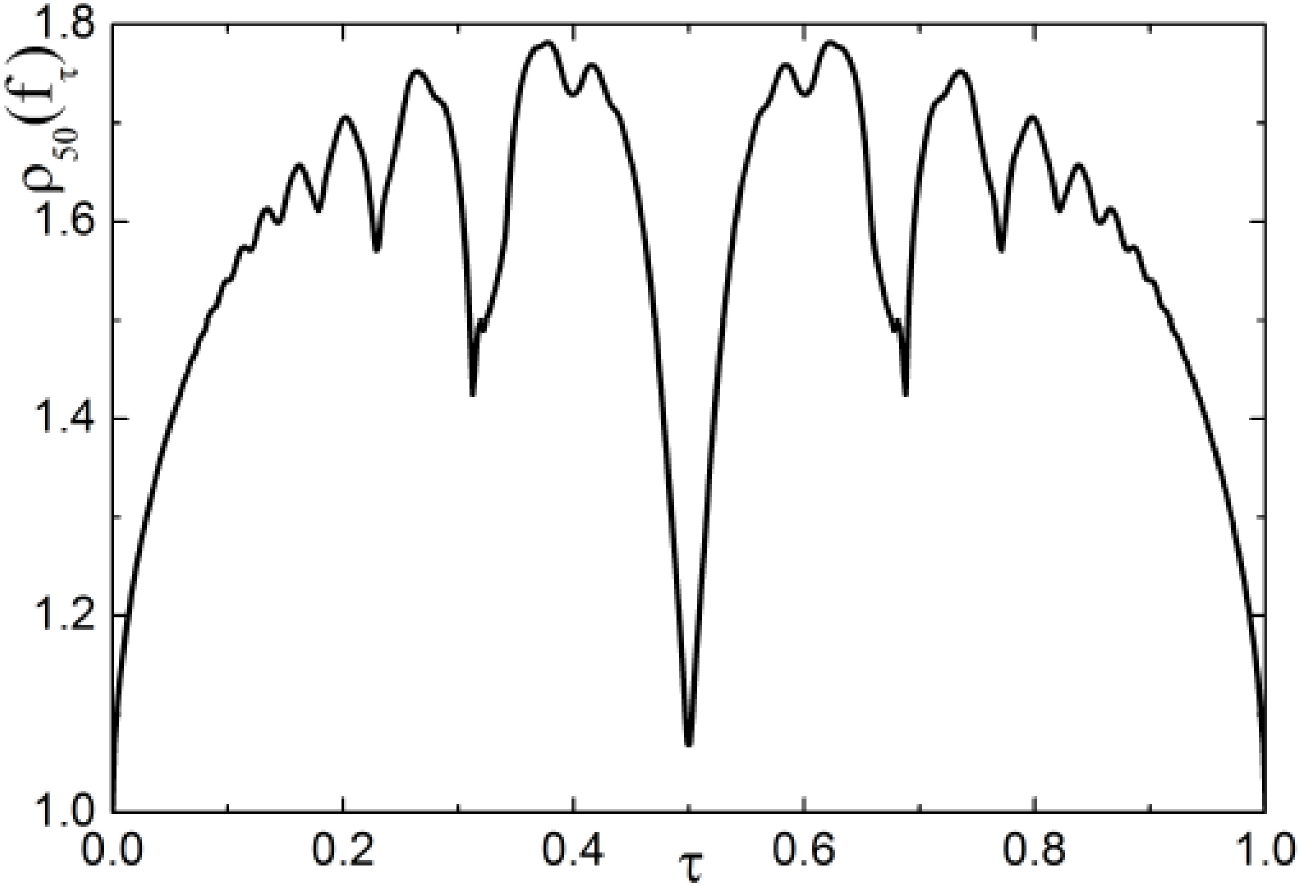
$$1 \le P \le 2$$

$$\sqrt{2} \le P \le 2$$

$$1.7 \le P \le 12^{1/4}$$

$$1.7818 \le P \le 1.82$$

$$f_\tau(z) = \sum_{k=0}^{\infty} \exp[i\pi\tau(k^2+k)]\ z^k$$



The maximum values are $\rho_{50}(f_{0.3780}) = \rho_{50}(f_{0.8780}) \approx 1.7818$, which coincide with the best known lower bound for P up to the precision of the computation.

Two ***nontrivial*** second-order nonlinear eigenvalue problems

(1) First Painleve transcendent

$$y''(x) = [y(x)]^2 + x.$$

Solution $y(x)$ must *choose* between two possible asymptotic behaviors as x gets large and negative:

$$y(x) \sim \pm\sqrt{-x} \quad (x \to -\infty)$$

Example of a *difficult* choice ...

Two possible asymptotic behaviors

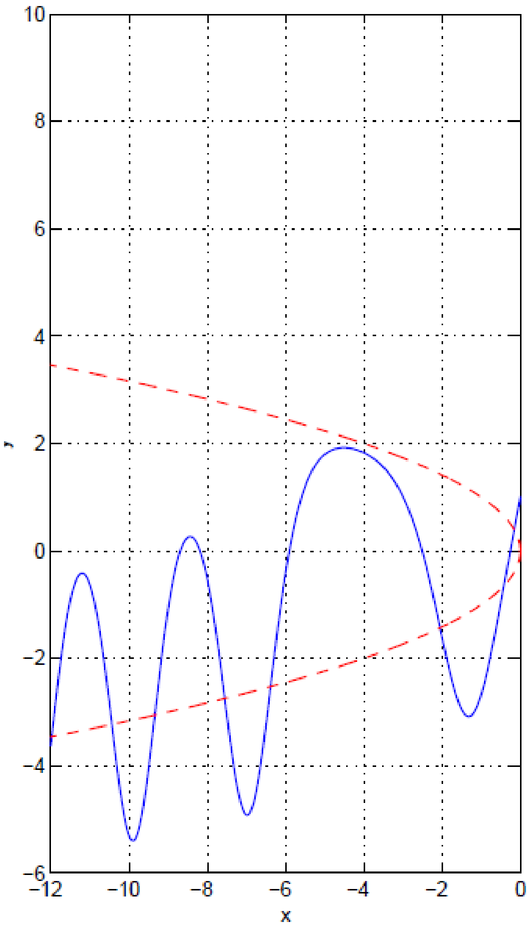
Lower branch is ***stable***:

$$y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos\left[\tfrac{4}{5}\sqrt{2}(-x)^{5/4} + d\right] \quad (x \to -\infty)$$

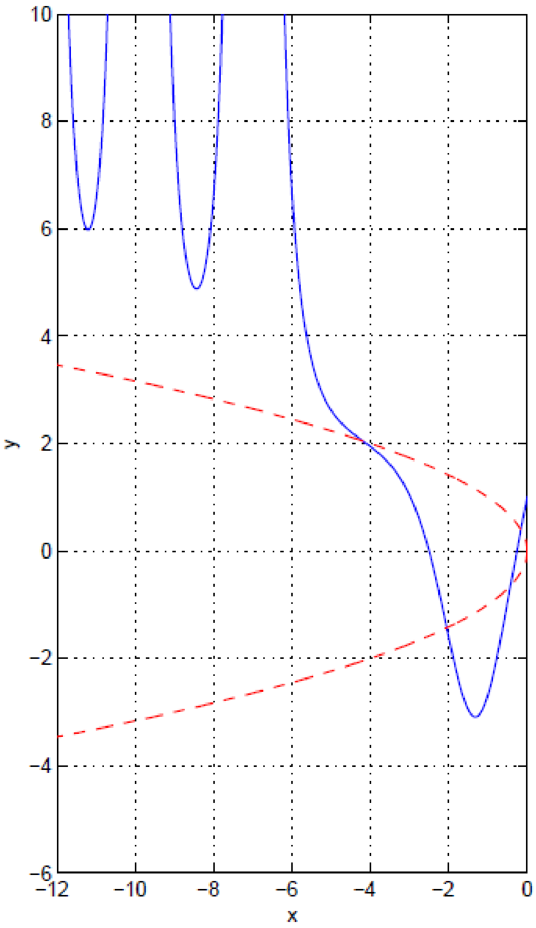
Upper branch is ***unstable***:

$$y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp\left[\pm\tfrac{4}{5}\sqrt{2}(-x)^{5/4}\right] \quad (x \to -\infty)$$

Two possible kinds of solutions:



Stable



Unstable

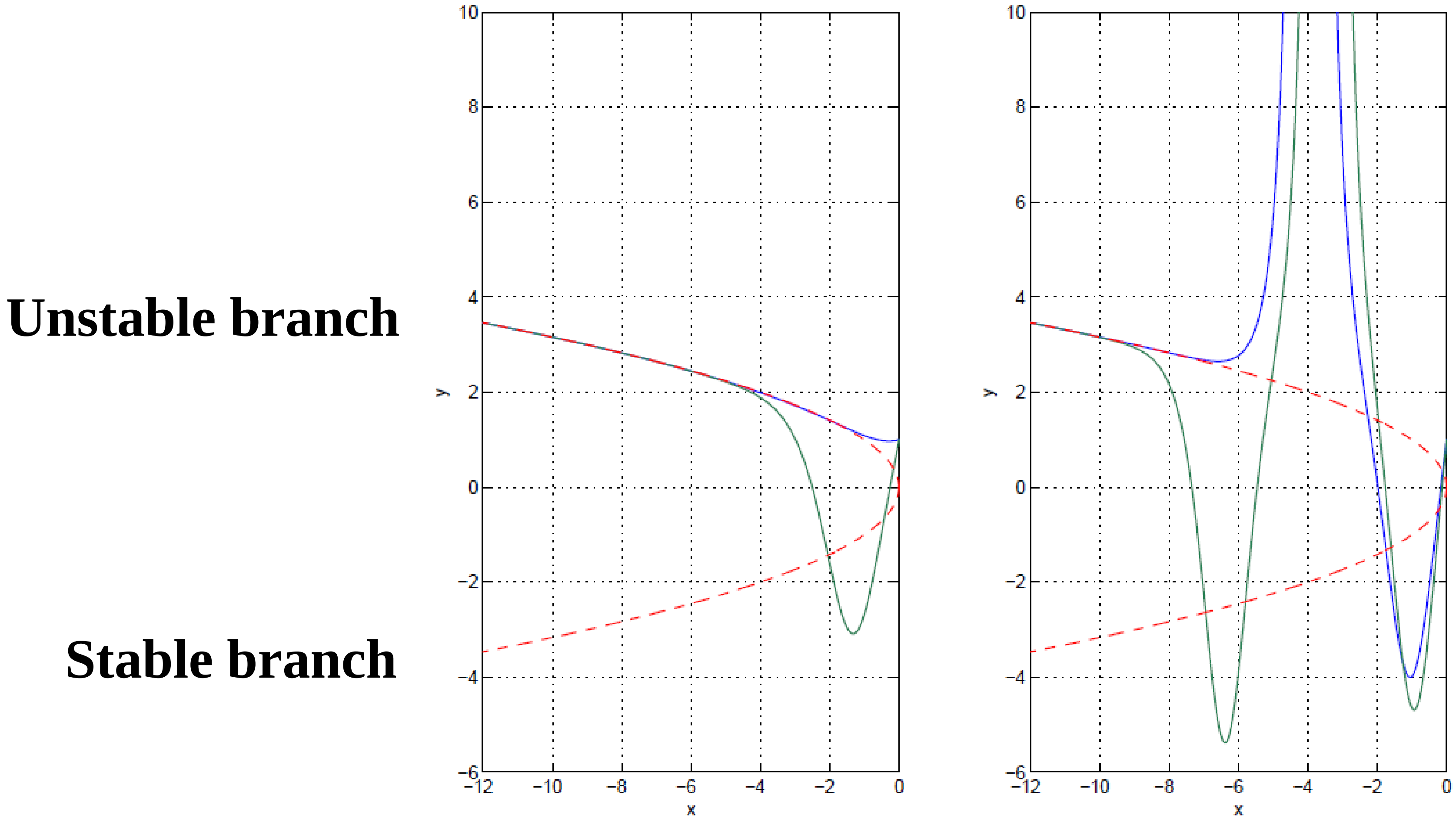
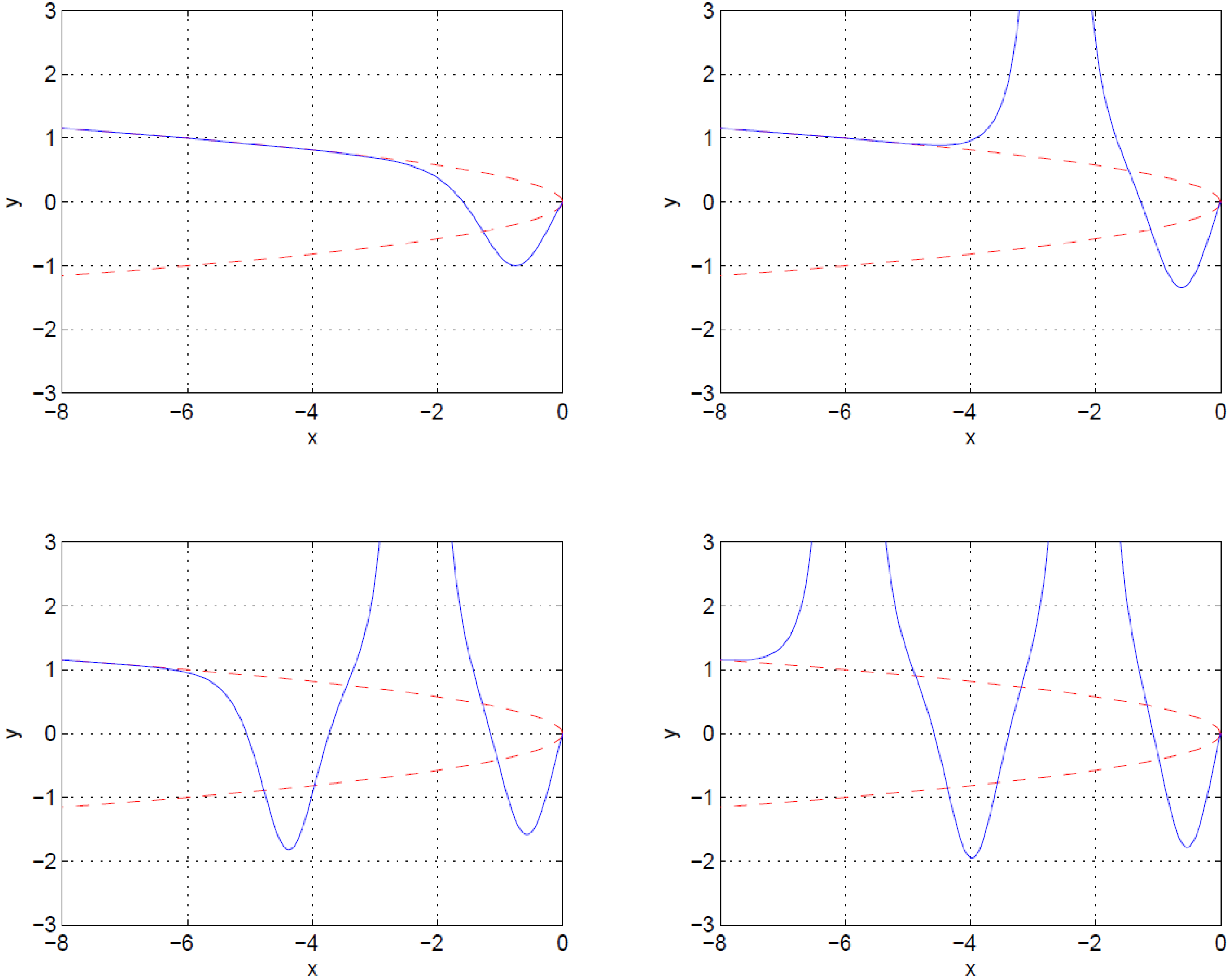


FIG. 6: Eigencurve solutions to the first Painlevé trenscendent. The eigencurves pass through $y(0) = 1$ and the slopes of the curves at $x = 0$ are the eigenvalues a_n. As $x \to -\infty$, the eigencurves approach $+\sqrt{-x}$ exponentially rapidly. Left panel: first two eigencurves corresponding to the eigenvalues $a_1 = 0.231955$ and $a_2 = 3.980669$. The a_1 curve approaches $+\sqrt{-x}$ from above and the a_2 curve approaches $+\sqrt{-x}$ from below. Right panel: The second two eigencurves for the Painlevé transcendent corresponding to the eigenvalues $a_3 = 6.257998$ and $a_4 = 8.075911$. Note that the second pair of eigenvalues passes through one double pole before approaching the curve $+\sqrt{-x}$.

First four eigenfunctions (separatrices)



Numerical calculation of eigenvalues

$y(0) = 1$ and $y'(0) = a$. There is a discrete set of eigencurves whose initial positive slopes are $a_1 = 0.231955$, $a_2 = 3.980669$, $a_3 = 6.257998$, $a_4 = 8.075911$, $a_5 = 9.654843$, $a_6 = 11.078201$, $a_7 = 12.389217$, $a_8 = 13.613878$, $a_9 = 14.769304$, $a_{10} = 15.867511$, $a_{11} = 16.917331$, $a_{12} = 17.925488$.

$$a_n \sim C n^{3/5} \quad (n \to \infty), \qquad C \approx 4.284031379$$

Analytical calculation of eigenvalues

$$a_n \sim C n^{3/5} \quad (n \to \infty),$$

$$C = \frac{1}{\sqrt{3}} \left[\frac{12\sqrt{\pi}\,\Gamma(11/6)\,2^{1/3}}{\Gamma(4/3)} \right]^{3/5}$$

Obtained by using WKB to calculate the large eigenvalues of the

cubic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{3}ix^3$$

(Do you remember
the cubic ***PT***-symmetric
Hamiltonian?!)

(2) Second Painleve transcendent

$$y''(x) = [y(x)]^3 + xy(x)$$

Now, both solutions

$$y(x) \sim \pm\sqrt{-x} \quad (x \to -\infty)$$

are unstable and $y(x) = 0$ is stable.

Unstable

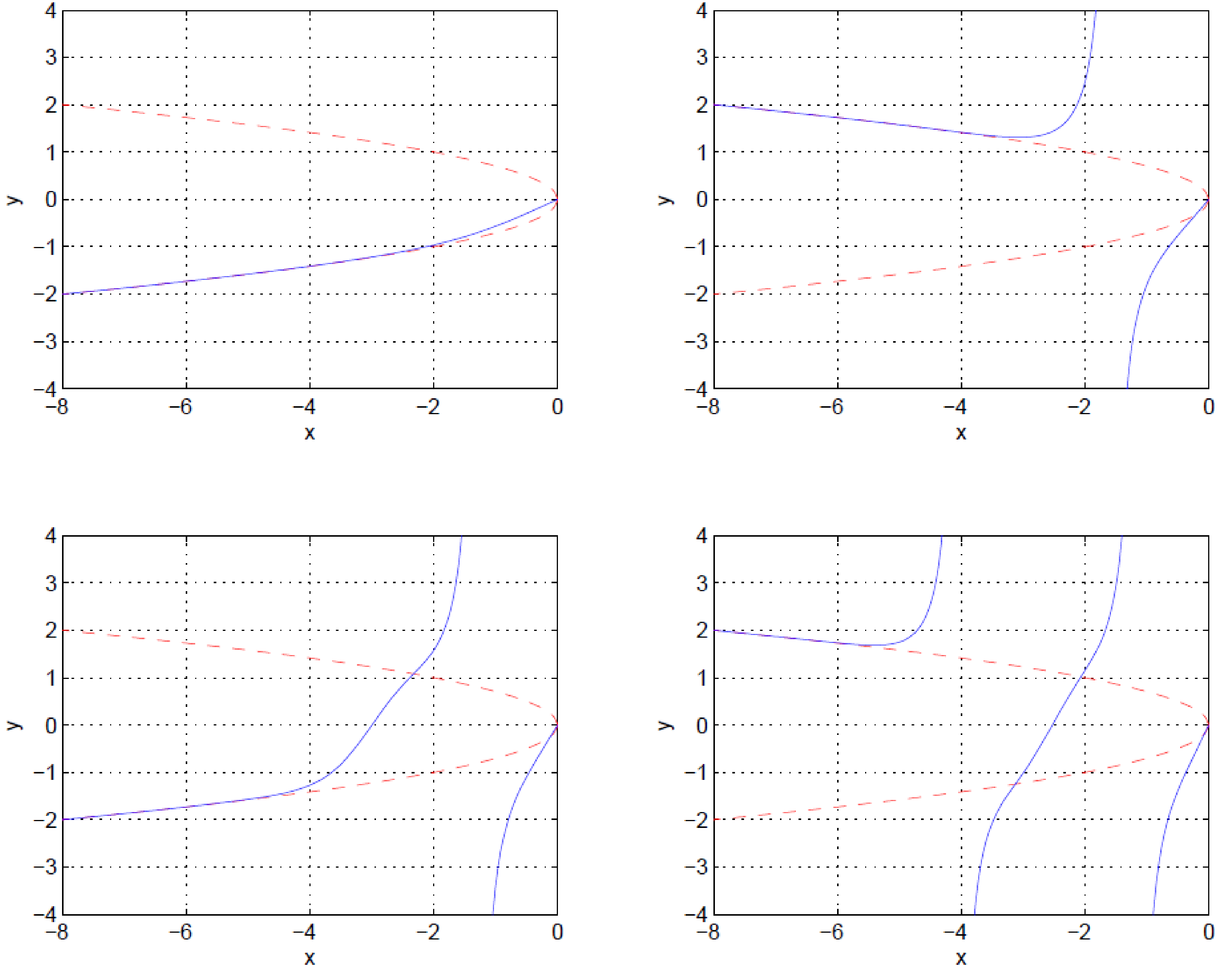
Stable

Unstable

Unstable

Stable

Unstable



Numerical and analytical calculation of eigenvalues

$$a_n \sim D n^{2/3}$$

$$D \approx 1.659221145$$

$$D = \frac{1}{\sqrt{2}}\left[\frac{2\sqrt{\pi}\,\Gamma(7/4)}{\Gamma(5/4)}\right]^{2/3}$$

CMB and J. Komijani
(in preparation)

Obtained by using WKB to calculate the large eigenvalues of the

quartic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{4}x^4$$

(Do you remember the quartic upside-down ***PT***-symmetric Hamiltonian?!)

CMB and J. Komijani, in preparation

We hope we have opened a window to a new area of asymptotic analysis

Thanks for listening!