Metric entropy and stochastic laws of invariant measures for elliptic functions

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Plan of the talk

1. Basic properties of 'critically tame' elliptic functions
2. The results
3. Idea of the proofs (the main ingredients of the proofs)
    - thermodynamic formalism for conformal graph directed Markov systems
    - nice sets for holomorphic maps of the Riemann surfaces
    - L.S. Young's towers
    - stochastic properties of the return map
    - metric entropy

Definition

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be an elliptic function and $c \in \mathrm{Crit}(f)$. We say that f is critically tame if the following conditions are satisfied:

- if $c \in F(f)$- Fatou set, then there exists an attracting or parabolic cycle of period p, $O(z_0) = \{z_0, f(z_0), ..., f^{p-1}(z_0)\}$ such that ω limit set $\omega(c) = O(z_0)$.
- if $c \in J(f)$ - Julia set, then one of the following holds:
  - $\omega(c)$ is a compact subset of $\mathbb{C}$ such that $c \notin \omega(c)$; (but $c \in \omega(c')$ where $c' \in \mathrm{Crit}(f)$)
  - c is eventually mapped onto some pole;
  - $\lim_{n\to\infty} f^n(c) = \infty$

Proposition

Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a critically tame elliptic function.

Then for $z\in J(f)\setminus \mathrm{Sing}^-(f)$ there is a sequence of integer $\{n_k\}$ such that

$$\lim_{k\to\infty} |(f^{n_k})'(z)| = \infty,$$

where

- $\Omega(f)$ - the set of parabolic periodic points
- $\mathrm{Crit(f) = \{z \in \mathbb{C}:\ f'(z) = 0\}}$
- $\mathrm{Sing}^-(f) := \bigcup_{n\geq 0} f^{-n}(\Omega(f)\cup \mathrm{Crit}(J(f))\cup f^{-1}(\infty)),$

Corollary

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function. Then

- f has neither Siegel discs nor Herman rings,

  so every component of the Fatou set is contained in a basin of attraction of attracting or parabolic periodic point.

- f has no Cremer points

  so every periodic point is either attracting, parabolic or repelling.

Conformal measure

Fix $t \geq 0$. Let G and H be non-empty open subsets of $\overline{\mathbb{C}}$. Let $f : G \to H$ be a meromorphic map.

A pair (m_G, m_H) of Borel finite measures on G and H respectively is called spherical t-conformal pair of measures for the map $f : G \to H$, if

$$m_H(f(A)) = \int_A |f^*|^t \, dm_G$$

for every Borel set $A \subset G$ such that $f|_A$ is injective.

If both measures m_G and m_H are restrictions of the same Borel finite measure m defined defined on $G \cup H$, we refer to m as t- conformal measure the map $f : G \to H$.

Conformal measure

- If $G \subset \mathbb{C}$ and in the two formulas above the spherical derivative f^* is replaced by the Euclidean derivative f', the pair (m_G, m_H) is called Euclidean t-conformal measures for the map $f : G \to H$.
- m_s denotes a spherical t-conformal measure $m_s(f(A)) = \int_A |f^*|^t \, dm_s$
- m_e denotes an Euclidean t-conformal measure $m_e(f(A)) = \int_A |f'|^t dm_e$
- $\frac{dm_e}{dm_s}(z) = (1 + |z|^2)^t$
- m_e is σ-finite, m_s is finite

Theorem A- Urbański and Kotus

Let f be a non-constant elliptic function. Then
$$\dim_H(J(f)) > \frac{2q}{q+1} \geq 1$$
where q is the maximal multiplicity of poles of f.

Theorem B - Urbański and Kotus

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function.

- If $h = \dim_H(J(f)) = 2$, then $J(f) = \overline{\mathbb{C}}$.
- If $h < 2$, then
    1. h- dimensional Hausdorff measure $\mathrm{H}_s^h(J(f)) = 0$.
    2. h -dimensional packing measure $\Pi_s^h(J(f)) > 0$.
    3. $\Pi_s^h(J(f)) = \infty$ if and only if $\Omega(f) \neq \emptyset$, $\Omega(f)$ is the set of parabolic periodic points.

Theorem C - Urbański and Kotus

Suppose that f is critically tame elliptic function, denote $h = \dim_H(J(f))$. Then there exist:

- a unique atomless h-conformal measure m for $f : J(f) \setminus \{\infty\} \to J(f)$ where m is ergodic and $m(Tr(f)) = 1$; $Tr(f) \subset J(f)$ denotes the set of all transitive points of f
- if f has no parabolic periodic points, then $0 < \Pi_s^h(J(f)) < \infty$ and m and Π_s^h are equivalent.

Definition

The measure μ is called *ergodic* if for $\forall G \in \mathcal{B}$ s.t. $f^{-1}(G) = G$ one has $\mu(G) = 0$ or $\mu(G^c) = 0$.

Theorem D - Urbański and Kotus

Suppose that f is critically tame elliptic function, denote $h = \dim_H(J(f))$. Then there exist:

- there exists a non-atomic, σ-finite, ergodic and invariant measure μ for f, equivalent to the measure m. Additionally, μ is unique up to a multiplicative constant and is supported on $J(f)$.
- the Jacobian $D_\mu f = \frac{d\mu \circ f}{d\mu}$ has a real analytic extension on a neighborhood of $J(f) \setminus (\overline{\mathrm{PC}(f)} \cup f^{-1}(\infty))$ in $\mathbb{C}$.

II - The results

Definitions

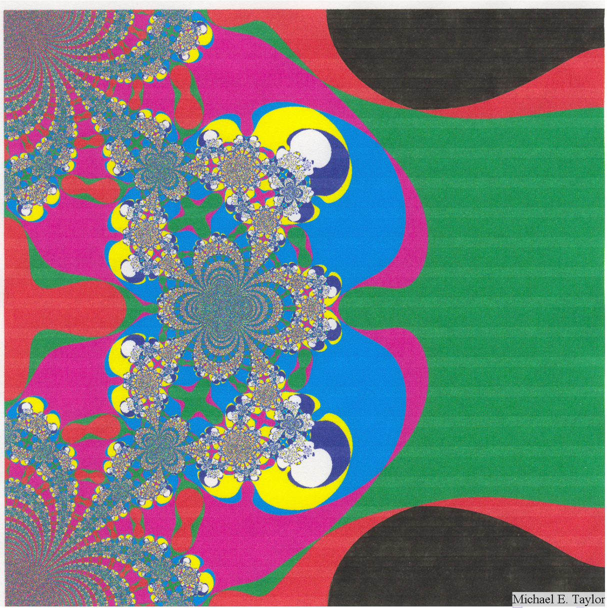
- Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be critically tame elliptic functions,
- f has no parabolic periodic points
- and $\mathrm{Crit}_\infty(f) = \emptyset$
- then f is called of finite character.

Theorem E - Urbański and Kotus

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame elliptic of finite character then

- μ_h finite.
- in particular if Julia set is equal to the entire complex plane $\mathbb{C}$, then there exists a unique Borel probability f-invariant measure μ equivalent to the planar Lebesgue measure on $\mathbb{C}$.

$\wp$- Weiestrass function



$\wp$ - Weiestrass function

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Theorem 1(a) - Decay of correlation - Urbański and Kotus

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an elliptic function of finite character and if μ is the probability f-invariant measure equivalent to the h-conformal measure m, then for the dynamical system (f, μ) the following holds.

Fix $\alpha \in (0, 1]$ and a bounded function $g : J(f) \to \mathbb{R}$ which is Hölder continuous with respect to the Euclidean metric on $J(f)$ with the exponent α. Then for every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that

$$\left| \int \psi \circ f^n \cdot g d\mu - \int g dmu \int \psi d\mu \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on α.

Theorem 1(b) - The Central Limit Theorem - Urbański and Kotus

The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$, i.e. for which there is no square integrable function η for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution. Equivalently for every $t \in \mathbb{R}$,

$$\mu\left(\{x \in X : \frac{1}{\sqrt{n}} S_n g(x) \le t\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-u^2/2\sigma^2\right) du.$$

II - The new results

Theorem 1(c)-The Law of Iterated Logarithm- Urbański and Kotus

The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu)$. This means that there exists a real positive constant A_g such that such that μ_ϕ almost everywhere

$$\limsup_{n\to\infty} \frac{S_n g - n\int g d\mu}{\sqrt{n \log\log n}} = A_g.$$

Theorem 2 - Urbański and Kotus

If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame map of finite type, μ_h is the corresponding Borel probability f-invariant measure equivalent to the h-conformal measure m,then a metric entropy

$$h_{\mu_h}(f) < +\infty.$$

The main ingredients of the proof of Theorem 1

A) Thermodynamic formalism of graph directed Markov system

B) Nice sets for analytic maps

C) Young's tower technique

D) Stochastic properties of the return map

A - Graph Directed Markov System

Subshifts of finite type

- Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers
- let E be a countable alphabet (either finite or infinite set)
- $E^{\mathbb{N}}$ -a coding space
- Let $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ be the shift map, i.e. cutting off the first coordinate. It is given by the formula $\sigma\big((\omega_n)_{n=1}^{\infty}\big) = \big((\omega_{n+1})_{n=1}^{\infty}\big)$.
- We consider a 0-1 matrix $A : E \times E \to \{0, 1\}$. Set $E_A^{\mathbb{N}} = \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } i \geq 1\}$.-a space of admissible sequences
- Let $E_A^n := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } 1 \leq i \leq n-1\}$ $n \geq 1$ and $E_A^* := \bigcup_{n=0}^{\infty} E_A^n$.
- The elements of the sets $E_A^{\mathbb{N}}$ and E_A^* are called A-admissible.

Subshifts of finite type

- $E_A^{\mathbb{N}}$ is a closed subset of $E^{\mathbb{N}}$, invariant under the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$. The later means that $\sigma(E_A^{\mathbb{N}}) \subset E_A^{\mathbb{N}}$.
- The matrix A is said to be finitely irreducible if there exists a finite set $\Lambda \subset E_A^*$ such that for all $i, j \in E$ there exists a path $\omega \in \Lambda$ for which $i\omega j \in E_A^*$.

Iterated Function System/Graph Directed Markov System

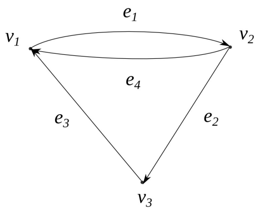
A Graph Directed Markov System consists of

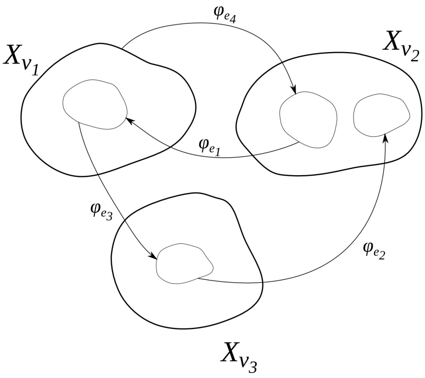
- a directed multigraph (E, V) with a countable set of edges E and a finite set of vertices V,
- an incidence matrix $A : E \times E \to \{0, 1\}$,
- two functions $i, t : E \to V$ such that $t(a) = i(b)$ whenever $A_{ab} = 1$.
- a family of non-empty compact metric spaces $\{X_v\}_{v \in V}$,
- a number $\beta \in (0, 1)$, and
- for every $e \in E$, a 1-to-1 contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\leq \beta$.

The set $\mathcal{S} = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e \in E}$ is called a Graph Directed Markov System (GDMS).

A - Graph Directed Markov System

$\omega = e_2 e_3 e_1 e_4 \ldots$





A - Graph Directed Markov System

Limit set of GDMS

- GDMS is called an iterated function system if V the set of vertices, is a singleton and $A(E \times E) = \{1\}$.
- For each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by ω:

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$$

- For $\omega \in E_A^\infty$, the images $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)}) \neq \emptyset$.
- Since for every $n \geq 1$,

$$\mathrm{diam}\big(\phi_{\omega|_n}(X_{t(\omega_n)})\big) \leq \beta_S^n \mathrm{diam}\big(X_{t(\omega_n)}\big) \leq s^n \max\{\mathrm{diam}(X_v) : v \in V\},$$

we conclude that the intersection $\bigcap_{n\geq 1} \phi_{\omega|_n}(X_{t(\omega_n)})$ is a singleton and we denote its only element by $\pi(\omega)$.

Limit set of GDMS

- In this way we have defined the projection map π

$$\pi : E_A^\infty \to \bigoplus_{v\in V} X_v$$

  from the coding space E_A^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v.

- The set

$$J = J_S := \pi(E_A^\infty)$$

  will be called the limit set $\mathcal{J}_S$ of the Graph Directed Markov System $\mathcal{S} = \{\phi_e\}_{e\in E}$ (GDMS).

A - Graph Directed Markov System

Topological Pressure $P(t)$

Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible GDMS. For every $t \geq 0$ let $Z_n(t) = \sum_{\omega\in E_A^n} ||\phi'_\omega||^t$. The limit

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(t)$$

exists and is called the topological pressure of the system $\mathcal{S}$

Properties of $P(t)$

- $\theta = \inf\{t \geq 0 : \ \mathrm{P}(t) < \infty\}$
- The topological pressure function $\mathrm{P}(t)$ is
  (1) convex and continuous on $(\theta, +\infty)$,
  (2) strictly decreasing on $(\theta, +\infty)$,
  (3) $\lim_{t\to+\infty} P(t) = -\infty$
- $\mathrm{P}(\theta) = +\infty$ if and only if E is infinite.

Bowen's Parameter

- The number

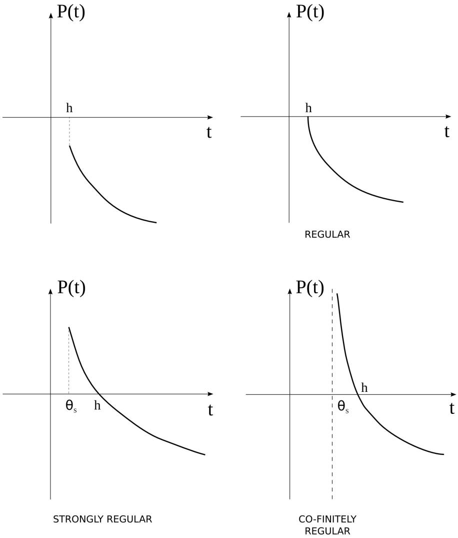
$$h := \inf\{t \geq 0 : \mathrm{P}(t) \leq 0\}$$

  is called the Bowen's parameter of the system $\mathcal{S}$. .

- If $\mathrm{P}(t) = 0$ for some $t \geq 0$, then $t = h$.

- We say that the system $\mathcal{S}$ is
  - regular if $\mathrm{P}(h) = 0$,
  - strongly regular if there exists $t \geq 0$ such that $0 < \mathrm{P}(t) < +\infty$, and
  - co-finitely regular if $\mathrm{P}(\theta) = +\infty$.

A - Graph Directed Markov System- topological pressure



Finer Geometrical Properties of CGDMS

Now assume that the alphabet E is finite and keep the incidence matrix A (finitely) irreducible.

- Fix $t \geq 0$. Consider the operator $\mathcal{L}_t$ given by the formula

$$\mathcal{L}_t g(\omega) := \sum_{i:\, A_{i\omega_1}=1} g(i\omega)|\phi_i'(\pi(\omega)|^t, \quad \omega \in E_A^{\mathbb{N}}, g \in C(E_A^{\mathbb{N}})$$

- the linear operator $\mathcal{L}_t$ acts continuously on $C(E_A^{\mathbb{N}})$- the Banach space of all real-valued continuous functions on E_A^{∞} endowed with the supremum norm.
- Let $\mathcal{L}_t^* : C^*(E_A^{\mathbb{N}}) \to C^*(E_A^{\mathbb{N}})$ be its dual operator.
- Denote by M_A the space of all Borel probability measures on $E_A^{\boldsymbol{N}}$ and consider a map $m \mapsto \frac{\mathcal{L}_t^* m}{\mathcal{L}_t^* m(\mathbb{1})} \in M_A$,

Finer Geometrical Properties of CGDMS

- Since this map is continuous in the weak-star-topology on M_A and since M_A is a compact (because E is finite) convex subset of the locally convex topological vector space $C^*(E_A^{\mathbb{N}})$, it follows from the Schauder-Tichonov Theorem that the map

$$m \mapsto \frac{\mathcal{L}_t^* m}{\mathcal{L}_t^* m(\mathbb{1})} \in M_A$$

has a fixed point $\tilde{m}_t$ and put $\lambda_t = \mathcal{L}_t^* \tilde{m}_t(\mathbb{1}) > 0$. We then have

$$\mathcal{L}_t^* \tilde{m}_t = \lambda_t \tilde{m}_t$$

- Let $m_t = \tilde{m}_t \circ \pi^{-1}$. We call m_t the t-conformal measure for the system $\mathcal{S}$.

A - Graph Directed Markov System

Theorem -Hausdorff dimension of the limit set

If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is strongly regular finitely irreducible GDMS, then $\dim_H(J_{\mathcal{S}}) = h > \theta_{\mathcal{S}},$ h- is a Bowen parameter.

Theorem

If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible system then

- $\mathcal{S}$ is regular if and only if there exists a Borel probability measure ν on $E_A^{\mathbb{N}}$ such that $\mathcal{L}_h^* \nu = \nu$.
- In addition, if $\mathcal{L}_t^* \nu = \nu$, for some $t \geq 0$ and some Borel probability measure ν, then $t = h$ and $\nu = \tilde{m}_h$.

Theorem

If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible strongly regular GDMS, then the metric entropy $h_{\tilde{\mu}_h}(\sigma)$ of the dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$ with respect to the σ-invariant measure $\tilde{\mu}_h$ is finite.

B - Nice sets for analytic maps

- Nice sets naturally appeared in dynamical systems in the context of self-map of an interval (Yoccoz's puzzles)
- They were adapted to holomorphic endomorphisms of the Riemann sphere, by J. Rivera-Letelier.
- Nice sets became an important tool in the complete treatment of Collect-Eckmann rational functions given by F. Przytycki and J. Rivera-Letelier.
- In the context of meromorphic (both transcendental and rational maps of the complex plane to the Riemann sphere) existence of nice sets was provided by N. Dobbs.
- We define nice sets for holomorphic maps of the Riemann surfaces (one of which is an open subset of the other). We need such generality in order to deal with the projected maps: $F : \mathbb{T} \setminus \Pi(f^{-1}(\infty)) \to \mathbb{T}$.
- **Nice sets give naturally rise to graph directed Markov systems.**

Iteration of analytic maps of compact Riemann surface

- Let Y be compact Riemann surface and let X be an open subset of Y.
- Let $f : X \to Y$ be an analytic map.
- We say that $y \in Y$ is a regular point of f^{-1} if for every $r > 0$ small enough and every connected component C of $f^{-1}(B(y,r))$ the restriction of $f_{|C} : C \to B(y,r)$ is a homepmorphism from C onto $B(y,r)$.
- Otherwise we say that y is a singular point of f^{-1} and we denote by $\mathrm{Sing}(f^{-1})$ the set of all such singular points.
- We also set

$$\mathrm{PS}(f) = \bigcup_{n=0}^{\infty} (\mathrm{Sing}(f^{-1}))$$

with the convention that $f(\{z\}) = \emptyset$ if $z \in Y \setminus X$.

Iteration of analytic maps of compact Riemann surface

- We say that a point $z \in X$ belongs to the Fatou set $F(f)$ if there is an open neighbourhood U of z such that all the iterates $f^n : U \to Y$, $n \geq 1$, are well-defined and contains a subsequence forming a normal family.

- The Julia set $J(f)$ is defined as $Y \setminus F(f)$.

  Clearly $J(f)$ is a closed subset of Y and 'completely invariant'.

B - Nice sets for analytic maps

Definition

A non-empty open set $V \subset Y$ is said to be a nice set for the analytic map $f : X \to Y$ if the following conditions are satisfied

(a) $\overline{V}$ is compact

(b) V has finitely many connected components

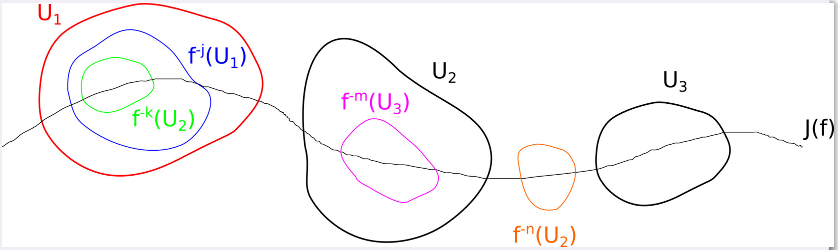
(c) If U is a connected component of V, then U is simply connected and there exists W, an open connected simply connected subset of Y such that $\overline{W} \cap \overline{\mathrm{PS(f)}} = \emptyset$

(d) $V \cap \bigcup_{n=0}^{\infty} f^n(\partial V) = \emptyset$

Proposition

- Suppose that V is a nice set for a holomorphic map $f : X \to Y$.
- Let U and W be two distinct components of V.
- If $j, k \geq 0$ are two integers and A and B are connected components respectively of $f^{-j}(U)$ and $f^{-k}(W)$,

then either $A \cap B = \emptyset$ or $A \subset B$ or $B \subset A$.



Nice sets and graph directed Markov systems

- For given a nice set V let $\mathcal{C}_1^\infty(V)$ be the family of all connected components of the set $V \cap \bigcup_{n=1}^\infty f^{-n}(V)$.
- For $U \in \mathcal{C}_1^\infty(V)$ there exists a unique connected component U^* of a nice set V and a unique holomorphic inverse branch $f^{-n(U)}$ such that $f^{-n(U)}(U^*) = U$
- If $f : X \to Y$ is an analytic map and V is a nice set for f, then $\mathcal{S}_V = \{f^{-n(U)} : U^* :\to U\}_{\{U \in C_1^\infty(V)\}}$ is a graph directed Markov system.

Theorem - existence of nice sets

- Let Y be a compact Riemann surface, let X be a non-empty open subset of Y and let $f : X \to Y$ be an analytic map with 'the Standard Property'.
- Fix F, a finite subset of $J(f) \setminus \overline{\mathrm{PS}(f)}$ such that $F \cap \bigcup_{n=1}^{\infty} f^n(F) = \emptyset$. Fix also $\kappa > 1$.

Then for every $r \in (0, \frac{1}{4} \min\{\rho(a,b) : a \neq b,\ a, b \in F\})$ small enough there exists a nice set $U = U_r$ with the following properties:

(a) $B(F, r) \subset U \subset B(F, \kappa r)$,

(b) If W is a connected component of U, then $W \cap F$ is a singleton.

Application to critically tame elliptic functions

- Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a critically tame elliptic function
- Let $\mathbb{T} = \mathbb{C}/\sim_f$ (the torus generated by the lattice Λ of f).
- $B(f) = f^{-1}(\infty) \cup (\mathrm{Crit}(f) \cap J(f))$ is infinite
- $\Pi : \mathbb{C} \to \mathbb{T}$ be the canonical projection, $\hat{\mathbb{T}} := \Pi(\mathbb{C} \setminus f^{-1}(\infty))$

$$\begin{array}{ccc} \mathbb{C} \setminus f^{-1}(\infty) & \xrightarrow{f} & \mathbb{C} \\ \Pi \downarrow & & \downarrow \Pi \\ \hat{\mathbb{T}} & \xrightarrow{\hat{f}} & \mathbb{T}. \end{array}$$

- Then $B(\hat{f}) = \Pi(B(f))$ is finite

B - Nice sets for analytic maps

Limit set of Graph Directed Markov System

- There exists a nice set - a neighbourhood V of $B(\hat{f})$ and a corresponding Graph Directed Markov System $\mathcal{S}_V = \{\hat{f}_U^{-n(U)} : U^* :\to U\}_{\{U \text{ component } of \ V \cap \bigcup_{n=1}^{\infty} \hat{f}^{-n}(V)\}}$.

  U^* is a component of a nice set V

  the points in $B(\hat{f}) = \Pi(B(f))$ are the vertices of $\mathcal{S}_V$

  the branches of $\hat{f}^{-n(U)}$ are edges of $\mathcal{S}_V$.

- The system $\mathcal{S}_V$ is strongly regular, J_V - limit set of $\mathcal{S}_V$.

- $\dim_H(J_V) = \dim_H(J(f))$.

C- Iterated Function Systems on $\mathbb{C}$ associated to $c \in Crit(f) \cap J(f)$

Properties of IFS

For $c \in Crit(f) \cap J(f)$ we define an iterated function system)(IFS) $\mathcal{S}_c = \{g^{-n} : V_c \to V_c\}$ by " lifting " $\mathcal{S}_V$ - graph directed Markov systems i.e.

- $V_c = \Pi^{-1}(V_{\Pi(c)})$, where $V_{\Pi(c)}$ is a component of a nice set V containing $\Pi(c)$.
- g^{-n} is "a lift " of some $\hat{f}^{-n} : V_{\Pi(c)} \to V_{\Pi(c)}$, $\hat{f}^{-n} \in \mathcal{S}_V$,
- Let J_c be the limit set of $\mathcal{S}_c$. J_c contains all transitive points from V_c, so $m_h(J_c) > 0$ and $\dim_H(J_c) = \dim_H(J(f))$.
- the system $\mathcal{S}_c$ is strongly regular.

C- Iterated Function Systems on $\mathbb{C}$ associated to $c \in Crit(f) \cap J(f)$

Properties of IFS

- For every point $c \in J(f) \cap Crit(f)$ the greatest common divisor of all return time numbers is equal to 1.
- For all $c \in J(f) \cap \mathrm{Crit}(f)$ the system $\mathcal{S}_c$ is regular and $dim_H(J_c) = h$, so the system $\mathcal{S}_c$ admits a unique normalized h-conformal measure ν_c.
- We define an induced map i.e. the return time map $F : J_c \to J_c$ by $F(g^{-n}(z)) = z$ and consider the system (F, ν_c).
- The standard distortion considerations show then that ν_c and $m|_{J_c}$ are equivalent.
- Thus $m(J_c) > 0$ and $\nu_c = (m(J_c))^{-1} m|_{J_c}$.
- Our goal is to show that the induced system (F, ν_c) satisfies the assumptions of L. S. Young's theorems

Definition

- Let $(\Delta_0, \mathcal{B}_0, m_0)$ be a measure space with a finite measure m_0,
- let $\mathcal{P}_0$ be a countable measurable partition of Δ_0 and
- let $T_0 : \Delta_0 \to \Delta_0$ be a measurable map such that, for every $\Delta' \in \mathcal{P}_0$ the map $T_0 : \Delta' \to \Delta_0$ is a bijection onto Δ_0.
- We assume that the partition $\mathcal{P}_0$ is generating, i.e. for every $x, y \in \Delta_0$ there exists $s \geq 0$ such that $T_0^s(x)$, $T_0^s(y)$ are in different elements of the partition $\mathcal{P}_0$.
- We denote by $s = s(x, y)$ the smallest integer with this property and we call it a separation time for the pair x, y.

Definition

- We assume also that for each $\Delta' \in \mathcal{P}_0$ the map $(T_{0|\Delta'})^{-1}$ is measurable and that $Jac_{m_0}(T_0)$, the Jacobian of T_0 with respect to the measure m_0, is well–defined and positive a.e. in Δ'.
- The following distortion property is assumed to be satisfied. With some constants $0 < \beta < 1$ and $C > 0$,

$$\left| \frac{Jac_{m_0} T_0(x)}{Jac_{m_0} T_0(y)} - 1 \right| \leq C\beta^{s(x,y)}$$

  for all $\Delta' \in \mathcal{P}_0$ and all $x, y \in \Delta'$.
- We have also a function $R : \Delta_0 \to \mathbb{N}$ ("return time") which is constant on each element of the partition $\mathcal{P}_0$.
- We assume that the greatest common divisor of the values of R is equal to 1.

Definition

- Finally, let $\Delta = \{(z,n) \in \Delta_0 \times \mathbb{N} \cup \{0\} : 0 \le n < R(z)\}$ where each point $z \in \Delta_0$ is identified with $(z,0)$. We extend T_0 to T, which acts on Δ as follows.

$$T(z,n) = \begin{cases} (z, n+1) & \text{if} \quad n+1 < R(z) \\ (T_0(z), 0) & \text{if} \quad n+1 = R(z) \end{cases}$$

- The measure m_0 is spread over the whole space Δ by putting

$$\tilde{m}_{|\Delta_0} = m_0 \quad \text{and} \quad \tilde{m}_{|\Delta' \times \{j\}} = m_{0|\Delta'} \circ \pi_j^{-1}$$

for all $\Delta' \in \mathcal{P}_0$, where $\pi_j(z,0) = (z,j)$.

- Thus, the measure $\tilde{m}$ is finite if and only if $\int_{\Delta_0} R dm_0 < \infty$.
- We refer to the pentapole $\mathcal{Y} = (\Delta_0, m_0, T_0, \mathcal{P}_0, R)$ as a Young tower.

Theorem - L.S. Young

If $\mathcal{Y} = (\Delta_0, m_0, T_0, \mathcal{P}_0, R))$ is a Young tower and $\int R dm_0 < \infty$, then

- there exists a unique probability T–invariant measure ν, absolutely continuous with respect to $\tilde{m}$.
- The Radon-Niokodym derivative $d\nu/d\tilde{m}$ is bounded from below by a positive constant.
- The dynamical system (T, ν) is exact, thus ergodic.

D - Young's tower

Theorem - L.S. Young

Let $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R))$ be a Young tower. Then the following hold.

1. If $m_0(R > n) = O(\theta^n)$ for some $0 < \theta < 1$, then there exists $0 < \tilde{\theta} < 1$ such that for all functions $\psi \in L^\infty$ and $g \in C_\beta(\Delta)$ we have,

$$|C_n(\psi, g)| = \left| \int (\psi \circ T^n) g d\nu - \int \psi d\nu \int g d\nu \right| = O(\tilde{\theta}^n).$$

2. If $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 1$ (in particular, if $m_0(R > n) = O(\theta^n)$), then the Central Limit Theorem is satisfied for all functions $g \in C_\beta(\Delta)$, that are not cohomologous to a constant in $L^2(\nu)$.

D - Young's tower

Theorem

Let

- $\mathcal{Y} = (\Delta_0, T_0, \mathcal{P}_0, R))$ be a Young tower
- and if $m_0(R > n) = O(n^{-\alpha})$ with some $\alpha > 4$ (in particular, if $m_0(R > n) = O(\theta^n))$,

then the Law of Iterated Logarithm holds for all functions $g \in C_\beta(\Delta)$, that are not cohomologous to a constant in $L^2(\nu)$.

Application to critically tame elliptic functions

- $\mathcal{Y} = (\Delta_0, m_0, T_0, \mathcal{P}_0, R)$ - Young's tower
- The space Δ_0 is J_c -the limit set of the iterated function system $\mathcal{S}_c$.
- The partition $\mathcal{P}_0$ consists of the sets $\Delta_n = g^{-n}(J_c)$, $g^{-n} \in \mathcal{S}_c$.
- The measure m_0 is the h-conformal measure m restricted J_c.

Application

- the map $T_0 : \Delta_0 \to \Delta_0$ is just the map F where $F(g^{-n}(z)) = z$
- the function R is the return time
- the partition $\mathcal{P}_0$ is generating (follows either from the contracting property of graph directed Markov systems)
- the greatest common divisor of all the return times equals to 1
- therefore the map $T : \Delta \to \Delta$ admits a probability T-invariant measure ν which is absolutely continuous with respect to $\tilde{m}$ and

$$|C_n(\psi, g)| = \left| \int (\psi \circ T^n) g d\nu - \int \psi d\nu \int g d\nu \right| = O(\tilde{\theta}^n).$$

the Central Limit Theorem and the Law of Iterated Logarithm are true.

Application

- Now consider $H : \Delta \to \mathbb{C}$, the natural projection from the abstract tower Δ to the complex plane $\mathbb{C}$ given by the formula
$$H(z,n) = f^n(z)$$
Then $H \circ T = f \circ H$.
- Since ν is F–invariant then the measure $\nu \circ H^{-1}$ is f–invariant.
- But the measure μ is f–invariant ergodic and equivalent to the conformal measure m.
- Hence, $\nu \circ H^{-1}$ is absolutely continuous with respect to the ergodic measure μ. Invariance and ergodicity of $\nu \circ H^{-1}$ yield thus $\nu \circ H^{-1} = \mu$.
- Thus the system (f, μ) has required stochastic properties stated in Theorem 1.

Definition

- Let $T : X \to X$ be a measure-preserving endomorphism of a probability space $(X, \mathcal{F}, \mu)$,
- let $\mathcal{A} = \{A_k\}_{k \geq 1}$ be a countable measurable partition of X.
- Define new partitions of X:

$$\mathcal{A}^n := \bigvee_{i=0}^{n-1} T^{-i}\mathcal{A} = T^{-(n-1)}\mathcal{A} \vee \cdots \vee T^{-1}\mathcal{A} \vee \mathcal{A}$$

  where $\mathcal{A} \vee \mathcal{B} := \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$
- The quantity

$$h_\mu(T, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{A}^n) = \lim_{n\to\infty} \frac{1}{n} \sum_{A \in \mathcal{A}^n} -\mu(A) \log \mu(A),$$

  is called the entropy of T with respect to $\mathcal{A}$

Definition

The measure-theoretic entropy of T is defined by

$$h_\mu(T) := \sup\{h_\mu(T, \mathcal{A}) : \mathcal{A} \text{ is a finite partition of } X\}.$$

Theorem 2 - Urbański and Kotus

- If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame map of finite type,
- μ_h is the corresponding Borel probability f-invariant measure equivalent to the h-conformal measure m,
- then a metric entropy $h_{\mu_h}(f) < +\infty$.

Corollary

- If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a critically tame elliptic function with $J(f) = \mathbb{C}$ and $\mathrm{Crit}_\infty(f) = \emptyset$,
- μ is the (unique) Borel probability f–invariant measure on $\mathbb{C}$ equivalent to the planar Lebesgue measure on $\mathbb{C}$.
- then $\mathrm{h}_\mu(f) < +\infty$

Theorem - Abramov

If $T : X \to X$ is an ergodic measure preserving transformation of a probability space $(X, \mathcal{F}, \mu)$, then for every set $K \in \mathcal{F}$ with $0 < \mu(K) < +\infty$, we have that

$$\mathrm{h}_{\mu_K}(T_K) = \frac{1}{\mu(K)} h_\mu(T).$$

where

- $T_K(x) := T^{\tau_K(x)}(x)$ is an induced map
- $\tau_K(x) := \min\{n \geq 1 : \ T^n(x) \in K\}$.
- $\mu_K := \mu_{|K}(\mu(K))^{-1}$

Krengel's Entropy

If $T: X \to X$ is an conservative ergodic measure preserving transformation of a measure space $(X, \mathcal{F}, \mu)$, then for all sets F and G in $\mathcal{F}$ with $0 < \mu(F), \mu(G) < +\infty$, we have that $\mathrm{h}_{\mu_F}(T_F) = \mathrm{h}_{\mu_G}(T_G)$.

- This common value is called the Krengel' entropy of the map $T: X \to X$ and is denoted simply by $\mathrm{h}_\mu(T)$.
- If μ is a probability measure, it coincides with the standard entropy of T with respect to μ.

The proof of Theorem 2

- For $c \in Crit(f) \cap J(f)$ we defined Iterated Function System $\mathcal{S}_c = \{g^{-n} : V_c \to V_c\}$, J_c is the limit set od $\mathcal{S}_c$.
- For all $c \in J(f) \cap \mathrm{Crit}(f)$ the system $\mathcal{S}_c$ is regular and $dim_H(J_c) = h$, so the system $\mathcal{S}_c$ admits a unique normalized h-conformal measure ν_c and $\nu_c = (m(J_c))^{-1} m|_{J_c}$.
- the return time map $F : J_c \to J_c$ is defined by $F(g^{-n}(z)) = z$.

The proof of Theorem 2

- Abramov's formula gives $h_{\nu_c}(F) = \frac{1}{\mu(J_c)} h_\mu(f)$, where f is critically tame elliptic function
- If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible strongly regular GDMS, then the metric entropy $h_{\tilde{\mu}_h}(\sigma)$ of the dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$ with respect to the σ-invariant measure $\tilde{\mu}_h$ is finite.
- $h_{\nu_c}(F) = h_{\tilde{\mu}_h}(\sigma) < +\infty$, so $h_\mu(f) = h_{\nu_c}(F) \cdot \mu(J_c) < +\infty$.