Entire functions in the Eremenko-Lyubich class which have bounded Fatou components

(joint work with Núria Fagella and Lasse Rempe-Gillen)

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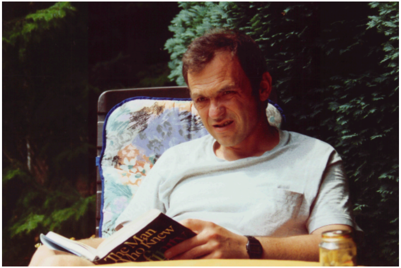

Perspectives of Modern Complex Analysis

Bedlewo, July 2014

Joensuu, 1991

Kommern, 1993

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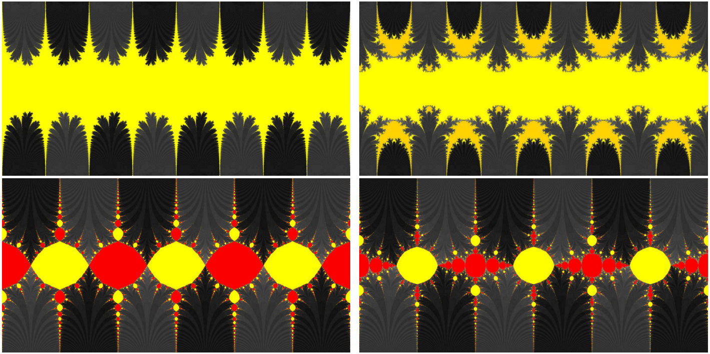
Examples of Julia sets in the family $f(z) = \cos(\alpha z + \beta)$



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May now use similar techniques as for hyperbolic polynomials.

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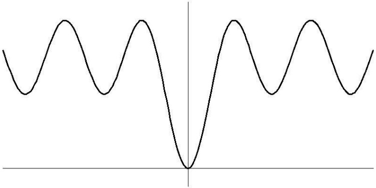

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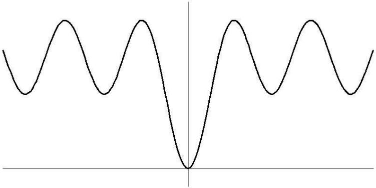

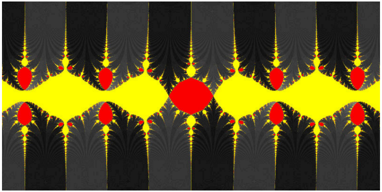

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Proof uses previous theorem and well-known

Lemma: A compact subset of the Riemann sphere is locally connected if and only if the following two conditions are satisfied:

(a) the boundary of each complementary component is locally connected,

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Local connectivity

Theorem 2: Let $f \in \mathcal{B}$ be hyperbolic with no asymptotic values. Suppose that there exists N such that every component of $F(f)$ contains at most N critical points, counting multiplicity. Then $J(f)$ is locally connected.

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Lemma: A compact subset of the Riemann sphere is locally connected if and only if the following two conditions are satisfied:

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Corollary: Let $f \in \mathcal{S}$ be hyperbolic with no asymptotic values. Suppose the critical values are all in different components of $F(f)$ and that there is a uniform bound on the multiplicity of the critical points. Then $J(f)$ is locally connected.

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Example: There exists a hyperbolic function $f \in \mathcal{S}$ with no asymptotic values and exactly two critical values such that

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Laguerre-Pólya functions

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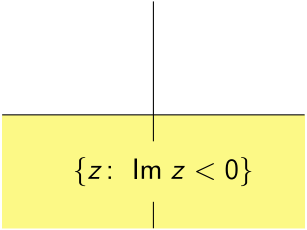

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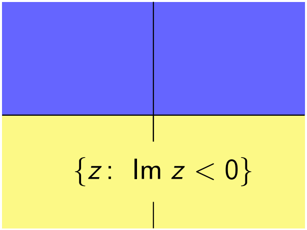

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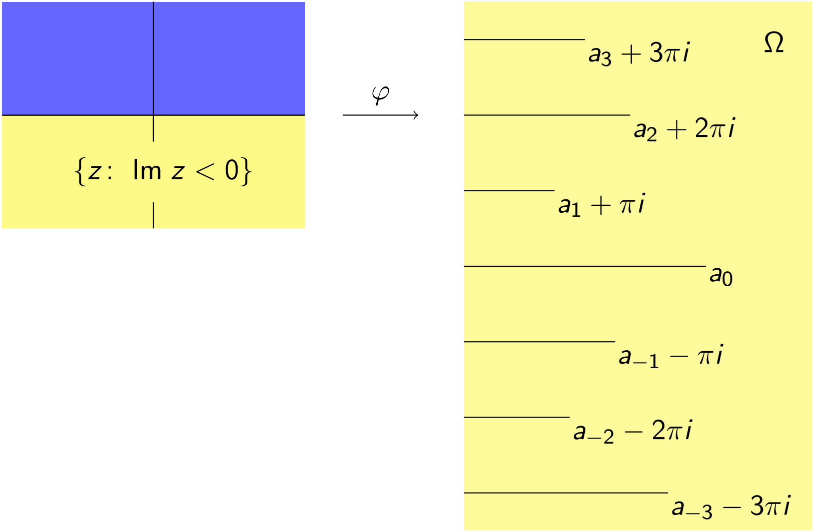

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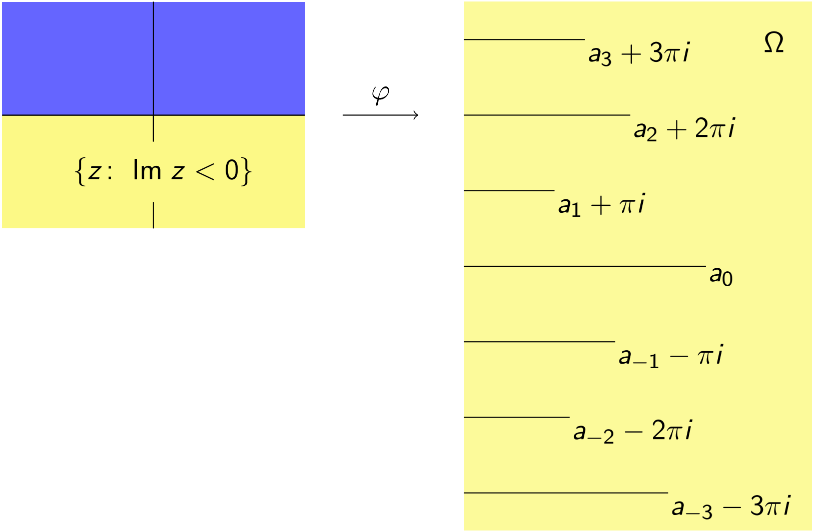

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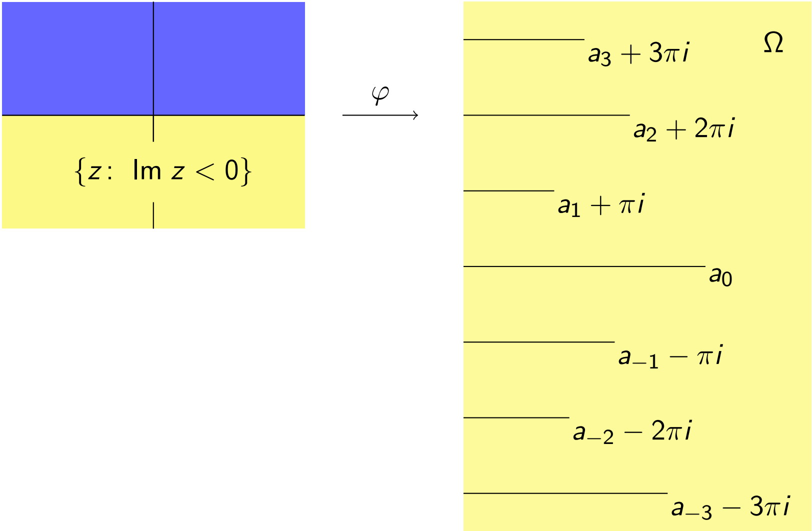

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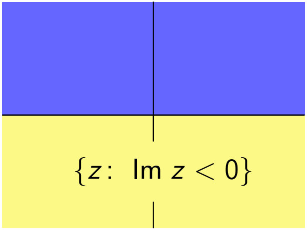

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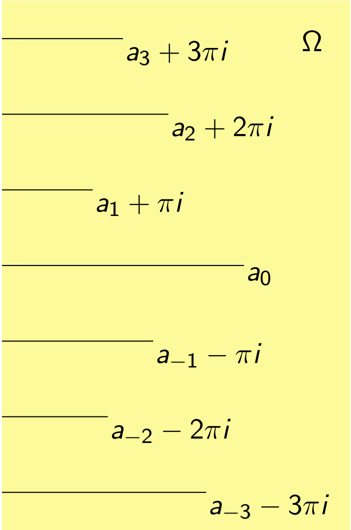
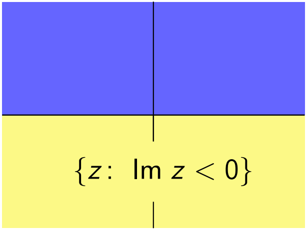


Choose conformal map φ from lower half-plane onto "comb domain" Ω.

Laguerre-Pólya functions



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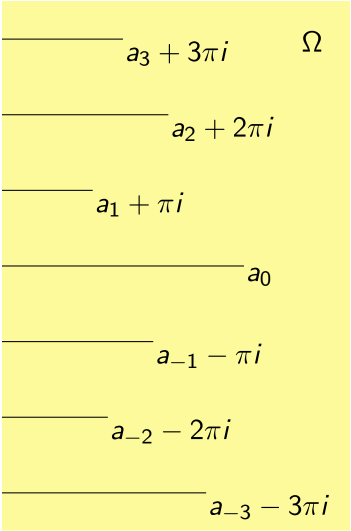

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Extend $\exp\circ\varphi$ to entire function f by Schwarz reflection principle.

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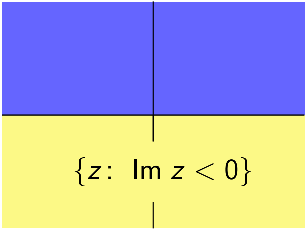

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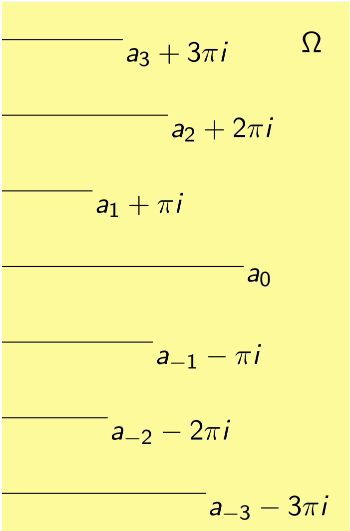

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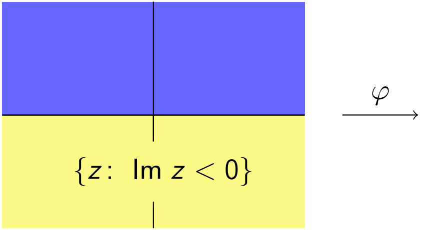

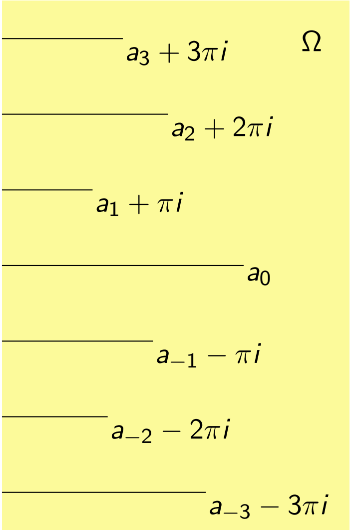

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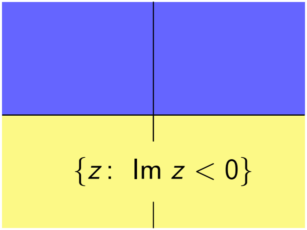

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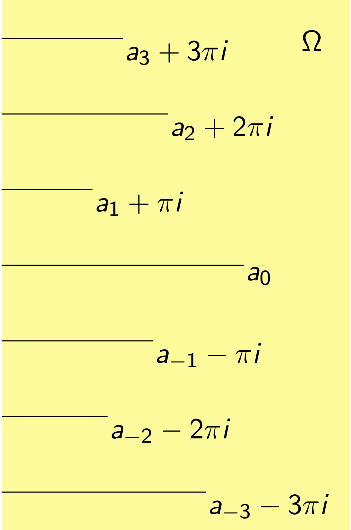

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Method of MacLane, Vinberg, Eremenko-Sodin, Eremenko-Yuditskii

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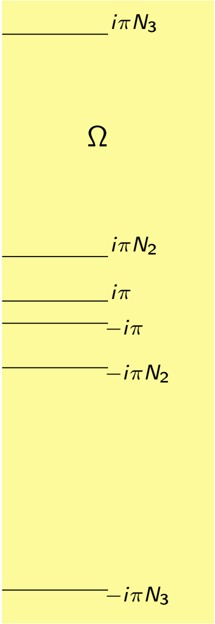

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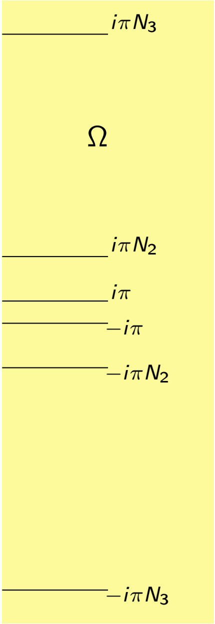

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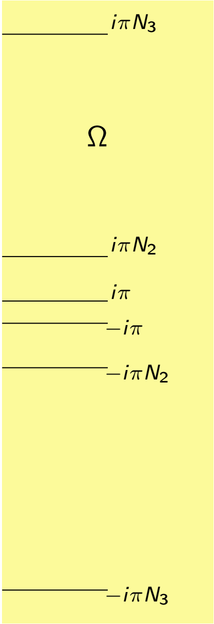
Laguerre-Pólya functions



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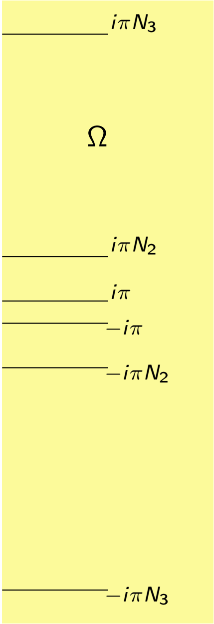

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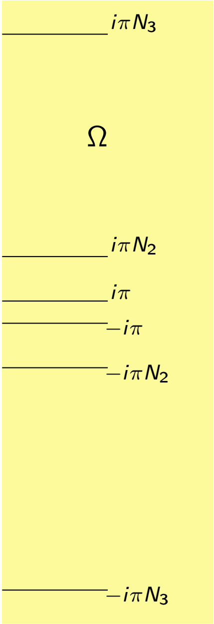

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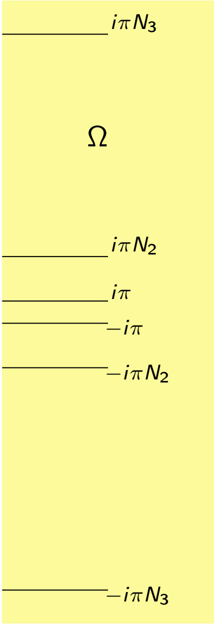

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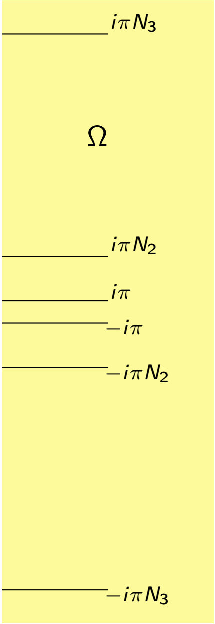

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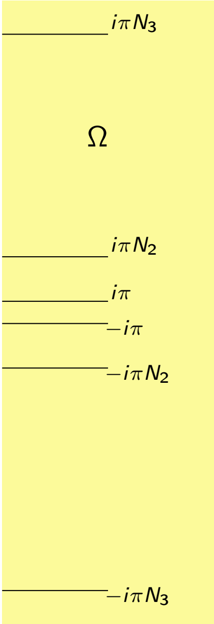
Laguerre-Pólya functions



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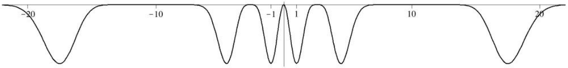

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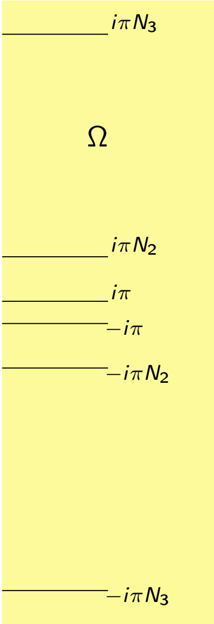

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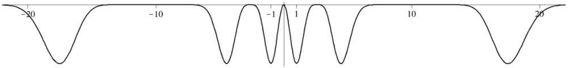

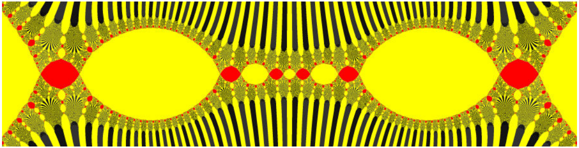

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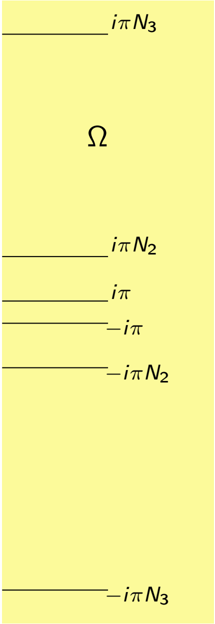

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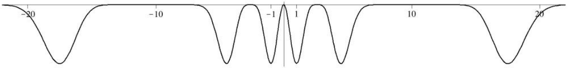

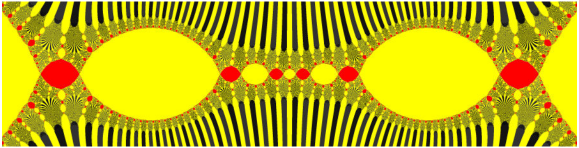

Obtain very large bounded Fatou components.

Laguerre-Pólya functions

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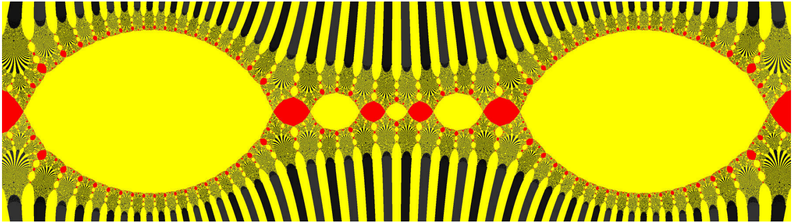

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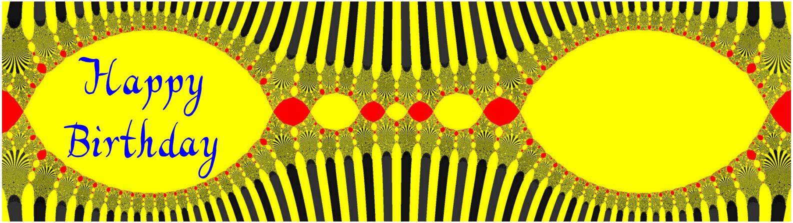

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