FUNCTIONAL CENTRAL LIMIT THEOREM FOR ADDITIVE FUNCTIONALS OF α-STABLE PROCESSES

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Abstract. Let β be a standard Brownian motion, let X be an α-stable process, and let $f = \widehat{\mu}$ be the Fourier transform of a discrete measure. It is shown that weakly in $C([0,+\infty))$,

$$\eta^{\alpha/2}\int_0^t f(\eta X_s)\mathrm{d}s \Rightarrow \sqrt{C_{f,\alpha}}\beta_t \qquad \text{as } \eta\to+\infty,$$

or equivalently

$$\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} f(X_s)\mathrm{d}s \Rightarrow \sqrt{C_{f,\alpha}}\beta_t \qquad \text{as } \lambda\to+\infty.$$

1. Introduction

The paper is concerned with assymptotic properties of the additive functionals $\mathcal{A}^X_t(f) := \int_0^t f(X_s)\mathrm{d}s$, $t\ge 0$, of real-valued processes X. Limit theorems for additive functionals of Markov processes have a long history. The study of this problem was initiated by Kallianpur and Robbins [9] for Brownian motion and bounded integrable functions f. Darling and Kac [6] considered a much more general class of Markov processes and f with compact support. The case where X is a Brownian motion is also described in detail in the book of Bingham, Goldie, and Teugels [3] (our very limited list of references includes also [2], [10], [14], [15], [16], [17], [18]). A very powerful martingale method useful for a large class of ergodic Markov processes had been invented by Papanicolaou, Stroock, and Varadhan [13] and was later developed by Kipnis and Varadhan [11] (see also the monograph by Jacod and Shiryev [8]). There are also some results for non-Markov processes (see e.g. [12]).

Let us start with some elementary consideration. Namely, let us assume that there is a local-time L^X of X and that for f the density occupation formula holds

$$\mathcal{A}^X_t(f) = \int_{\mathbb{R}} f(x)L^X_t(x)\mathrm{d}x, \qquad t\ge 0.$$

Then, for any $\eta>0$,

$$\eta\mathcal{A}^{\eta X}_t(f) = \eta\int_{\mathbb{R}} f(\eta x)L^X_t(x)\mathrm{d}x = \int_{\mathbb{R}} f(y)L^X_t\left(y\eta^{-1}\right)\mathrm{d}y.$$

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Therefore, if f is integrable and L^X_t is continuous in x one can expect the following asymptotic result

$$\lim_{\eta\to+\infty} \eta\mathcal{A}^{\eta X}_t(f) = \int_{\mathbb{R}} f(x)\mathrm{d}x L^X_t(0), \qquad \mathbb{P}-a.s.$$

In fact if X is a Brownian motion, then, see [14], using the inverse Itô formula one can extend the definition of the additive functional $\mathcal{A}^X_t(f)$ for Borel sign measures f on $\mathbb{R}$ with finite total variation $\|f\|_{\mathrm{Var}}$, and show that $\eta\mathcal{A}^{\eta X}_t(f)$ converges $\mathbb{P}$-a.s to $f(\mathbb{R})L^X_t(0)$. Moreover, see [14], if $\int_{\mathbb{R}}|x|\|f\|_{\mathrm{Var}}(\mathrm{d}x)<\infty$, then

$$\sqrt{\eta}\left(\eta\mathcal{A}^{\eta X}_t(f) - f(\mathbb{R})L^X_t(0)\right)$$

converges weakly as $\eta\to+\infty$ to $2\sqrt{c_f}\beta_{L^X_t(0)}$, where

$$c_f := \int_{\mathbb{R}} \left(f((-\infty,x)) - f(\mathbb{R})\chi_{(0,+\infty)}\right)^2(x)\mathrm{d}x,$$

and β is an independent of X Brownian motion.

Recall that a symmetric α-stable Lévy process X is self-similar, i.e. for any $\lambda>0$ the process $(X_{\lambda t})_{t\geq 0}$ has the same law as $(\lambda^{1/\alpha}X_t)_{t\geq 0}$. Hence, for any $\gamma>0$, the convergence in law of $\left(\eta^\gamma\mathcal{A}^{\eta X}_t(f)\right)_{t\geq 0}$ as $\eta\to+\infty$ and of $\left(\lambda^{\gamma/\alpha-1}\mathcal{A}^X_{\lambda t}(f)\right)_{t\geq 0}$ as $\lambda\to+\infty$ are equivalent.

In the case of the α-stable process and if f is integrable and sufficiently "nice" (e.g. bounded) the situation is clear. It is easy to check that if $\alpha<1$, more generally in d-dimensional case if $\alpha<d$, then $\mathcal{A}^X_\lambda(f)$ converges pointwise and in L^1 to an integrable random variable $\int_0^{+\infty} f(X_s)\mathrm{d}s$, and consequently for any $\varepsilon>0$, the process $\mathcal{A}^X_{\lambda t}$ converge in law in $C([\varepsilon,+\infty))$ and its limit is constant in time. If $\alpha>1$ then the local time L^X of X exists (see e.g. [1]) and the processes $\left(\lambda^{1/\alpha-1}\mathcal{A}^X_{\lambda t}(f)\right)$ converges in law to $\left(L^X_t(0)f(\mathbb{R})\right)$, as expected. If $\alpha=1$, more generally if $\alpha=d$, $d=1,2$, and X is d-dimensional, then for any $\varepsilon>0$ the processes $\left((\log\lambda)^{-1}\mathcal{A}^X_{\lambda t}(f)\right)$ converge in law in $C([\varepsilon,+\infty))$. The limit is constant in time and is of the form $Const\, f(\mathbb{R})\rho$, where ρ is an exponential random variable, see [5] Lemma 2.12 and Theorem 2.13.

In this paper we are interested in the case when f is not integrable. Quite a lot is known in the case of a Brownian motion. If f is square integrable, and its Hilbert transform is integrable, Yamada (see [17], [18]) showed that uniformly in t on bounded intervals, $\lambda^{-\frac{1}{2}}\mathcal{A}^X_{\lambda t}(f)\to H_f\gamma_t$ in probability, as $\lambda\to+\infty$, where γ is a process depending on the Brownian motion X, $H_f=\frac{1}{\pi}\int_{\mathbb{R}}\mathcal{H}^{-1}f(x)\mathrm{d}x$ and $\mathcal{H}^{-1}$ is the inverse Hilbert transform. Using self similarity of the Brownian motion one obtains that $\eta\mathcal{A}^{\eta X}_t(f)$ converges weakly (uniformly in t on bounded intervals) to $H_f\gamma_t$ as $\eta=\sqrt{\lambda}\to+\infty$.

Assume now that

$$f(x) = \sum_{j=1}^m \left(a_j\mathrm{e}^{\mathrm{i}p_jx} + \overline{a_j}\mathrm{e}^{-\mathrm{i}p_jx}\right) \tag{1}$$

where $m\in\mathbb{N}$, $p_1,\ldots,p_m\in\mathbb{R}\setminus\{0\}$, and $a_1,\ldots,a_m\in\mathbb{C}$. Then, again for a Brownian motion B, the martingale method introduced in the classical work by Papanicolaou, Stroock, and

Varadhan [13] yields that uniformly in t on bounded intervals $\eta\mathcal{A}^{\eta X}_t(f)$ converges weakly to $\sqrt{C_f}\beta_t$, where (see e.g. [14]), $C_f = 4\sum_{j=1}^m |a_j|^2|p_j|^{-2}$ and β is a Brownian motion. In fact the convergence can by shown in a larger class of functions f (see e.g. [11] or [14]). The aim of this paper is to establish a similar result for an α stable process X. We will use the direct yet elementary moment method as in e.g. [5] or [7]. It seems that a similar result can be deduced from [11] for f given by (1).

2. Main result

In what follows X is a real-valued symmetric α-stable Lévy process with $\alpha \in (0,2]$. Recall that $|x|^\alpha$ is the Fourier exponent of X, that is $\mathbb{E}\,\mathrm{e}^{\mathrm{i}xX_t} = \mathrm{e}^{-t|x|^\alpha}$ for $t \ge 0$ and $x \in \mathbb{R}$.

Definition 2.1. We say that a function $f\colon \mathbb{R} \mapsto \mathbb{R}$ is of class $\mathcal{D}_\alpha$ if $f(x) = \widehat{\mu}(x) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}xy}\mu(\mathrm{d}x)$, where μ is a complex-valued measure of finite total variation $\|\mu\|_{\mathrm{Var}}$, satisfying $\overline{\mu}(-\Gamma) = \mu(\Gamma)$, $\Gamma \in \mathcal{B}(\mathbb{R})$, $\mu(\{0\}) = 0$, and

$$\int_{U_n} \prod_{k=1}^{n} |\sum_{i=1}^{k} x_i|^{-\alpha} \|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x) < \infty, \qquad \forall\, n \in \mathbb{N}, \tag{2}$$

where $U_n := \{x \in \mathbb{R}^n : \sum_{i=1}^k x_i \neq 0, k = 1,\ldots,n\}$.

Remark 2.2. Let $f = \widehat{\mu} \in \mathcal{D}_\alpha$. Note that $\int_{\{x=y\}} \frac{\|\mu\|_{\mathrm{Var}}(\mathrm{d}x)\|\mu\|_{\mathrm{Var}}(\mathrm{d}y)}{|x|^\alpha} < \infty$. Moreover,

$$C_{f,\alpha} := 2\int_{\{x+y=0\}} \frac{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}{|x|^\alpha} = 2\int_{\{x=y\}} \frac{\mu(\mathrm{d}x)\overline{\mu(\mathrm{d}y)}}{|x|^\alpha} \tag{3}$$

is real and non-negative.

Example 2.3. Let f be given by (1). Then $f \in \mathcal{D}_\alpha$ for any $\alpha \in (0,2]$. For, $f = \widehat{\mu}$, where $\mu = \sum_{j=1}^m \left(a_j\delta_{p_j} + \overline{a_j}\delta_{-p_j}\right)$ clearly satisfies (2). Note that

$$C_{f,\alpha} = 4\sum_{j=1}^{m} \frac{|a_j|^2}{|p_j|^\alpha}.$$

Example 2.4. Clearly $\cos = \frac{1}{2}\,\widehat{(\delta_{-1} + \delta_1)}$ and $\sin = \frac{1}{2}\,\widehat{\left(\frac{1}{\mathrm{i}}\delta_1 - \frac{1}{\mathrm{i}}\delta_{-1}\right)}$. In both cases $C_{f,\alpha} = 2$.

The main result of this paper is the following functional central limit theorem valid for α-stable processes with $\alpha \in (0,2]$. Its proof has been postponed till Section 4.

Theorem 2.5. *Assume that $f \in \mathcal{D}_\alpha$, $C_{f,\alpha}$ is given by (3) and β is a standard Brownian motion. Then, weakly in $C([0,+\infty))$,*

$$\eta^{\alpha/2}\mathcal{A}^{\eta X}_t(f) \Rightarrow \sqrt{C_{f,\alpha}}\beta_t \quad \text{as } \eta \to +\infty, \tag{4}$$

$$\frac{1}{\sqrt{\lambda}}A^X_{\lambda t}(f) \Rightarrow \sqrt{C_{f,\alpha}}\beta_t \quad \text{as } \lambda \to +\infty. \tag{5}$$

Remark 2.6. Since $\eta X_s = X_{\eta^\alpha s}$ in law, we have,

$$\eta^{\alpha/2}\mathcal{A}_t^{\eta X}(t) \overset{\text{in law}}{=} \eta^{\alpha/2}\int_0^t f(X_{\eta^\alpha t}) = \eta^{-\alpha/2}\int_0^{\eta^\alpha t} f(X_s)\mathrm{d}s$$

and hence (4) and (5) are equivalent.

3. Moment estimates

Let $f=\widehat{\mu}\in\mathcal{D}_\alpha$ and let X be an α-stable process. In this section we are concerned with the behavior of the moments of $\eta^{\alpha/2}\,\mathcal{A}_t^{\eta X}(f)$.

Let us note first that for $t\ge 0$, $\eta>0$, $n=1,2,\ldots$, and $x=(x_1,\ldots,x_n)$, we have

$$\begin{aligned}
&\eta^{n\alpha/2}\,\mathbb{E}\left(\mathcal{A}_t^{\eta X}(f)\right)^n\\
&= n!\int_{\mathbb{R}^n}\mu^{\otimes^n}(\mathrm{d}x)\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\eta^{n\alpha/2}\mathbb{E}\,\mathrm{e}^{\mathrm{i}\eta x_1X_{t_1}+\ldots+\mathrm{i}\eta x_nX_{t_n}}\\
&= n!\int_{\mathbb{R}^n}\mu^{\otimes^n}(\mathrm{d}x)\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\eta^{n\alpha/2}\\
&\qquad\times\mathbb{E}\,\mathrm{e}^{\mathrm{i}\eta x_1\left(X_{t_1}-X_{t_2}\right)+\mathrm{i}\eta(x_1+x_2)\left(X_{t_2}-X_{t_3}\right)+\ldots+\mathrm{i}\eta(x_1+\ldots+x_n)X_{t_n}}\\
&= n!\int_{\mathbb{R}^n}\mu^{\otimes^n}(\mathrm{d}x)\int_0^t \mathrm{d}t_n\int_{t_n}^{t}\mathrm{d}t_{n-1}\ldots\int_{t_2}^{t}\mathrm{d}t_1\eta^{n\alpha/2}\\
&\qquad\times\mathrm{e}^{-(t_1-t_2)|\eta x_1|^\alpha-(t_2-t_3)|\eta(x_1+x_2)|^\alpha-\ldots-t_n|\eta(x_1+\ldots+x_n)|^\alpha}\\
&= n!\int_{\mathbb{R}^n}\int_{\mathcal{T}(n,t)}\eta^{n\alpha/2}\mathrm{e}^{-s_1|\eta x_1|^\alpha-s_2|\eta(x_1+x_2)|^\alpha-\ldots-s_n|\eta(x_1+\ldots+x_n)|^\alpha}\mu^{\otimes^n}(\mathrm{d}x)\mathrm{d}\mathbf{s},
\end{aligned}\tag{6}$$

where $\mathrm{d}\mathbf{s}:=\mathrm{d}s_1\ldots\mathrm{d}s_n$, and

$$\mathcal{T}(n,t):=\left\{\mathbf{s}=(s_1,\ldots,s_n)\colon s_i\ge 0,\ \sum_{i=1}^n s_i\le t\right\}.$$

Clearly, $\mu^{\otimes^n}$ is supported on

$$\mathbb{R}^n_*:=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n\colon x_i\neq 0, i=1,\ldots,n\}.$$

Write

$$\Delta_n:=\begin{cases}\left\{x\in\mathbb{R}^n_*\colon x_{2k-1}+x_{2k}=0,\ k=1,\ldots,\frac{n}{2}\right\} & \text{if } n \text{ is even},\\ \emptyset & \text{if } n \text{ is odd.}\end{cases}$$

Lemma 3.1. (i) *For all $t>0$ and $n\in\mathbb{N}$,*

$$\int_{\mathbb{R}^n}\sup_{\eta\ge 1}\int_{[0,t]^n}\eta^{n\alpha/2}\mathrm{e}^{-s_1|\eta x_1|^\alpha-s_2|\eta(x_1+x_2)|^\alpha-\ldots-s_n|\eta(x_1+\ldots+x_n)|^\alpha}\mathrm{d}\mathbf{s}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)<\infty,\tag{7}$$

$$\lim_{\eta\to+\infty}\int_{\mathbb{R}^n\setminus\Delta_n}\int_{[0,t]^n}\eta^{n\alpha/2}\mathrm{e}^{-s_1|\eta x_1|^\alpha-s_2|\eta(x_1+x_2)|^\alpha-\ldots-s_n|\eta(x_1+\ldots+x_n)|^\alpha}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)\mathrm{d}\mathbf{s}=0.\tag{8}$$

(ii) For all $t > 0$ and even $n \in \mathbb{N}$,

$$\lim_{\eta\to+\infty} n! \int_{\Delta_n}\int_{\mathcal{T}(n,t)} \eta^{n\alpha/2} \mathrm{e}^{-s_1|\eta x_1|^\alpha - s_2|\eta(x_1+x_2)|^\alpha - \ldots - s_n|\eta(x_1+\ldots+x_n)|^\alpha} \mu^{\otimes^n}(\mathrm{d}x)\mathrm{d}\mathbf{s}$$
$$= (C_{f,\alpha}t)^{\frac{n}{2}} d_n,$$

where d_n is the n^{th} moment of the normal distribution $\mathcal{N}(0,1)$.

Proof of (i) For $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$ define

$$\mathcal{K}_n(x) := \left\{ k \le n \colon \sum_{i=1}^k x_i \ne 0 \right\}.$$

Observe that if $x \in \mathbb{R}^n_*$, then

$$\#\mathcal{K}_n(x) \ge \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Moreover, for n even

$$\Delta_n = \left\{ x \in \mathbb{R}^n_* \colon \#\mathcal{K}_n(x) = \frac{n}{2} \right\}. \tag{10}$$

Obviously, the integral in (7) can be estimated by

$$I(t) := (t\vee 1)^n \sup_{\eta\ge 1} \int_{\mathbb{R}^n} \sup_{\eta\ge 1} \eta^{\frac{\alpha}{2}n - \alpha\#\mathcal{K}_n(x)} \prod_{k\in\mathcal{K}_n(x)} \left| \sum_{i=1}^k x_i \right|^{-\alpha} \|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x).$$

By (9) and (2), $I(t) < +\infty$, and (7) follows. Since $x \in \mathbb{R}^n \backslash \Delta_n$ implies $\mathcal{K}_n(x) > \frac{n}{2}$ identity (8) follows from (2). □

Proof of (ii) For an even n set

$$H_{n,t}(\eta) := n! \int_{\Delta_n}\int_{\mathcal{T}(n,t)} \eta^{n\alpha/2} \mathrm{e}^{-s_1|\eta x_1|^\alpha - s_2|\eta(x_1+x_2)|^\alpha - \ldots - s_n|\eta(x_1+\ldots+x_n)|^\alpha} \mu^{\otimes^n}(\mathrm{d}x)\mathrm{d}\mathbf{s}.$$

We have

$$\begin{aligned} H_{n,t}(\eta) = n! &\int_{\left\{\substack{s_2+s_4+\ldots+s_n\le t \\ s_j\ge 0}\right\}} \mathrm{d}s_2\mathrm{d}s_4\ldots\mathrm{d}s_n \int_{\left\{\substack{x_{2j-1}+x_{2j}=0 \\ j=1,\ldots,n/2}\right\}} \mu^{\otimes n}(\mathrm{d}x) \\ &\times \int_{\left\{\substack{s_1+s_3+\ldots+s_{n-1}\le t-(s_2+s_4+\ldots+s_n) \\ s_j\ge 0}\right\}} \mathrm{d}s_1\mathrm{d}s_3\ldots\mathrm{d}s_{n-1}\eta^{n\alpha/2}\mathrm{e}^{-s_1|\eta x_1|^\alpha - s_3|\eta x_3|^\alpha - \ldots - s_{n-1}|\eta x_{n-1}|^\alpha} \\ = n! &\int_{\left\{\substack{s_2+s_4+\ldots+s_n\le t \\ s_j\ge 0}\right\}} \mathrm{d}s_2\mathrm{d}s_4\ldots\mathrm{d}s_n I_n(\eta, t-(s_2+s_4+\ldots+s_n)), \end{aligned}$$

where

$$I_n(\eta,c) := \int_{\Delta_n} \mu^{\otimes n}(\mathrm{d}x) \int_{\{s_1+s_3+\ldots+s_{n-1}\le c\}} \mathrm{d}s_1\mathrm{d}s_3\ldots\mathrm{d}s_{n-1}\eta^{\frac{n\alpha}{2}}\mathrm{e}^{-s_1|\eta x_1|^\alpha}\mathrm{e}^{-s_3|\eta x_3|^\alpha}\ldots\mathrm{e}^{-s_{n-1}|\eta x_{n-1}|^\alpha}.$$

We have to show that

$$\lim_{\eta\to+\infty} H_{n,t}(\eta) = n!\left(\frac{C_{f,\alpha}}{2}\right)^{\frac{n}{2}} \int_{\left\{\substack{s_2+s_4+\ldots+s_n\le t\\ s_j\ge 0}\right\}} \mathrm{d}s_2\mathrm{d}s_4\ldots\mathrm{d}s_n$$
$$= \frac{n!}{(\frac{n}{2})!2^{\frac{n}{2}}}(C_{f,\alpha}t)^{\frac{n}{2}} = d_n(C_{f,\alpha}t)^{\frac{n}{2}}.$$

Note that by (2), $I_n(\eta,c)$ is uniformly bounded for $c\in[0,t]$. Thus it is enough to show that for any $c>0$,

$$\lim_{\eta\to+\infty} I_n(\eta,c) = \left(\frac{C_{f,\alpha}}{2}\right)^{\frac{n}{2}}. \tag{11}$$

To this end observe that for any $c>0$,

$$\int_{\{y_1+y_2=0\}} \eta^\alpha \int_0^c \mathrm{e}^{-r_1|\eta y_1|^\alpha}\mu(\mathrm{d}y_1)\mu(\mathrm{d}y_2) = \int_{\{y_1+y_2=0\}} \frac{1-\mathrm{e}^{-c|\eta y_1|^\alpha}}{|y_1|^\alpha}\mu(\mathrm{d}y_1)\mu(\mathrm{d}y_2). \tag{12}$$

As $\eta\to+\infty$, the integrand converges pointwise to $\frac{1}{|y_1|^\alpha}$ and is bounded by $\frac{1}{|y_1|^\alpha}$. Hence the expression in (12) is bounded and converges to $C_{f,\alpha}/2$. Thus (11) holds for $n=2$. Next we use induction. Suppose that (11) holds for $n-2$. We will show that it also holds for n. Since

$$I_n(\eta,c) = I_{n-2}(\eta,c)\frac{C_{f,\alpha}}{2} + J_n(\eta,c),$$

where

$$J_n(\eta,c) := \int_{\Delta_n}\mu^{\otimes n}(\mathrm{d}x)\int_{\{s_1+\ldots+s_{n-3}\le c\}} \mathrm{d}s_1\ldots\mathrm{d}s_{n-3}\eta^{\frac{(n-2)\alpha}{2}}\mathrm{e}^{-s_1|\eta x_1|^\alpha}\ldots\mathrm{e}^{-s_{n-3}|\eta x_{n-3}|^\alpha}$$
$$\times\left[\int_0^{c-(s_1+\ldots+s_{n-3})}\eta^\alpha \mathrm{e}^{-s_{n-1}|\eta x_{n-1}|^\alpha}\mathrm{d}s_{n-1} - \frac{1}{|x_{n-1}|^\alpha}\right],$$

the proof will be completed as we show that $J_n(\eta,c)\to 0$ as $\eta\to+\infty$. We have

$$\begin{aligned}|J_n(\eta,c)| &\le \int_{\Delta_n}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)\int_{\{s_1+\ldots+s_{n-3}\le c\}}\mathrm{d}s_1\ldots\mathrm{d}s_{n-3}\\
&\qquad\times\eta^{\frac{(n-2)\alpha}{2}}\mathrm{e}^{s_1|\eta x_1|^\alpha}\ldots\mathrm{e}^{s_{n-3}|\eta x_{n-3}|^\alpha}\frac{1}{|x_{n-1}|^\alpha}\mathrm{e}^{(-c+s_1+\ldots+s_{n-3})|\eta x_{n-1}|^\alpha}\\
&\le \int_{\Delta_n}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)\int_{\{s_1+\ldots+s_{n-3}\le c/2\}}\mathrm{d}s_1\ldots\mathrm{d}s_{n-3}\\
&\qquad\times\eta^{\frac{(n-2)\alpha}{2}}\mathrm{e}^{s_1|\eta x_1|^\alpha}\ldots\mathrm{e}^{s_{n-3}|\eta x_{n-3}|^\alpha}\frac{1}{|x_{n-1}|^\alpha}\mathrm{e}^{-(c/2)|\eta x_{n-1}|^\alpha}\\
&\quad+\int_{\Delta_n}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)\int_{\{s_1+\ldots+s_{n-3}> c/2\}}\mathrm{d}s_1\ldots\mathrm{d}s_{n-3}\\
&\qquad\times\eta^{\frac{(n-2)\alpha}{2}}\mathrm{e}^{s_1|\eta x_1|^\alpha}\ldots\mathrm{e}^{s_{n-3}|\eta x_{n-3}|^\alpha}\frac{1}{|x_{n-1}|^\alpha}\\
&\le \int_{\Delta_n}\frac{1}{|x_1|^\alpha\ldots|x_{n-1}|^\alpha}\mathrm{e}^{-(c/2)|\eta x_{n-1}|^\alpha}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x)\\
&\quad+\int_{\Delta_n}\frac{1}{|x_1|^\alpha\ldots|x_{n-1}|^\alpha}\mathrm{e}^{-(c/n)|\eta x_1|^\alpha}\|\mu\|^{\otimes n}_{\mathrm{Var}}(\mathrm{d}x),\end{aligned}$$

where in the last estimate we have used the fact that if $s_1+\ldots+s_{n-3}>c/2$, then at least one s_j is greater than c/n. By the dominated convergence theorem and assumption (2) both terms appearing in the estimate above converge to 0. □

4. Proof Theorem 2.5

Assume that $\{\mathcal{X}_n\}$ is a sequence of random elements in $C([0,+\infty))$. Let us recall (see [4], Theorems 8.1, 12.3) that the sequence of laws $\mathcal{L}(\mathcal{X}(n))$ converges weakly in $C([0,+\infty))$ to the law $\mathcal{L}(\mathcal{X})$ of a random element $\mathcal{X}$ in $C([0,+\infty))$ if

(i) for all $0\le t_1<t_2<\ldots<t_k<+\infty$, $\mathcal{L}\left(\mathcal{X}_{t_1}(n),\ldots,\mathcal{X}_{t_k}(n)\right)$ converges weakly to $\mathcal{L}\left(\mathcal{X}_{t_1},\ldots,\mathcal{X}_{t_k}\right)$,

(ii) for any $T\in(0,+\infty)$ there is a constant $C<+\infty$ such that for all n and $t,s\in[0,T]$,

$$\mathbb{E}\left(\mathcal{X}_t(n)-\mathcal{X}_s(n)\right)^4\le C\,|t-s|^2.$$

Convergence of finite dimensional distributions Since Gaussian laws are determined by their moments, the convergence of finite dimensional distributions follows from the convergence of all moments. Thus it suffices to show that for any $k,n_1,\ldots,n_k\in\mathbb{N}$ and any $0=t_0<t_1<\ldots<t_k$, we have

$$\lim_{\eta\to+\infty}\eta^{\frac{\alpha}{2}\sum_{j=1}^k n_j}\mathbb{E}\prod_{j=1}^k\left(\mathcal{A}^{\eta X}_{t_j}(f)-\mathcal{A}^{\eta X}_{t_{j-1}}(f)\right)^{n_j}=(C_{f,\alpha})^{\sum_{j=1}^k\frac{n_j}{2}}\prod_{j=1}^k d_{n_j}(t_j-t_{j-1})^{n_j/2}. \tag{13}$$

Observe that if $r < s$ and $\mathcal{F}_r := \sigma(X_u \colon u \le r)$ then, by the definition of f, and since X has stationary independent increments, we have

$$
\begin{aligned}
&\eta^{n\alpha/2}\mathbb{E}\left(\left(\mathcal{A}^{\eta X}_s(f) - \mathcal{A}^{\eta X}_r(f)\right)^n \,\middle|\mathcal{F}_r\right)\\
&\quad = \eta^{n\alpha/2}\mathbb{E}\left(\left(\int_r^s f(\eta(X_u - X_r) + \eta X_r)\mathrm{d}u\right)^n \,\middle|\mathcal{F}_r\right) \qquad (14)\\
&\quad = J_\eta(n, s-r, X_r),
\end{aligned}
$$

where

$$
J_\eta(n,t,y) := \eta^{n\alpha/2}\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \mathrm{e}^{i\eta X_u x}\mathrm{e}^{i\eta y x}\mu(\mathrm{d}x)\mathrm{d}u\right)^n .
$$

As in the proof of (6) we obtain

$$
J_\eta(n,t,y) = n!\int_{\mathbb{R}^n}\int_{\mathcal{T}(n,t)} \mathrm{e}^{\mathrm{i}\eta y(x_1+\ldots+x_n)}\mathrm{e}^{-s_1|\eta x_1|^\alpha - s_2|\eta(x_1+x_2)|^\alpha - \ldots - s_n|\eta(x_1+\ldots+x_n)|^\alpha}\mu^{\otimes n}(\mathrm{d}x)\mathrm{d}\mathbf{s}.
$$

Therefore, for any y, by Lemma 3.1 (i) we obtain that if n is odd, then $\lim_{\eta\to\infty} J_\eta(n,t,y) = 0$, and if n is even, then, since $x_1 + \ldots + x_n = 0$ on Δ_n, by Lemma 3.1 (ii) we have

$$
\begin{aligned}
&\lim_{\eta\to+\infty} J_\eta(n,t,y)\\
&\quad = \lim_{\eta\to\infty} n!\int_{\Delta_n}\int_{\mathcal{T}(n,t)} \eta^{n\alpha/2}\mathrm{e}^{-s_1|\eta x_1|^\alpha - s_2|\eta(x_1+x_2)|^\alpha - \ldots - s_n|\eta(x_1+\ldots+x_n)|^\alpha}\mu^{\otimes n}(\mathrm{d}x)\mathrm{d}\mathbf{s} \qquad (15)\\
&\quad = (C_{f,\alpha}t)^{\frac{n}{2}}\, d_n.
\end{aligned}
$$

Moreover, by Lemma 3.1(i), $J_\eta(n,t,y)$ is bounded. To obtain (13) we use induction on k. If $k = 1$ then

$$
\eta^{\frac{\alpha}{2}n_1}\mathbb{E}\left(\ \mathcal{A}^{\eta X}_{t_1}(f) - \mathcal{A}^{\eta X}_{t_0}(f)\right)^{n_1} = \mathbb{E}\, J_\eta(n_1, t_1 - t_0, X_{t_0})
$$

and, since J_η is bounded, (13) follows from (15). Suppose that (13) holds for $k-1$, then

$$
\begin{aligned}
&\eta^{\frac{\alpha}{2}\sum_{j=1}^k n_j}\mathbb{E}\prod_{j=1}^k\left(\ \mathcal{A}^{\eta X}_{t_j}(f) - \mathcal{A}^{\eta X}_{t_{j-1}}(f)\right)^{n_j}\\
&\quad = \eta^{\frac{\alpha}{2}\sum_{j=1}^{k-1} n_j}\mathbb{E}\prod_{j=1}^{k-1}\left(\ \mathcal{A}^{\eta X}_{t_j}(f) - \mathcal{A}^{\eta X}_{t_{j-1}}(f)\right)^{n_j} J_\eta(n_k, t_k - t_{k-1}, X_{t_k})\\
&\quad = \eta^{\frac{\alpha}{2}\sum_{j=1}^{k-1} n_j}\mathbb{E}\prod_{j=1}^{k-1}\left(\ \mathcal{A}^{\eta X}_{t_j}(f) - \mathcal{A}^{\eta X}_{t_{j-1}}(f)\right)^{n_j} d_{n_k}(C_{f,\alpha}(t_k - t_{k-1}))^{\frac{n_k}{2}} + R_\eta,
\end{aligned}
$$

where

$$
R_\eta := \eta^{\frac{\alpha}{2}\sum_{j=1}^{k-1} n_j}\mathbb{E}\prod_{j=1}^{k-1}\left(\ \mathcal{A}^{\eta X}_{t_j}(f) - \mathcal{A}^{\eta X}_{t_{j-1}}(f)\right)^{n_j}\left[J_\eta(n_k, t_k - t_{k-1}, X_{t_{k-1}}) - d_{n_k}(C_{f,\alpha}(t_k - t_{k-1}))^{\frac{n_k}{2}}\right]
$$

By the Schwarz inequality

$$|R_\eta| \le \sqrt{\mathbb{E}\left(\eta^{\frac{\alpha}{2}\sum_{j=1}^{k-1} n_j} \prod_{j=1}^{k-1} \left(\mathcal{A}_{t_j}^{\eta X}(f) - \mathcal{A}_{t_{j-1}}^{\eta X}(f)\right)^{n_j}\right)^2} \times \sqrt{\mathbb{E}\left(J_\eta(n_k, t_k - t_{k-1}, X_{t_{k-1}}) - d_{n_k}(C_{f,\alpha}(t_k - t_{k-1}))^{\frac{n_k}{2}}\right)^2}.$$

Since J_η is bounded, the right hand side of the inequality above tends to zero by (15).

Tightness in $C([0,T])$. Let $0 \le r < s \le T$. By (14),

$$\mathbb{E}\, \eta^{2\alpha} \left(\mathcal{A}_s^{\eta X}(f) - \mathcal{A}_r^{\eta X}(f)\right)^4 = \mathbb{E}\, J_\eta(4, s-r, X_r). \tag{16}$$

Using (15) and (2) we obtain

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} |J_\eta(4, s-r, y)| \\
&\le 4!(s-r)^2 \int_{\mathbb{R}^4 \cap \{x:\, x_1+x_2=0\}} \|\mu\|_{\mathrm{Var}}^{\otimes 4}(\mathrm{d}x) \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_3 \eta^{2\alpha} \mathrm{e}^{-s_1|\eta x_1|^\alpha - s_3|\eta x_3|^\alpha} \\
&\quad + 4!(s-r)^2 \int_{\mathbb{R}^4 \cap \{x:\, x_1+x_2\neq 0\}} \|\mu\|_{\mathrm{Var}}^{\otimes 4}(\mathrm{d}x) \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \eta^{2\alpha} \mathrm{e}^{-s_1|\eta x_1|^\alpha - s_2|\eta(x_1+x_2)|^\alpha} \\
&\le 4!(s-r)^2 \left(\int_{\mathbb{R}^4 \cap \{x:\, x_1+x_2=0\}} \frac{1}{|x_1|^\alpha\, |x_3|^\alpha} \|\mu\|_{\mathrm{Var}}^{\otimes 4}(\mathrm{d}x) \right. \\
&\qquad \left. + \int_{\mathbb{R}^4 \cap \{x:\, x_1+x_2\neq 0\}} \frac{1}{|x_1|^\alpha\, |x_1+x_2|^\alpha} \|\mu\|_{\mathrm{Var}}^{\otimes 4}(\mathrm{d}x) \right) \\
&\quad \le C(s-r)^2,
\end{aligned}$$

and hence (16) gives the desired estimate. □

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