Entire functions in the Eremenko-Lyubich class which have bounded Fatou components

(joint work with Núria Fagella and Lasse Rempe-Gillen)

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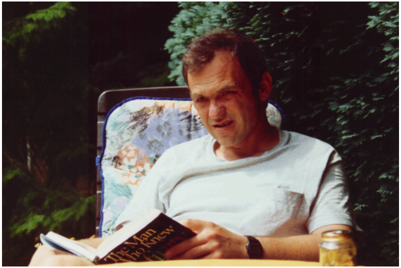


Perspectives of Modern Complex Analysis

Bedlewo, July 2014

Joensuu, 1991

Kommern, 1993

Complex dynamics

$f: \mathbb{C} \to \mathbb{C}$ entire, $f^n = f \circ f \circ \cdots \circ f$

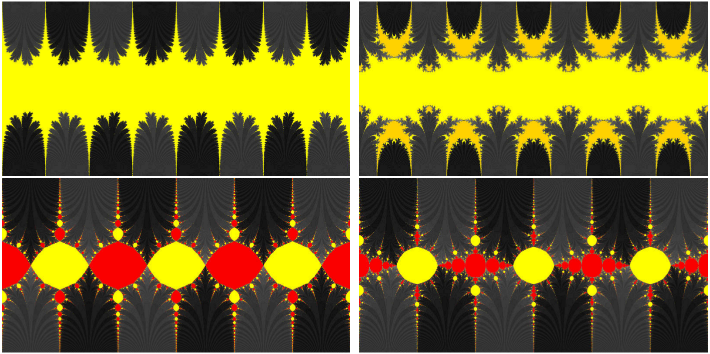
$F(f) = \{z \in \mathbb{C}\colon \{f^n\}$ normal in $z\}$ = Fatou set

$J(f) = \mathbb{C} \setminus F(f) = \{z \in \mathbb{C}\colon \{f^n\}$ not normal in $z\}$ = Julia set

$I(f) = \{z \in \mathbb{C}\colon f^n(z) \to \infty\}$ = escaping set

Eremenko 1989: $I(f) \neq \emptyset$ and $\partial I(f) = J(f)$

Examples of Julia sets in the family $f(z) = \cos(\alpha z + \beta)$



Singularities of the inverse

w critical value $:\Leftrightarrow \exists \xi \in \mathbb{C}: \ f'(\xi) = 0$ and $f(\xi) = w$

($\xi :=$ critical point)

w asymptotic value $:\Leftrightarrow \exists$ curve γ with $\gamma(t) \to \infty$ and $f(\gamma(t)) \to w$

($\gamma :=$ asymptotic curve)

$\operatorname{sing}(f^{-1})$ = set of singularities of f^{-1}

= set of critical values and asymptotic values of f

$\mathcal{B} = \{f : \operatorname{sing}(f^{-1})$ is bounded$\}$

=  -  class

= Eremenko-Lyubich class

$\mathcal{S} = \{f : \operatorname{sing}(f^{-1})$ is finite$\}$

=  class

= Speiser class

Eremenko-Lyubich: $f \in \mathcal{B} \ \Rightarrow \ I(f) \subset J(f)$

E-L, Goldberg-Keen: $f \in \mathcal{S} \ \Rightarrow \ f$ has no wandering domains, i.e., if U_0 is a component of $F(f)$ and U_n is the component containing $f^n(U_0)$, then there exist $m \neq n$ with $U_m = U_n$

Hyperbolicity

z_0 attracting periodic point $:\Leftrightarrow \exists p\colon f^p(z_0) = z_0$ and $|(f^p)'(z_0)| < 1$

U_j = immediate attracting basin

$:=$ component of $F(f)$ containing $z_j = f^j(z_0)$

$\subseteq$ attracting basin

$:= \{z\colon f^{np}(z) \to z_j\}$

Fatou: $\displaystyle\bigcup_{j=0}^{p-1} U_j \cap \operatorname{sing}(f^{-1}) \neq \emptyset$

Definition: $f \in \mathcal{B}$ is called *hyperbolic* if $\overline{\operatorname{sing}(f^{-1})}$ is contained in attracting basins.

f hyperbolic $\Leftrightarrow$ $P(f) := \overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing}(f^{-1}))} \subset F(f)$

$\Leftrightarrow$ $\exists K \subset F(f)$ compact, $f(K) \subset \operatorname{int}(K)$, $\operatorname{sing}(f^{-1}) \subset K$

f hyperbolic $\Rightarrow$ $F(f)$ consists of finitely many attracting basins

Hyperbolicity

Theorem 1: For hyperbolic $f \in \mathcal{B}$ and a periodic component D of $F(f)$ the following are equivalent:

(a) the orbit of D contains no asymptotic curves and only finitely many critical points

(b) D is bounded

(c) D is a Jordan domain

Idea of proof: Choose $K \subset F(f)$ compact with $f(K) \subset \text{int}(K)$ and $\text{sing}(f^{-1}) \subset K$.

Put $W := \mathbb{C} \backslash K$ and $V := f^{-1}(W) \subset W$.

With hyperbolic metric ρ_W define

$$D_W f(z) := \text{hyperbolic derivative} := \lim_{\zeta \to z} \frac{\rho_W(f(\zeta), f(z))}{\rho_W(\zeta, z)}.$$

Rempe-Gillen: $D_W f(z) \geq \lambda > 1$ for $z \in V$.

May now use similar techniques as for hyperbolic polynomials.

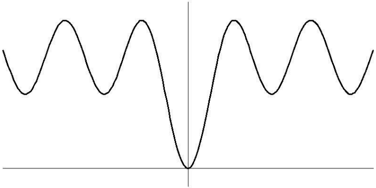
Hyperbolicity

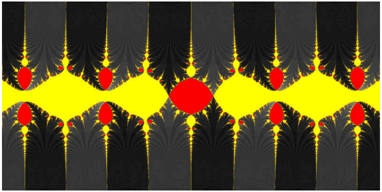
Corollary 1: Let $f \in \mathcal{S}$ be hyperbolic with no asymptotic values. Suppose the critical values are all in different components of $F(f)$. Then all components of $F(f)$ are bounded.

Corollary 2: Let $f \in \mathcal{S}$ be hyperbolic with no asymptotic values and exactly two critical values. Then either all components of $F(f)$ are bounded, or all components of $F(f)$ are unbounded.

Corollary 2 does not hold for functions with three critical values:

$$f(z) = \frac{1}{4}\left(3 - \cos\sqrt{\left(\operatorname{arcosh}^2 3 + \pi^2\right) z^2 - \operatorname{arcosh}^2 3}\right)$$





Local connectivity

Theorem 2: Let $f \in \mathcal{B}$ be hyperbolic with no asymptotic values. Suppose that there exists N such that every component of $F(f)$ contains at most N critical points, counting multiplicity. Then $J(f)$ is locally connected.

Proof uses previous theorem and well-known

Lemma: A compact subset of the Riemann sphere is locally connected if and only if the following two conditions are satisfied:

(a) the boundary of each complementary component is locally connected,

(b) for every positive ε there are only finitely many complementary components of spherical diameter greater than ε.

Corollary: Let $f \in \mathcal{S}$ be hyperbolic with no asymptotic values. Suppose the critical values are all in different components of $F(f)$ and that there is a uniform bound on the multiplicity of the critical points. Then $J(f)$ is locally connected.

Local connectivity

The hypothesis on the multiplicity of the critical points is essential:

Example: There exists a hyperbolic function $f \in \mathcal{S}$ with no asymptotic values and exactly two critical values such that

(a) the critical values are superattracting fixed points,

(b) every Fatou component of f is bounded by a Jordan curve,

(c) the Julia set of f is not locally connected.

The example is from another famous class of functions, namely the

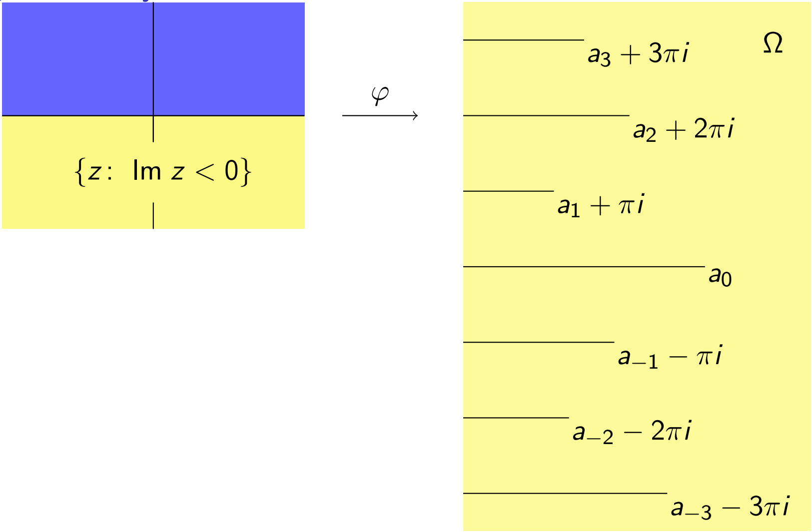
 -  class = closure of real polynomials with real zeros

$$= \left\{ e^{-az^2+bz+c} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{z/x_k} : a, b, c, x_k \in \mathbb{R}, a \geq 0 \right\}$$

= Laguerre-Pólya class

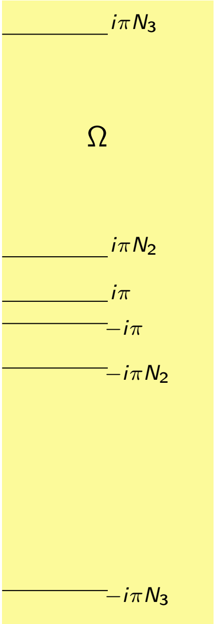
$= \mathcal{LP}$

Laguerre-Pólya functions



Choose conformal map φ from lower

Laguerre-Pólya functions



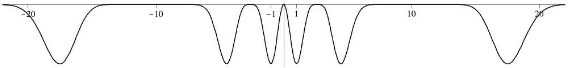
Choose Ω as sketched, with a rapidly increasing sequence (N_k) of odd numbers, where $N_1 = 1$.

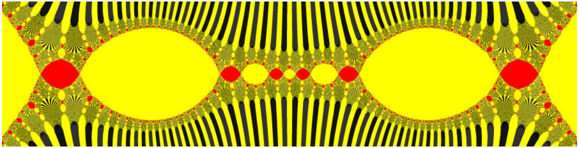
Obtain $f \in \mathcal{LP}$ with critical values -1 and 0.

Critical points corresponding to $\pm i\pi N_k$ are simple and give critical value -1.

Critical points corresponding to gaps between $i\pi N_{k-1}$ and $i\pi N_k$ have multiplicity $N_k - N_{k-1} - 1$ and give critical value 0.

Look at graph and Julia set ($N_2 = 5$, $N_3 = 25$):





Obtain very large bounded Fatou components.

Laguerre-Pólya functions

The "large" Fatou components can be made arbitrarily large by choosing (N_k) rapidly increasing.

These components have preimages intersecting the unit disk $\mathbb{D}$, since $\mathbb{D} \cap J(f) \neq \emptyset$.

These preimages are also large, and thus intersect $\{z \colon |z| = 2\}$, provided N_k increases rapidly.

Have infinitely many Fatou components intersecting both $\{z \colon |z| = 1|\}$ and $\{z \colon |z| = 2\}$; that is, infinitely many Fatou components of spherical diameter greater than $\frac{1}{2}$.

This contradicts the local connectivity criterion stated earlier.