Time regularity of solutions to linear equations with Lévy noise in infinite dimensions

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Abstract

The existence of strong and weak càdlàg versions of a solution to a linear equation in a Hilbert space H, driven by a Lévy process taking values in a Hilbert space $U \hookleftarrow H$ is established. The so-called cylindrical càdlàg property is investigated as well. A special emphasis is put on infinite systems of linear equations driven by independent Lévy processes.

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1. Introduction

This paper is concerned with the time regularity of a solution to the following linear evolution equation

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}Z, \qquad X(0) = 0, \tag{1}$$

where A generates a C_0-semigroup S on a Hilbert space H and Z is a Lévy process taking values in a Hilbert space U.

Note that (1) includes linear parabolic and hyperbolic equations perturbed by an infinite dimensional stochastic process. To work with the solution having Markov property one has to assume (see [19], Chapter 1) that Z is a Lévy process. A good theory for linear equations is necessary to study the nonlinear problems of the type

$$\mathrm{d}X = (AX + F(X))\,\mathrm{d}t + \mathrm{d}Z, \qquad X(0) = x \in H.$$

These problems are even more important from the application point of view.

It is not difficult to formulate necessary and sufficient conditions under which the solution to (1) exists in H and/or is mean square continuous; see Proposition 2.6. However for a deeper study of the solution it is necessary to find out how regular are its trajectories and this is the main purpose of the present study.

Since the case of Z being a Wiener process is rather well understood (see [7, 8, 11, 3]) we assume that Z is without the Gaussian part. Moreover if the process Z takes values in the space $U \hookrightarrow H$, then, under rather mild assumptions on the semigroup S, the solution has a càdlàg modification due to the classical Kotelenez results (see [14, 15], and [10]). This is obviously the best possible time regularity result for an equation driven by a jump noise. It is however important to consider the case when Z lives in a bigger space $U \hookleftarrow H$; see e.g. [1]. For instance to establish the strong Feller property of the corresponding transition semigroup the inclusion $U \hookleftarrow H$ is often very helpful. We will limit our considerations to this case only. Obviously to have the solution taking values in H we will have to assume that the semigroup S exhibits some regularising property. Namely we assume that for each $t > 0$, $S(t)$ has a (unique) extension to a bounded linear operator, denoted also by $S(t)$, from the space in which Z lives to H. We assume that the solution X takes values in H and is given in the so-called mild form:

$$X(t) = \int_0^t S(t-s)\mathrm{d}Z(s), \qquad t \geq 0.$$

Path properties of X depend heavily on the jump measure ν of Z. Namely if $\nu(U\setminus H) > 0$, then regardless the generator A, even if X is square integrable in H, there is a non empty set of vectors $z \in H$ such that for any $T > 0$, with non zero, or even in many cases with probability 1, the trajectories of the real-valued process $\langle X(t), z\rangle_H$, $t \in [0,T]$, are unbounded; see [4] and [19], Proposition 9.25. Therefore we will assume that the Lévy measure ν of Z satisfies

$$\nu(U \setminus H) = 0. \tag{2}$$

One obtains an interesting class of equations, including the perturbed heat equation, assuming that A is a negative definite operator with eigenvectors (e_n), forming an orthonormal and complete basis in H, and with the corresponding, positive eigenvalues (γ_n). If in addition

$$Z = \sum_{n=1}^{\infty} Z_n e_n, \tag{3}$$

where (Z_n) are independent real-valued Lévy processes each with a Lévy measure μ_n, then equations of this form will be called of the *diagonal type*. In fact then

$$X(t) = \sum_{n=1}^{\infty} X_n(t) e_n,$$

where

$$\mathrm{d}X_n = -\gamma_n X_n \mathrm{d}t + \mathrm{d}Z_n, \qquad X_n(0) = 0.$$

One can thus identify X with the sequence of processes (X_n) and the space H with l^2. Due to the independence of the processes Z_n, the Lévy measure ν of Z is always supported on the set sum $\bigcup_n \mathbb{R}e_n$ and obviously hypothesis (2) is satisfied.

Condition (2) imposed on the Lévy measure of the noise, does not guarantee that X has càdlàg trajectories in H. In fact it was shown recently, for a large class of equations satisfying (2), including a subclass of diagonal ones, that their solutions live in H but do not allow a càdlàg modification; see [2], and also Theorem 2.3, Propositions 4.1, 4.3, and Corollary 4.2 of the present paper. Moreover, it was shown by Liu and Zhai [17], see also Remark 3.5, for the diagonal systems with Z_n being independent α-stable processes that if X is càdlàg in H then necessarily the process Z takes values in H.

These results are in sharp contrast with what one could expect from the results on the equations with Gaussian noise. The càdlàg property for the equations with Lévy perturbations is much less frequent than the continuity of the trajectories in the Gaussian case; see [7, 8, 11, 3]. This is one of the reasons that in the jump case more sophisticated concepts of regularity might be useful. In fact we will deal with the following properties.

Definition 1.1. *i*) We say that a process X has a *càdlàg modification* if there is a modification $\tilde{X}$ of X with càdlàg trajectories; that is right continuous and having left limit at any point.
ii) We say that an H-valued process X has a *weakly càdlàg modification* if there is a modification $\tilde{X}$ of X such that for any $z \in H$, the real-valued process $\langle \tilde{X}(\cdot), z\rangle_H$ has càdlàg trajectories.
iii) An H-valued process X is *cylindrical càdlàg* if for any $z \in H$, the real-valued process $\langle X(\cdot), z\rangle$ has a càdlàg modification.
iv) Let $V \hookrightarrow H$. An V-valued process X is V-*cylindrical càdlàg* if for all $v^* \in V^*$ the real-valued process $v^*X(\cdot)$ has a càdlàg modification.

We have the following obvious implications

$$i) \Longrightarrow ii) \Longrightarrow iii) \Longleftarrow iv)$$

The càdlàg property is fundamental for establishing the strong Markov property of the solution and for various localisation procedures. It allows to formulate and study the exit time τ_D from a given set D. The existence of a weakly càdlàg modification ensures at least the local boundedness of trajectories; see Theorem 2.3(ii). For solutions which are cylindrically càdlàg or V-cylindrically càdlàg, the exit times are meaningful for a large class of cylindrical sets of the form $D = \{x \in H \colon \Pi x \in G\}$ or $D = \{x \in V \colon \Pi x \in G\}$ where Π is a finite projection in H or in V, respectively.

We will describe now the main results of the paper. Results on the càdlàg property are formulated and discussed in Section 3. Our main result is Theorem 3.1. We restrict our attention to the case when the operator A generates an analytic semigroup and thus to parabolic type of equations. The case of hyperbolic equations leads to the group, rather than to semigroup S, and therefore requires the noise taking values in the state space. We show that the class of equations having càdlàg solutions in H but with the noise process evolving outside H is non empty and of some interest; see e.g. Example 3.4. Thus, the phenomenon encountered in the case of α-stable perturbations by Liu and Zhai [17], and described above, is not valid in general. Our results rely on the classical Chentsov work [5] or [9] and on formulae for moments of integrals with respect to Poissonian random measures gathered in the preliminaries. As a byproduct we obtain also the estimates of the type

$$\mathbb{E} \sup_{0\le t\le T} |X(t)|_H^p < \infty.$$

We note that the classical result of Kinney see [13] or [19], Theorem 3.23, on càdlàg version of a Markov process would require that

$$\lim_{t\downarrow 0}\sup_{x\in H}\mathbb{P}\left(\left|S(t)x+\int_0^t S(t-s)\mathrm{d}Z(s)-x\right|_H>r\right)=0, \qquad \forall\, r>0. \tag{4}$$

Obviously (4) is satisfied if and only if $A=0$. Assume (4). Then putting $x=0$ we obtain

$$\lim_{t\downarrow 0}\mathbb{P}\left(\left|\int_0^t S(t-s)\mathrm{d}Z(s)\right|_H>r\right)=0, \qquad \forall\, r>0.$$

Therefore $|S(t)x-x|_H$ has to converge to 0 uniformly in x from the whole space H. It is impossible as $\sup_{x\in H}|S(t)x-x|_H=\infty$ if $S(t)$ is not the identity.

We do not treat in this paper an interesting question under which conditions the solution X has a càdlàg or weakly càdlàg modification in a bigger space $\tilde{H}\hookleftarrow H$. For some answers to these questions we refer the reader to [16].

Results on the three remaining càdlàg properties are contained in Sections 4–6 and are established only for the diagonal systems. They are rather technical and involve long formulae. Extensions to the general case however would be possible if needed.

The main result on weak càdlàg property is formulated as Proposition 4.1 and is of negative character. It states sufficient conditions for non-existence of weakly càdlàg solutions in terms of the tails of the measures μ_n. Its proof is based on the general if and only if conditions for weak càdlàg property, formulated in terms of the orthogonal expansions and established in the preliminaries as Theorem 2.3. It turns out that also weak càdlàg property implies severe restrictions on the parameters of the equations. In particular the case of independent and identically distributed coordinates of the noise process (cylindrical noise) is excluded. At this moment we do not have any example of linear system (1) whose solution has no càdlàg but only a weakly càdlàg modification.

Fortunately the cylindrical càdlàg property takes place under much weaker assumptions and in particular the coordinates of the noise can be identically distributed. Sufficient conditions for this property are formulated in Theorems 5.1 and 5.2. It is worth noting that to show the cylindrical càdlàg property of X one cannot apply directly the Chentsov result (see Remark 8.1) but one needs a new way of circumventing that obstacle.

Results on V-cylindrical càdlàg property are presented in Section 6. The main result, Theorem 5.2, provides sufficient conditions for that property. As a consequence of Theorem 5.2 and Proposition 4.3 we see that even if the embedding $V \hookrightarrow H$ is compact, then V-cylindrical càdlàg property does not ensure the existence of a weakly càdlàg modification. Therefore the implication

$$ii) \Longleftarrow iv)$$

does not hold in general.

We complete this introduction with some open questions.

Question 1 We have conditions for the existence of càdlàg modifications and conditions which exclude the existence of a weakly càdlàg modification. Find applicable conditions leading to weak but not strong càdlàg property. In particular find example of a linear equation whose solution admits a weakly càdlàg but not càdlàg modification.

Question 2 In our investigation we rely on the Chentsov criteria with the exponent $p = 2$; see Corollary 2.2. Find extensions of our results for $p \neq 2$ or more generally using the function g appearing in Theorem 2.1 not of the power type.

Question 3 Find conditions for cylindrical càdlàg properties in the non-diagonal case.

Question 4 Find conditions for the càdlàg or cylindrical càdlàg property in case of the noise with tails (see Corollaries 3.2 and 5.3). In particular in diagonal case assume that $Z_n = \sigma_n L_n$ where L_n are independent symmetric α-stable processes and

$$\sum_{n=1}^{\infty} \frac{\sigma_n^{\alpha}}{\gamma_n} < \infty.$$

Is it true that the process X is cylindrical càdlàg?

2. Preliminary results

2.1. Càdlàg criteria in metric spaces

We first recall some basic facts on path regularity of stochastic processes in metric spaces. They can be attributed to N.N. Chentsov, but the exposition is based on the book of Gihman and Skorohod [9].

Let $\xi = (\xi(t),\, t \in [0,T])$ be a separable process taking values in a metric space (U,ρ). We extent ξ on $\mathbb{R}$ putting $\xi(t) = \xi(0)$ for $t < 0$ and $\xi(t) = \xi(T)$ for $t \geq T$. The following result holds (see [9], Lemma 3 and Theorem 1 of Chapter 3).

Theorem 2.1. *Assume that there are an increasing function $g\colon (0,\infty) \mapsto (0,\infty)$ and a function $q\colon (0,\infty)\times(0,\infty) \mapsto (0,\infty)$ such that for all $C, h > 0$,*

$$\mathbb{P}\left\{[\rho(\xi(t),\xi(t-h)) > Cg(h)] \cap [\rho(\xi(t),\xi(t+h)) > Cg(h)]\right\} \leq q(C,h),$$

and

$$G := \sum_{n=1}^{\infty} g(T2^{-n}) < \infty, \qquad Q(C) := \sum_{n=1}^{\infty} 2^n q(C, T2^{-n}) < \infty.$$

Then with probability 1, ξ has no discontinuities of the second kind, and for any $N > 0$,

$$\mathbb{P}\left\{\sup_{t,s\in[0,T]} \rho(\xi(t),\xi(s)) > N\right\} \leq \mathbb{P}\left\{\rho(\xi(0),\xi(T)) > \frac{N}{2G}\right\} + Q\left(\frac{N}{2G}\right).$$

Corollary 2.2. *Assume that there are $p, r, K > 0$ such that for all $t \in [0,T]$ and $h > 0$,*

$$\mathbb{E}\left[\rho\left(\xi(t),\xi(t-h)\right)\rho\left(\xi(t),\xi(t+h)\right)\right]^p \leq Kh^{1+r}. \tag{5}$$

Then with probability 1, ξ has no discontinuities of the second kind. Moreover, for any $1 \leq q < 2p$,

$$\mathbb{E}\sup_{t,s\in[0,T]} \left(\rho(\xi(t),\xi(s))\right)^q \leq (2G)^q\,\mathbb{E}\left(\rho(\xi(T),\xi(0))\right)^q + R, \tag{6}$$

where $0 < r' < r$,

$$G = \sum_{n=1}^{\infty} (T2^{-n})^{r'/(2p)} < \infty, \tag{7}$$

and $R := 1 + \frac{q}{2p-q}\frac{K(2G)^{2p}T^{1+r-r'}}{1-2^{r'-r}}$.

Proof Let $0 < r' < r$. Then the assumptions of Theorem 2.1 hold with

$$g(h) := h^{r'/(2p)} \qquad \text{and} \qquad q(C,h) := \frac{K}{C^{2p}} h^{1+r-r'}.$$

By Chebyshev's inequality

$$\begin{aligned}
&\mathbb{P}\left\{[\rho(\xi(t),\xi(t-h)) > Cg(h)] \cap [\rho(\xi(t),\xi(t+h)) > Cg(h)]\right\} \\
&\quad \le \mathbb{P}\left\{\rho(\xi(t),\xi(t-h))\rho(\xi(t),\xi(t+h)) > C^2 g^2(h)\right\} \\
&\quad \le \frac{Kh^{1+r}}{C^{2p}g^{2p}(h)} = \frac{Kh^{1+r-r'}}{C^{2p}}.
\end{aligned}$$

Note that in this case G is given by (7), and

$$\begin{aligned}
Q\left(\frac{N}{2G}\right) &= \sum_{n=1}^{\infty} 2^n \frac{K(2G)^{2p}}{N^{2p}} (T2^{-n})^{1+r-r'} = \frac{K(2G)^{2p}}{N^{2p}} T^{1+r-r'} \sum_{n=1}^{\infty} 2^{-n(r-r')} \\
&= \frac{K(2G)^{2p} T^{1+r-r'}}{1-2^{r'-r}} N^{-2p}.
\end{aligned}$$

To show (6) take $q \ge 1$. Since

$$\mathbb{E} \sup_{t,s\in[0,T]} (\rho(\xi(t),\xi(s)))^q = q\int_0^\infty \mathbb{P}\left\{\sup_{t,s\in[0,T]} \rho(\xi(t),\xi(s)) \ge N\right\} N^{q-1}\mathrm{d}N,$$

Theorem 2.1 yields

$$\mathbb{E} \sup_{t,s\in[0,T]} (\rho(\xi(t),\xi(s)))^q \le (2G)^q \mathbb{E}\left(\rho(\xi(T),\xi(0))\right)^q + 1 + q\int_1^\infty Q\left(\frac{N}{2G}\right) N^{q-1}\mathrm{d}N,$$

and consequently (6). $\square$

2.2. Criteria for càdlàg properties in Hilbert spaces

Let now (X_n) be a sequence of real-valued càdlàg processes defined on a finite time interval $[0,T]$, and let (e_n) be an orthonormal basis of a Hilbert space H. Assume that for each $t \in [0,T]$,

$$X(t) = \sum_{n=1}^{\infty} X_n(t) e_n,$$

where the series converges in probability or equivalently $\mathbb{P}$-a.s. The first part of the theorem below is taken from the paper by Liu and Zhai [17]. It will not be used in the paper but it is included for a more complete picture. In the proofs we follow suggestions by A. Jakubowski [12].

Theorem 2.3. (i) *Process X has a càdlàg modification if and only if*

$$\mathbb{P}\left(\lim_{N\to\infty}\sup_{t\in[0,T]}\sum_{n=N}^{\infty} X_n^2(t)=0\right)=1. \tag{8}$$

(ii) *Process X has a weakly càdlàg modification if and only if*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{n=1}^{\infty} X_n^2(t)<\infty\right)=1. \tag{9}$$

Proof Since each X_n is càdlàg, (8) implies that X has a càdlàg modification. Therefore we need to show that (8) follows from the existence of a càdlàg modification $\tilde{X}$ of X. To see this note that on a dense set $Q\subset[0,T]$, $X=\tilde{X}$, $\mathbb{P}$-a.s. Consequently, since each X_n is càdlàg,

$$\langle \tilde{X}(t), e_n\rangle_H = X_n(t) = \langle X(t), e_n\rangle_H, \qquad \forall\, t\in[0,T],\ \mathbb{P}-a.s.$$

Therefore there is an $\Omega_0\subset\Omega$ such that $\mathbb{P}(\Omega_0)=1$ and for any $\omega\in\Omega_0$,

$$[0,T]\ni t\mapsto \tilde{X}(t;\omega)\in H$$

is càdlàg and $\langle\tilde{X}(t;\omega),e_n\rangle_H = X_n(t;\omega)$. By càdlàg property, for any $\omega\in\Omega_0$, the set $\{X(t;\omega)\colon t\in[0,T]\}$ is compact in H. Therefore the desired conclusion follows from the following criterion for a relative compactness of a bounded set $\mathcal{K}$ in H;

$$\lim_{N\to\infty}\sup_{x\in\mathcal{K}}\sum_{n=N}^{\infty}\langle x, e_n\rangle_H = 0.$$

To see that (9) implies the weakly càdlàg property of X take a $z\in H$. One has to show that

$$[0,T]\ni t\mapsto \langle X(t), z\rangle_H\in\mathbb{R}$$

is càdlàg $\mathbb{P}$-a.s. Let $\tilde{\Omega}$ be the set of all $\omega\in\Omega$ such that

$$\sup_{t\in[0,T]}\sum_{n=1}^{\infty} X_n^2(t;\omega)<\infty.$$

Then

$$X(t;\omega)=\sum_{n=1}^{\infty} X_n(t;\omega)e_n$$

is a bounded H-valued mapping of $t \in [0, T]$. Moreover,

$$\langle X(t;\omega), e_k\rangle_H = X_k(t;\omega), \qquad t \in [0, T],$$

are càdlàg functions. Let $t \in [0, T)$ and let $t_m \downarrow t$. Then $H \ni z \mapsto \langle X(t_m;\omega), z\rangle_H$, $m = 1, 2, \ldots$, is a sequence of linear functionals, converging on a dense set. Since their norms are bounded $\langle X(t_m;\omega), z\rangle_H$, $m = 1, 2, \ldots$, converges for any $z \in H$. Since

$$\lim_{m\to\infty} \langle X(t_m;\omega), z\rangle_H = \langle X(t;\omega), z\rangle_H$$

holds on a dense set of $z \in H$ it holds for any $z \in H$. Therefore $X(\cdot;\omega)$ is weakly right continuous.

Now let $t \in (0, T]$ and let $t_m \uparrow t$. Then

$$\lim_{m\to\infty} \langle X(t_m;\omega), e_k\rangle_H = X_k(t-;\omega), \qquad k \in \mathbb{N}.$$

Boundedness of $X(t_m;\omega)$, $m \in \mathbb{N}$, implies that the weak limit of $X(t_m;\omega)$ exists. Since it holds for any sequence $t_m \uparrow t$, the weak left limit $X(t-;\omega)$ exists.

Assume now that $\tilde{X}$ is a weakly càdlàg modification of X. We will show (9). To do this observe that

$$\tilde{X}_n(t;\omega) = \langle \tilde{X}(t;\omega), e_n\rangle_H, \qquad t \in [0, T],\ n = 1, 2, \ldots,$$

are càdlàg functions. Moreover, for any $t \in [0, T]$,

$$\mathbb{P}\left(\omega \in \Omega\colon \tilde{X}_n(t;\omega) = X_n(t;\omega)\right) = 1.$$

Since both processes $\tilde{X}_n$ and X_n are càdlàg, therefore there is a set $\tilde{\Omega} \subset \Omega$ such that $\mathbb{P}(\tilde{\Omega}) = 1$ and

$$\tilde{X}_n(t;\omega) = X_n(t;\omega), \qquad \forall\, t \in [0, T],\ \forall\, n,\ \forall\, \omega \in \tilde{\Omega}.$$

Since

$$\sup_{t\in[0,T]} \sum_{n=1}^{\infty} \tilde{X}_n^2(t;\omega) < \infty, \qquad \mathbb{P} - a.s.,$$

(9) holds. $\square$

2.3. Moments of stochastic integral

In the proposition below $\widehat{\pi}$ is a compensated Poisson random measure on a measurable space E with the intensity measure ν.

Proposition 2.4. *Assume that deterministic real-valued measurable functions $f_1, \ldots, f_4$ have finite forth moments with respect to ν. Then we have*

$$\begin{aligned}
&\mathbb{E} \int_E f_1(x)\widehat{\pi}(\mathrm{d}x) \int_E f_2(x)\widehat{\pi}(\mathrm{d}x) \int_E f_3(x)\widehat{\pi}(\mathrm{d}x) \int_E f_4(x)\widehat{\pi}(\mathrm{d}x) \\
&\quad = \int_E f_1(x)f_2(x)\nu(\mathrm{d}x) \int_E f_3(x)f_4(x)\nu(\mathrm{d}x) \\
&\qquad + \int_E f_1(x)f_3(x)\nu(\mathrm{d}x) \int_E f_2(x)f_4(x)\nu(\mathrm{d}x) \\
&\qquad + \int_E f_1(x)f_4(x)\nu(\mathrm{d}x) \int_E f_2(x)f_3(x)\nu(\mathrm{d}x) \\
&\qquad + \int_E f_1(x)f_2(x)f_3(x)f_4(x)\nu(\mathrm{d}x).
\end{aligned}$$

A simple proof of the proposition above can be obtained by consecutive differentiation of the characteristic function

$$\begin{aligned}
F(x_1, x_2, x_3, x_4) &= \mathbb{E}\exp\left\{ i \sum_{j=1}^{4} x_j \int_E f_j(x)\widehat{\pi}(\mathrm{d}x) \right\} \\
&= \exp\left\{ \int_E \left(\mathrm{e}^{i\sum_{j=1}^{4} x_j f_j(x)} - i\sum_{j=1}^{4} x_j f_j(x) - 1 \right) \nu(\mathrm{d}x) \right\}.
\end{aligned}$$

In the paper we will also need the following special case of Proposition 2.4. Namely, let L be a real-valued purely jump Lévy process with a Lévy measure μ. We assume that μ has finite moments up to order 4. Write

$$m_j := \int_{\mathbb{R}} |y|^j \mu(\mathrm{d}y), \qquad j = 1, \ldots, 4.$$

Then $L(t)$, $t \geq 0$, can be written as the sum of a drift term mt and a pure jump Lévy martingale with the Lévy exponent

$$\Psi(z) = \int_{\mathbb{R}} \left(1 + izy - e^{izy}\right) \mu(\mathrm{d}y). \tag{10}$$

For our purposes we can assume that the drift term vanishes and consequently that the Lévy exponent of L is given by (10).

Proposition 2.5. *For any $T < \infty$ and continuous deterministic functions f_1, f_2,*

$$\mathbb{E}\left(\int_0^T f_1(s)\mathrm{d}L(s)\right)^2\left(\int_0^T f_2(s)\mathrm{d}L(s)\right)^2$$
$$= 2m_2^2\left(\int_0^T f_1(s)f_2(s)ds\right)^2 + m_2^2\int_0^T f_1^2(s)ds\int_0^T f_2^2(s)ds$$
$$+ m_4\int_0^T f_1^2(s)f_2^2(s)ds.$$

2.4. Criteria for evolution of OU processes in H

In the proposition below we are concerned with X given by (1). Let π be the random jump measure of Z, and let

$$\widehat{\pi}(\mathrm{d}s, \mathrm{d}z) := \pi(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s$$

be the compensated random measure. By the Lévy–Khinchin representation formula (see e.g. [19], Theorem 6.8) the process Z can be written as follows

$$Z(t) = mt + \int_0^t\int_{\{|z|_U>1\}} z\pi(\mathrm{d}s, \mathrm{d}z) + \int_0^t\int_{\{|z|_U\le 1\}} z\widehat{\pi}(\mathrm{d}s, \mathrm{d}z), \qquad t \ge 0. \quad (11)$$

Proposition 2.6. *Assume that Z is a Lévy process in a Hilbert space $U \hookleftarrow H$ with representation* (11). *Assume that the Lévy measure ν of Z satisfies $\nu(U \setminus H) = 0$ and that the drift term $m = 0$. Then we have the following.*
(i) Process Z takes values in H if and only if

$$\int_H |z|_H^2 \wedge 1\nu(\mathrm{d}z) < \infty, \qquad t \ge 0.$$

(ii) Process X takes values in H if and only if for any $t > 0$,

$$\int_0^t\int_H |S(s)z|_H^2 \wedge 1\mathrm{d}s\nu(\mathrm{d}z) < \infty,$$

$$\int_0^t\int_U \left[\chi_B(v) - \chi_B(S(s)v)\right] S(s)v\nu(\mathrm{d}v)ds \in H,$$

where $B := \{x \in U\colon |x|_U \le 1\}$.

(iii) Assume that $\int_{\{|z|_U>1\}}|z|_U\nu(\mathrm{d}z)<\infty$ *and that* $\int_0^t S(s)a\;ds\in H$, $t>0$, *where*

$$a=\int_{\{|z|_U>1\}} z\nu(\mathrm{d}z).$$

Then X *has finite second moment in* H *if and only if*

$$\int_0^t\int_H |S(s)z|_H^2\,\mathrm{d}s\nu(\mathrm{d}z)<\infty,\qquad t\geq 0. \tag{12}$$

Moreover, if (12) *holds then* X *is mean square continuous in* H.

Proof The first part follows directly from the Lévy–Khinchin theorem (see e.g. [19], Theorems 4.23 and 6.8). In order to show the second part note that the random variable $X(t)$ is infinitely divisible with the Lévy measure

$$\nu_t:=\int_0^t S(s)\mathrm{d}s\circ\nu$$

and the drift

$$m_t=\int_0^t\int_U\left[\chi_B(v)-\chi_B(S(s)v)\right]S(s)v\nu(\mathrm{d}v)ds,$$

for more details see [6]. We have to check only that $m_t\in H$, and that

$$\int_H |z|_H^2\wedge 1\nu_t(\mathrm{d}z)<\infty.$$

Since

$$\int_H |z|_H^2\wedge 1\nu_t(\mathrm{d}z)=\int_0^t\int_H |S(s)z|_H^2\wedge 1\nu(\mathrm{d}z),$$

the desired conclusion holds.

Let

$$Z_0(t)=\int_0^t\int_U z\widehat{\pi}(\mathrm{d}z,\mathrm{d}s),\qquad t\geq 0.$$

Clearly (12) is an if and only if condition under which the process

$$\int_0^t S(t-s)\mathrm{d}Z_0(s),\qquad t\geq 0,$$

is square integrable in H. On the other hand $Z(t) = Z_0(t) + at$, where $a \in U$ and $\int_0^t S(t-s)a\mathrm{d}s \in H$. Therefore the desired equivalence holds. To see the continuity in probability assume that $t > s \geq 0$. Without any loss of generality we can assume that

$$Z(t) = \int_0^t \int_U z\widehat{\pi}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0.$$

Then

$$\begin{aligned}
& I(t,s) := \mathbb{E}\,|X(t) - X(s)|_H^2 \\
= \; & \mathbb{E}\left|\int_0^t S(t-r)\mathrm{d}Z(r) - \int_0^s S(s-r)\mathrm{d}Z(r)\right|_H^2 + \mathbb{E}\left|\int_s^t S(t-r)\mathrm{d}Z(r)\right|_H^2 \\
= \; & \int_0^s \int_H |(S(t-r) - S(s-r))\,z|_H^2\,\mathrm{d}r\nu(\mathrm{d}z) + \int_s^t \int_H |S(t-r)z|_H^2\,\mathrm{d}r\nu(\mathrm{d}z) \\
= \; & \int_0^s \int_H |(S(t-s) - I)\,S(r)z|_H^2\,\mathrm{d}r\nu(\mathrm{d}z) + \int_0^{t-s} \int_H |S(r)z|_H^2\,\mathrm{d}r\nu(\mathrm{d}z).
\end{aligned}$$

By (12) and the Lebesgue dominated convergence theorem $I(t,s) \to 0$ provided $t - s \to 0$ and s is in a bounded interval. $\square$

3. Càdlàg property

This section is concerned with the existence of a càdlàg modification of the solution X to (1). For some technical reason we will need to assume that the semigroup S is analytic on H. Without any loss of generality we may assume that S is exponentially stable. Then 0 belongs to the resolvent set of the generator A.

Let H_ρ, $\rho > 0$, be the domain of $(-A)^\rho$ equipped with the norm $|z|_\rho := |(-A)^\rho z|_H$. If $\rho < 0$, then H_ρ is the dual space to $H_{-\rho}$ where the duality is given by the identification $H = H^*$. The proof of the theorem below is postponed to Section 6.

Theorem 3.1. *Let X be the solution to* (1)*, where A is the generator of an exponentially stable analytic semigroup S on a Hilbert space H. Let Z be a Lévy process taking values in a Hilbert space $U = H_{-\rho}$ for a certain*

$\rho < 1/2$ and having representation (11). *Assume that the Lévy measure ν of Z satisfies $\nu(H_{-\rho} \setminus H) = 0$ and that*

$$\int_H \left(|z|_{-\rho}^2 + |z|_\varepsilon^4\right) \nu(\mathrm{d}z) < \infty$$

for a certain $\varepsilon > 0$.

Then X has a càdlàg modification in H and

$$\mathbb{E} \sup_{0 \le t \le T} |X(t)|_H^q < \infty, \qquad \forall\, T < \infty,\ \forall\, q \in [1, 4). \tag{13}$$

Using the standard localisation procedure we obtain the following.

Corollary 3.2. *Assume that there are $\rho < 1/2$ and $\varepsilon > 0$ such that Z takes values in $H_{-\rho}$, the Lévy measure ν of Z satisfies $\nu(H_{-\rho} \setminus H) = 0$, and*

$$\int_{\{|z|_{-\rho} \le R\}} |z|_\varepsilon^4 \nu(\mathrm{d}z) < \infty, \qquad \forall\, R > 0.$$

Then X has a càdlàg modification in H.

3.1. Diagonal case

Let us now consider the diagonal case of (1). Clearly we can assume that $H = l^2$, $\nu = \sum_{n=1}^\infty \mu_n$ where each μ_n is a Lévy measure of a one dimensional Lévy process $Z_n e_n$, (e_n) is the canonical basis of l^2, $Ae_n = -\gamma_n e_n$, and $\gamma_n > 0$, $n \in \mathbb{N}$. We assume that each Z_n has Lévy–Khinchin representation

$$Z_n(t) = \int_0^t \int_{\{|z|>1\}} z \pi_n(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z| \le 1\}} z \widehat{\pi}_n(\mathrm{d}s, \mathrm{d}z), \tag{14}$$

where π_n is a Poisson random measure with intensity measure μ_n satisfying $\int_{\mathbb{R}} |z|^2 \wedge 1 \mu_n(\mathrm{d}z) < \infty$.

Corollary 3.3. *In the diagonal case assume that there are $\rho < 1/2$ and $\varepsilon > 0$ such that*

$$\sum_{n=1}^\infty \int_{\mathbb{R}} \left(z_n^2 \gamma_n^{-2\rho} + z_n^4 \gamma_n^{4\varepsilon}\right) \mu_n(\mathrm{d}z_n) < \infty.$$

Then X has a càdlàg modification in $H = l^2$ and (13) *holds. If*

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} \left(z_n^2 \gamma_n^{-2\rho}\right) \wedge 1 \mu_n(\mathrm{d}z_n) < \infty$$

and

$$\sum_{n=1}^{\infty} \int_{\{|z_n| \le \gamma_n^{\rho} R\}} z_n^4 \gamma_n^{4\varepsilon} \mu_n(\mathrm{d}z_n) < \infty, \qquad \forall\, R > 0, \tag{15}$$

then X has a càdlàg modification in l^2.

In the following example we show that solution of the linear equation can be càdlàg in H although the noise process does not live in H.

Example 3.4. Let $Z_n = \sigma_n L_n$, $n \in \mathbb{N}$, where L_n are independent and identically distributed Lévy processes of type (14), and (σ_n) is a sequence of strictly positive numbers. Assume that the Lévy measure μ of L_n has finite moments up to order 4. Then the Lévy measure μ_n of $Z_n = \sigma L_n$ equals $\mu(\cdot/\sigma_n)$. Consequently, if there is an $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty} \left[\gamma_n^{\varepsilon-1}\sigma_n^2 + \gamma_n^{\varepsilon}\sigma_n^4\right] < \infty,$$

then by Corollary 3.3, X has a càdlàg modification in l^2. In particular, if $\gamma_n \asymp n^{\alpha}$, $\sigma_n \asymp n^{-\kappa}$, then X has a càdlàg modification in l^2 provided that $\alpha > 1 - 2\kappa$ and $\kappa > 1/4$. Assume now that each L_n is a Poisson process with intensity 1. Then $\mu = \delta_1$ and $\sigma_n L_n$ has a Lévy measure $\mu_n = \delta_{\sigma_n}$. Therefore $Z = (\sigma_n L_n)$ lives in l^2 if and only if

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} x_n^2 \wedge 1 \mu_k(\mathrm{d}x_n) = \sum_{n=1}^{\infty} \sigma_n^2 \wedge 1 < \infty.$$

Hence, if $\sigma_n \asymp n^{-\kappa}$, then Z takes values in l^2 if and only if $\kappa > 1/2$. Summing up, if $1/4 < \kappa < 1/2$ and $\alpha > 1 - 2\kappa$ then Z does not take values in l^2 but the solution X has a càdlàg modification in l^2.

Remark 3.5. It turns out that our result is not applicable to the case of α-stable noise considered by Liu and Zhai [17]; see discussion in Section 1. Namely, still in the diagonal case, assume that $Z = (Z_n)$ where $Z_n = \sigma_n L_n$,

$\sigma_n > 0$ and L_n are independent real-valued symmetric α-stable processes for a fixed $\alpha \in (0,2)$. Note that the Lévy measure μ_n of $\sigma_n L_n$ is given by $\mu_n(\cdot) = \mu(\sigma_n^{-1}\cdot)$, where μ is the Lévy measure of the symmetric α-stable process. Hence μ_n has the density $C\sigma_n^\alpha/|x|^{1+\alpha}$. Therefore the condition (15) has the form

$$\sum_{n=1}^\infty \sigma_n^\alpha \int_0^{\gamma_n^\rho R} \frac{z_n^4\gamma_n^{4\varepsilon}}{z_n^{1+\alpha}}\mathrm{d}z_n = \frac{R^{4-\alpha}}{4-\alpha}\sum_{n=1}^\infty \sigma_n^\alpha \gamma_n^{4\varepsilon+(4-\alpha)\rho} < \infty, \qquad \forall\, R>0.$$

Therefore (15) and the fact that $\gamma_n \to +\infty$ imply that $\sum_n \sigma_n^\alpha < \infty$. The last inequality is however if and only if condition under which Z takes values in l^2; see [21, 20].

4. Weakly càdlàg property in diagonal case

In the present section we are concerned with the weakly càdlàg property dealing only with the diagonal case. Results here are mainly of negative type.

Assume that A is negative definite self-adjoint with a compact resolvent, (e_n) is the orthonormal basis of eigenvectors of A, $(-\gamma_n)$ is the corresponding sequence of eigenvalues, and

$$Z = \sum_{n=1}^\infty Z_n e_n,$$

where (Z_n) are independent real-valued Lévy processes with the Lévy–Khinchin representation (14). Then

$$X(t) = \sum_{n=1}^\infty X_n(t) e_n,$$

where

$$\mathrm{d}X_n = -\gamma_n X_n \mathrm{d}t + \mathrm{d}Z_n, \qquad X_n(0) = 0.$$

Note that each process X_n is càdlàg. Taking into account Theorem 2.3 the following result provides, in particular, a necessary condition for the existence of the weakly càdlàg modification of X. The proof is in the spirit of [17].

Proposition 4.1. *If for each $r > 0$,*

$$\sum_{n=1}^\infty \int_{\{r\le |y|\}} \mu_n(\mathrm{d}y) = \infty,$$

then for every $T > 0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{n=1}^{\infty} X_n^2(t) < \infty\right) = 0.$$

In particular the process X does not have a weakly càdlàg modification.

Proof Note that for each n,

$$\sup_{t\in[0,T]} X_n^2(t-) \le \sup_{t\in[0,T]} X_n^2(t).$$

Hence

$$\sup_{t\in[0,T]} (\Delta Z_n(t))^2 = \sup_{t\in[0,T]} (\Delta X_n(t))^2 \le 4 \sup_{t\in[0,T]} X_n^2(t),$$

where $\Delta Z_n(t) := Z_n(t) - Z_n(t-)$. Taking into account the obvious estimate

$$4 \sup_{t\in[0,T]} \sum_{n=1}^{\infty} X_n^2(t) \ge 4 \sup_n \sup_{t\in[0,T]} X_n^2(t) \ge \sup_n \sup_{t\in[0,T]} (\Delta Z_n(t))^2 ,$$

we obtain

$$\sup_n \zeta_n \le 4 \sup_{t\in[0,T]} \sum_{n=1}^{\infty} X_n^2(t),$$

where

$$\zeta_n := \sup_{t\in[0,T]} (\Delta Z_n(t))^2 , \qquad n \in \mathbb{N}.$$

Since ζ_n are independent,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{n=1}^{\infty} X_n^2(t) < \infty\right) \le \mathbb{P}\left(\sup_n \zeta_n < \infty\right) = \lim_{r\uparrow\infty} \prod_{n=1}^{\infty} \mathbb{P}\left(\zeta_n \le r^2\right).$$

Let π_n be the Poisson random measure corresponding to Z_n. Then

$$\mathbb{P}\left(\zeta_n \le r^2\right) = \mathbb{P}\left(\pi_n([0,T] \times \{y\colon |y| > r\}) = 0\right) = \exp\left\{-T\mu_n\{y\colon r < |y|\}\right\}.$$

Therefore the desired conclusion follows from the estimate

$$\mathbb{P}\left(\sup_{t\in[0,T]}\sum_{n=1}^{\infty} X_n^2(t) < \infty\right) \le \lim_{r\uparrow\infty} \exp\left\{-T\sum_{n=1}^{\infty}\mu_n\{y\colon r < |y|\}\right\}. \qquad (16)$$

□

As a direct consequence of (16) we obtain the following result.

Corollary 4.2. *Let $Z_n(t) = \sigma_n L_n(b_n t)$, $t \geq 0$, where L_n are independent identically distributed Lévy processes of type* (14), *and $\sigma_n, b_n > 0$. If the Lévy measure μ of L_n has an unbounded support, and there is a $T > 0$ such that*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \sum_{n=1}^{\infty} X_n^2(t) < \infty\right) > 0,$$

then $\sigma_n \to 0$ or $b_n \to 0$.

The following result shows that the assumption $\sigma_n \to 0$ is in general not sufficient for the existence of a weakly càdlàg modification of X.

Proposition 4.3. *Let L_n be independent identically distributed Lévy processes, of type* (14), *each with the Lévy measure μ. Assume that μ has an unbounded support. Then there is a sequence $\sigma_n \downarrow 0$ such that*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \sum_{n=1}^{\infty} X_n^2(t) < \infty\right) = 0, \qquad \forall\, T > 0,$$

regardless the sequence (γ_n).

Proof Taking into account (16) it is enough to find a sequence $\sigma_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \mu\{y\colon \sigma_n^{-1} r < |y|\} = \infty, \qquad \forall\, r > 0.$$

Let $\psi(x) = \mu\{y\colon |y| > x\}$, $x > 0$. By the assumption ψ is a decreasing function from $(0, +\infty)$ to $(0, +\infty)$. Thus the result follows from the lemma below. □

Lemma 4.4. *Assume that $\psi\colon (0, +\infty) \mapsto (0, +\infty)$ is a decreasing function. Then there is a sequence (a_n) such that $a_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \psi(r a_n) = \infty, \qquad \forall\, r > 0.$$

Proof Let $N_k \uparrow \infty$ be a sequence such that

$$\sum_{n=N_k}^{N_{k+1}-1} \psi(k^2) \geq 1.$$

Let $a_n = k$ for $n \in [N_k, N_{k+1} - 1]$. Then for any $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \psi(ma_n) &= \sum_{k=1}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \psi(mk) \geq \sum_{k=m}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \psi \\ &\geq \sum_{k=m}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \psi(k^2) = \infty. \qquad \square\end{aligned}$$

5. Cylindrical càdlàg property in diagonal case

This section deals with diagonal case. As in the previous section we assume that $H = l^2$ and $X = (X_n)$ where

$$\mathrm{d}X_n = -\gamma_n X_n \mathrm{d}t + \mathrm{d}Z_n, \qquad X_n(0) = 0, \qquad n = 1, 2, \ldots, \tag{17}$$

γ_n, $n \in \mathbb{N}$, are strictly positive real numbers, and Z_n, $n \in \mathbb{N}$, are independent real-valued Lévy processes with the Lévy–Khinchin decomposition (14).

Given a sequence (β_n) of strictly positive numbers, set

$$l^2_\beta := \left\{ (x_n) \colon |(x_n)|^2_{l^2_\beta} := \sum_{n=1}^{\infty} x_n^2 \beta_n^2 < \infty \right\}.$$

Since (Z_n) are independent, it is known, see [19], that Z is a Lévy process in l^2_β if and only if

$$\begin{aligned}&\sum_{n=1}^{\infty} \int_{\mathbb{R}} |\beta_n x_n|^2 \wedge 1 \mu_n(\mathrm{d}x_n) \\ &\quad = \sum_{n=1}^{\infty} \left[\beta_n^2 \int_{(-\beta_n^{-1}, \beta_n^{-1})} x_n^2 \mu_n(\mathrm{d}x_n) + \mu_n \left(\mathbb{R} \setminus (-\beta_n^{-1}, \beta_n^{-1}) \right) \right] < \infty.\end{aligned}$$

Therefore, the assumption that each μ_n has a finite second moment ensures that Z is a Lévy process in a suitably chosen weighted l^2-space.

We assume that for any n, the Lévy measure μ_n of Z_n has finite moments up to order 4. Write

$$m_j(n) := \int_{\mathbb{R}} |x|^j \mu_n(\mathrm{d}x), \qquad j = 1, \ldots, 4. \tag{18}$$

Note that since the Lévy measures μ_n have finite first moments, the processes Z_n have trajectories with bounded variation. Subtracting the drifts we may assume that

$$Z_n(t) = \int_0^t \int_{\mathbb{R}} x \widehat{\pi}_n(\mathrm{d}s, \mathrm{d}x), \tag{19}$$

where π_n is the Poisson jump measure of Z_n and $\widehat{\pi}_n$ is the compensated measure. In this way each Z_n is a square integrable martingale. Obviously,

$$X_n(t) = \int_0^t \mathrm{e}^{-\gamma_n(t-s)} \mathrm{d}Z_n(s), \tag{20}$$

and

$$\mathbb{E}X_n^2(t) = m_2(n) \int_0^t \mathrm{e}^{-2\gamma_n(t-s)} \mathrm{d}s \le \frac{m_2(n)}{2\gamma_n}.$$

In this section we assume that

$$\sum_{n=1}^{\infty} \left(\frac{m_2(n)}{\gamma_n} + \frac{m_1(n)}{\gamma_n} \right) < \infty, \tag{21}$$

which is more that is needed to guarantee that the process $X := (X_n)$ restricted to any finite time interval $[0, T]$, is a square integrable random element in l^2 satisfying

$$\sup_{0 \le t \le T} \mathbb{E} \left| X(t) \right|_{l^2}^2 < \infty.$$

Taking modifications we can also assume that the real-valued processes X_n are càdlàg.

Note that X solves the linear equation (1) with a diagonal linear operator $A(x_n) = (-\gamma_n x_n)$, and with $Z = (Z_n)$.

The main results of the present section are the theorems below. Their proofs are however postponed to Sections 8 and 9, respectively. The first result covers the case where (Z_n) are identically distributed with the Lévy measure having finite moments up to order 4. This particular case requires $\gamma_n \to +\infty$. By [2] the solution X has no càdlàg modification unless $\mu = 0$, Moreover, if μ has an unbounded support, then by Proposition 4.3, X does not have even a weakly càdlàg modification.

Theorem 5.1. *Assume the estimate* (21), *and that there is an* $\varepsilon \in (0,1)$ *such that:*

$$\sup_n \left(m_1(n) + m_2(n) + m_4(n) + \gamma_n^{(\varepsilon-1)/2} m_2(n) + \gamma_n^{\varepsilon-1} m_4(n) \right) < \infty. \tag{22}$$

Then X *is cylindrical càdlàg. Moreover,*

$$\forall\, T < \infty,\ \forall\, q \in [1,4), \qquad \sup_{z \in l^2 : |z|_{l^2} \le 1} \mathbb{E} \sup_{t \in [0,T]} |\langle X(t), z \rangle_{l^2}|^q < \infty. \tag{23}$$

In addition for any $z \in l^2$*, the series*

$$\sum_{n=1}^{\infty} X_n(t) z_n = \langle X(t), z \rangle_{l^2}$$

converges in L^q *uniformly in* t *on each compact interval.*

The proofs of the difficult first parts of the theorem are postponed to the following sections. Here we sketch the proof of the final one.

Let $(\mathrm{Pr}_{n,m} z)_k = z_k$ if $m \ge k \ge n$ and 0 otherwise. For any $z \in l^2$ and for any $\delta > 0$ there is an $n_\delta \in \mathbb{N}$ such that

$$|\mathrm{Pr}_{n,m} z|_{l^2} \le \delta |z|_{l^2}, \qquad \forall\, m > n \ge n_\delta.$$

Moreover, if $\mathrm{Pr}_{n,m} z \ne 0$, then

$$\begin{aligned} \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{k=n}^{m} X_k(t) z_k \right|^q &= \mathbb{E} \sup_{t \in [0,T]} \left| \left\langle X(t), \frac{\mathrm{Pr}_{n,m} z}{|\mathrm{Pr}_{n,m} z|} \right\rangle \right|^q |\mathrm{Pr}_{n,m} z|^q \\ &\le |\mathrm{Pr}_{n,m} z|^q \sup_{v \in l^2 : |v|_{l^2} \le 1} \mathbb{E} \sup_{t \in [0,T]} |\langle X(t), v \rangle_{l^2}|^q . \end{aligned}$$

Consequently, the estimate in (23) guarantees that for any $z \in l^2$, for all $T > 0$ and $\varepsilon > 0$ there is an $n_{\varepsilon,T}$ such that for all $n_{\varepsilon,T} \le n \le m$,

$$\mathbb{E} \sup_{t \in [0,T]} \left| \sum_{k=n}^{m} X_k(t) z_k \right|^q \le \varepsilon.$$

5.1. l^2_β-cylindrical càdlàg property

Recall that l^2_β is a weighted l^2-space. Then $(l^2_\beta)^* \equiv l^2_{1/\beta}$, where $1/\beta = (1/\beta_n)$. Our next result is concerned with l^2_β-cylindrical càdlàg property and in particular it covers the case where $Z_n = \sigma_n L_n$, $n \in \mathbb{N}$, where L_n are identically distributed with the Lévy measure having finite moments up to order 4, and $\sigma_n \to 0$.

Theorem 5.2. *Assume that*

$$\lim_{n\to\infty} \left(m_1(n) + \ldots + m_4(n) + \gamma_n^{-1}\right) = 0. \tag{24}$$

Let moreover (β_n) be a sequence of positive numbers tending to $+\infty$ such that

$$\sum_{n=1}^{\infty} \left(\frac{\beta_n^2 m_2(n)}{\gamma_n} + \frac{\beta_n m_1(n)}{\gamma_n} \right) < \infty,$$

and for a certain $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \beta_n m_1(n)^2 \gamma_n^{-2} < \infty, \qquad \sup_n \beta_n (m_1(n))^{\frac{2\varepsilon}{2+\varepsilon}} < \infty,$$

$$\sup_n \beta_n \left[(m_2(n))^{\frac{\varepsilon}{1+\varepsilon}} + (\gamma_n^{-1})^{\frac{1-\varepsilon}{2}} \right] < \infty,$$

$$\sup_n \beta_n \left[(m_4(n))^{\frac{1}{2}} ((\gamma_n^{-1})^{1-\varepsilon} + (\gamma_n^{-1})^{1/2}) + (m_4(n))^{1/2(1-\frac{1}{1+2\varepsilon}} \right] < \infty.$$

Then for any $z \in (l^2_\beta)^* = l^2_{1/\beta}$, the process $\langle X(t), z\rangle_{l^2}$, $t \geq 0$, is well-defined and has a càdlàg modification. Moreover,

$$\forall\, T < \infty,\ \forall\, q \in [1,4), \qquad \sup_{z \in l^2_{1/\beta} : |z|_{l^2_{1/\beta}} \leq 1} \mathbb{E} \sup_{t\in[0,T]} |\langle X(t), z\rangle_{l^2}|^q < \infty.$$

5.2. Comments on localisation

It is of interest to extend the results to the case when the Lévy measures do not have finite moments, like for α-stable processes. However the localisation idea employed in Section 3 in the study of the càdlàg property does not lead to any interesting applications. In fact: assume that Z lives in the weighted space l^2_β, where β_n tends to 0. Let $B_\beta(0,R)$ be the ball in l^2_β of radius R, and let ν_R be the restriction of the Lévy measure ν of Z to $B_\beta(0,R)$. Then $\nu_R = \sum_n \mu_{n,R}$, where $\mu_{n,R}$ is the restriction of μ_n to $[-R/\beta_n, R/\beta_n]\, e_n$. Let $m_{j,R}(n)$ be the moment of order j of $\mu_{n,R}$. Then from Theorem 5.1 we have the following.

Corollary 5.3. *If for each $R > 0$,*

$$\sum_{n=1}^{\infty} \left(\frac{m_{2,R}(n)}{\gamma_n} + \frac{m_{1,R}(n)}{\gamma_n} \right) < \infty,$$

and there is an $\varepsilon \in (0,1)$ such that:

$$\sup_n \left(m_{1,R}(n) + m_{2,R}(n) + m_{4,R}(n) + \gamma_n^{(\varepsilon-1)/2} m_{2,R}(n) + \gamma_n^{\varepsilon-1} m_{4,R}(n) \right) < \infty.$$

Then X is cylindrical càdlàg.

Unfortunately, if $Z_n = \sigma_n L_n$ and L_n are independent α-stable, then the supremum appearing in the formulation of the corollary above is finite if and only if $\sum_{n=1}^{\infty} \sigma_n^{\alpha} < \infty$, which holds if and only if Z takes values in l^2; see Remark 3.5. Therefore the last question formulated at the end the introduction is open.

6. Proof of Theorem 3.1

Since $\int_U |z|_{-\rho}^2 \nu(\mathrm{d}z) < \infty$ and Z is given (11), we see that $Z(t) = Z_0(t) + (a+m)t$, where

$$Z_0(t) = \int_0^t \int_{H_{-\rho}} z\nu(\mathrm{d}z)\mathrm{d}s, \qquad a := \int_{\{|z|_{-\rho} > 1\}} z\nu(\mathrm{d}z)\mathrm{d}s \in H_{-\rho}.$$

Since

$$[0, +\infty) \ni t \mapsto \int_0^t S(t-s)(a+m)\mathrm{d}s \in H$$

is continuous, we may assume that $a + m = 0$. Thus

$$X(t) = \int_0^t S(t-s)\mathrm{d}Z(s) = \int_0^t \int_U S(t-s) z \widehat{\pi}(\mathrm{d}s, \mathrm{d}z).$$

We will use the following fact

$$\forall -\infty < \rho_1 < \rho < +\infty, \qquad C_{\rho_1,\rho_2} := \sup_{0<t} t^{\rho_2 - \rho_1} \|S(t)\|_{L(H_{\rho_1}, H_{\rho_2})} < \infty. \quad (25)$$

Since

$$|(S(h) - I)\, z|_H \le \int_0^h |AS(s)z|_H \,\mathrm{d}s \le \int_0^h \|(-A)^{1-\rho} S(s)\|_{L(H,H)} \mathrm{d}s |z|_{\rho},$$

as a consequence of (25) we have

$$|(S(h)-I)\,z|_H \le C_{0,\rho} h^\rho |z|_\rho, \qquad \rho>0,\ 0\le h<\infty,\quad z\in H_\rho. \tag{26}$$

For $0\le t-h<t<t+h\le T$ we have

$$\mathbb{E}\,|X(t+h)-X(t)|_H^2\,|X(t)-X(t-h)|_H^2 = \mathbb{E}\,|\xi_1+\xi_2+\xi_3|_H^2\,|\eta_1+\eta_2|_H^2\,,$$

where

$$\begin{aligned}
\xi_1 &:= \int_0^{t-h} (S(t+h-s)-S(t-s))\,\mathrm{d}Z(s),\\
\xi_2 &:= \int_{t-h}^{t} (S(t+h-s)-S(t-s))\,\mathrm{d}Z(s),\\
\xi_3 &:= \int_t^{t+h} S(t+h-s)\mathrm{d}Z(s),\\
\eta_1 &:= \int_0^{t-h} (S(t-s)-S(t-h-s))\,\mathrm{d}Z(s),\\
\eta_2 &:= \int_{t-h}^{t} S(t-s)\mathrm{d}Z(s).
\end{aligned}$$

Therefore, taking into account the inequality

$$|\langle\xi_1,\xi_2\rangle_H\langle\eta_1,\eta_2\rangle_H| \le \frac{1}{2}\left(|\xi_1|_H^2\,|\eta_1|_H^2+|\xi_2|_H^2\,|\eta_2|_H^2\right)$$

we obtain

$$\begin{aligned}
&\mathbb{E}\,|X(t+h)-X(t)|_H^2\,|X(t)-X(t-h)|_H^2\\
&\quad\le 3\mathbb{E}\left[|\xi_1|_H^2|\eta_1|_H^2+|\xi_2|_H^2|\eta_2|_H^2\right]+\mathbb{E}|\xi_1|_H^2\mathbb{E}|\eta_2|_H^2\\
&\qquad+\mathbb{E}|\xi_2|_H^2\mathbb{E}|\eta_1|_H^2+\mathbb{E}|\xi_3|_H^2\mathbb{E}|\eta_1|_H^2+\mathbb{E}|\xi_3|_H^2\mathbb{E}|\eta_2|_H^2.
\end{aligned}$$

Assume that $\kappa>0$ is such that $\kappa+\rho<1/2$. By (25) and (26), we have

$$\begin{aligned}
\mathbb{E}\,|\xi_1|_H^2 &= \int_0^{t-h}\int_H |(S(t+h-s)-S(t-s))\,z|_H^2\,\nu(\mathrm{d}z)\mathrm{d}s\\
&= \int_0^{t-h}\int_H |(S(h)-I)\,S(t-s)z|_H^2\,\nu(\mathrm{d}z)\mathrm{d}s\\
&= \int_0^{t-h}\int_H |(S(h)-I)\,S(t-s)z|_H^2\,\nu(\mathrm{d}z)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq C_{0,\kappa}^2 h^{2\kappa} C_{-\kappa,\rho}^2 \int_0^{t-h} (t-s)^{-2(\kappa+\rho)} \mathrm{d}s \int_H |z|_{-\rho}^2 \, \nu(\mathrm{d}z) \\
&\leq \frac{C_{0,\kappa}^2 C_{-\rho,\kappa}}{1-2(\kappa+\rho)} T^{1-2(\kappa+\rho)} \\
|h|^{2\kappa} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z) &=: C(\kappa) h^{2\kappa} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z).
\end{aligned}
$$

Similarly

$$
\mathbb{E}\, |\xi_2|_H^2 \leq C(\kappa) h^{2\kappa} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z)
$$

and

$$
\begin{aligned}
\mathbb{E}\, |\eta_1|_H^2 &= \mathbb{E} \left| \int_0^{t-h} \left(S(t-h-s+h) - S(t-h-s) \right) \mathrm{d}Z(s) \right|_H^2 \\
&\leq C(\kappa) h^{2\kappa} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z).
\end{aligned}
$$

Next

$$
\begin{aligned}
\mathbb{E}\, |\xi_3|_H^2 &= \int_t^{t+h} \int_H |S(t+h-s)z|_H^2 \, \nu(\mathrm{d}z)\mathrm{d}s = \int_0^h \int_H |S(s)z|_H^2 \, \nu(\mathrm{d}z)\mathrm{d}s \\
&\leq \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z) \int_0^h \|S(s)\|_{L(H_{-\rho},H)}^2 \mathrm{d}s \\
&\leq \frac{C_{-\rho,0}^2}{1-2\rho} h^{1-2\rho} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z) =: B(\rho) h^{1-2\rho} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z),
\end{aligned}
$$

and using the same arguments we obtain

$$
\mathbb{E}\, |\eta_2|_H^2 \leq B(\rho) h^{1-2\rho} \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z).
$$

Summing up, for any $\kappa > 0$ such that $2(\kappa + \rho) < 1$ there is a constant C independent of h such that

$$
\mathbb{E} \left(\sum_{i=1}^3 |\xi_i|_H^2 + \sum_{i=1}^2 |\eta_i|_H^2 \right) \leq C \left(h^{1-2\rho} + h^{2\kappa} \right) \int_H |z|_{-\rho}^2 \nu(\mathrm{d}z). \tag{27}
$$

We proceed to the calculation of the terms $\mathbb{E}|\xi_1|_H^2 |\eta_1|_H^2$ and $\mathbb{E}|\xi_2|_H^2 |\eta_2|_H^2$. To do this let (e_k) be an orthonormal basis of H. By Proposition 2.4,

$$
\mathbb{E}\, |\xi_1|_H^2 \, |\eta_1|_H^2 = \sum_{k,l} \mathbb{E} \langle \xi_1, e_k \rangle_H^2 \langle \eta_1, e_l \rangle_H^2 = J_1 + J_2 + J_3,
$$

where

$$\begin{aligned}
J_1 &= \sum_{j,k} \int_H \int_0^{t-h} \langle (S(t+h-s) - S(t-s))\, z, e_j \rangle_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&\qquad \times \int_H \int_0^{t-h} \langle (S(t-s) - S(t-h-s))\, z, e_k \rangle_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&= \int_0^{t-h} \int_H |(S(t+h-s) - S(t-s))\, z|_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&\qquad \times \int_H \int_0^{t-h} |(S(t-s) - S(t-h-s))\, z|_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&= \int_0^{t-h} \int_H |(S(h) - I)\, S(t-s) z|_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&\qquad \times \int_H \int_0^{t-h} |(S(h) - I)\, S(t-h-s) z|_H^2 \, \mathrm{d}s \nu(\mathrm{d}z) \\
&\leq h^{4\kappa} C_{-\rho,\kappa}^2 \left(\int_H |z|_H^2 \, \nu(\, dz) \right)^2 \\
&\qquad \times \int_0^{t-h} (t-s)^{-2(\rho+\kappa)} \mathrm{d}s \int_0^{t-h} (t-h-s)^{-2(\rho+\kappa)} \mathrm{d}s \\
&\leq C(1) h^{4\kappa} \left(\int_H |z|_H^2 \, \nu(\, dz) \right)^2 .
\end{aligned}$$

In the estimate above, $\kappa > 0$ is such that $\kappa + \rho < 1/2$ and $C(1)$ depends on T, κ and ρ. Note that J_2 equals

$$2 \left[\int_H \int_0^{t-h} \langle (S(t+h-s) - S(t-s))\, z, (S(t-s) - S(t-h-s))\, z \rangle_H \, \mathrm{d}s \nu(\mathrm{d}z) \right]^2$$

is less than or equal to

$$\begin{aligned}
&2 \left[\int_H \int_0^{t-h} |(S(h) - I)\, S(t-s) z|_H \, |(S(h) - I)\, S(t-h-s) z|_H \, \mathrm{d}s \nu(\mathrm{d}z) \right]^2 \\
&\quad \leq C(2) h^{4\kappa} \left(\int_H |z|_H^2 \, \nu(\, dz) \right)^2 .
\end{aligned}$$

Finally J_3 equals

$$\int_0^{t-h} \int_H |(S(t+h-s) - S(t-s))\, z|_H^2 \, |(S(t-s) - S(t-h-s))\, z|_H^2 \, \mathrm{d}s \nu(\mathrm{d}z)$$

is less than or equal to

$$
\begin{aligned}
&h^{4\zeta}\int_H |z|^4_\varepsilon\,\nu(\mathrm{d}z)C^2_{\varepsilon,\zeta}\int_0^{t-h}(t-s)^{-2(\zeta-\varepsilon)}(t-h-s)^{-2(\zeta-\varepsilon)}\mathrm{d}s\\
&\qquad\le C(3)h^{4\zeta}\int_H |z|^4_\varepsilon\,\nu(\mathrm{d}z),
\end{aligned}
$$

where $\zeta>0$ is such that $\zeta-\varepsilon<1/4$.

To estimate $\mathbb{E}\,|\xi_2|^2_H\,|\eta_2|^2_H$ we use similar calculations. Namely

$$
\mathbb{E}\,|\xi_2|^2_H\,|\eta_2|^2_H = I_1+I_2+I_3,
$$

where

$$
\begin{aligned}
I_1 &= \int_{t-h}^t\int_H |(S(t+h-s)-S(t-s))\,z|^2_H\,\mathrm{d}s\nu(\mathrm{d}z)\\
&\qquad\times\int_H\int_{t-h}^t |S(t-s)z|^2_H\,\mathrm{d}s\nu(\mathrm{d}z)\\
&\le \left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 C^2_{-\rho,\kappa}C^2_{-\rho,0}h^{2\kappa+1-2\rho}\frac{1}{1-2\rho}\int_{t-h}^t (t-s)^{-2(\rho+\kappa)}\mathrm{d}s\\
&\le \left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 C^2_{-\rho,\kappa}C^2_{-\rho,0}h^{2\kappa+1-2\rho+1-2(\rho+\kappa)}\frac{1}{1-2\rho}\frac{1}{1-2(\rho+\kappa)}\\
&\le C(4)\left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 h^{2-4\rho}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= 2\left[\int_H\int_{t-h}^t \langle (S(t+h-s)-S(t-s))\,z, S(t-s)z\rangle_H\,\mathrm{d}s\nu(\mathrm{d}z)\right]^2\\
&\le \left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 C^2_{-\rho,\kappa}C^2_{-\rho,0}\left(\int_{t-h}^t (t-s)^{-(\rho+\kappa)}(t-s)^{-\rho}\mathrm{d}s\right)^2 h^{2\kappa}\\
&\le \left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 C^2_{-\rho,\kappa}C^2_{-\rho,0}\frac{1}{(1-2\rho-\kappa)^2}h^{2\kappa+2(1-2\rho-\kappa)}\\
&\le C(5)\left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z)\right)^2 h^{2-4\rho}
\end{aligned}
$$

and

$$
I_3 = \int_{t-h}^t\int_H |(S(t+h-s)-S(t-s))\,z|^2_H\,|S(t-s)z|^2_H\,\mathrm{d}s\nu(\mathrm{d}z)
$$

$$
\begin{aligned}
&\leq \int_H |z|_\varepsilon^4\,\nu(\mathrm{d}z) \sup_{s\leq T} \|S(s)\|^2_{L(H_\varepsilon,H)} C^2_{\varepsilon,\delta} \int_{t-h}^t (t-s)^{-2(\delta-\varepsilon)} \mathrm{d}s h^{2\delta} \\
&\leq \int_H |z|_\varepsilon^4\,\nu(\mathrm{d}z) \sup_{s\leq T} \|S(s)\|^2_{L(H_\varepsilon,H)} C^2_{\varepsilon,\delta} \frac{1}{1-2(\delta-\varepsilon)} h^{2\delta+1-2(\delta-\varepsilon)} \\
&\leq C(6) \int_H |z|_\varepsilon^4\,\nu(\mathrm{d}z) h^{1+2\varepsilon}.
\end{aligned}
$$

Summing up there are constants B and $\tilde{\varepsilon} > 0$ such that

$$
\mathbb{E} \sum_{i=1}^2 |\xi_i|_H^2 |\eta_i|_H^2 \leq B h^{1+\tilde{\varepsilon}} \left[\int_H |z|_\varepsilon^4\,\nu(\mathrm{d}z) + \left(\int_H |z|^2_{-\rho}\,\nu(\mathrm{d}z) \right)^2 \right].
$$

Using the estimate above and (27) we can find constants $\tilde{\delta} >$ and R such that

$$
\mathbb{E}\, |X(t+h) - X(t)|_H^2\, |X(t) - X(t-h)|_H^2 \leq R h^{1+\tilde{\delta}},
$$

and the desired conclusion follows form Corollary 2.2, and the Bichteler–Jacod type estimates from [18].

7. Auxiliary result for the proofs of Theorems 5.1 and 5.2

In this section

$$
L_n(t) := \int_0^t \int_{\mathbb{R}} |x| \pi_n(\mathrm{d}s, \mathrm{d}x), \qquad n = 1, 2, \ldots,
$$

and (λ_n) is a sequence of strictly positive real numbers. Later $\lambda_n = \gamma_n \wedge (z_n^2)^{-1/\varepsilon}$, where (z_n) is a fixed sequence and $\varepsilon > 0$.

Obviously L_n are independent Lévy processes, and

$$
m_j(n) = \int_{\mathbb{R}} |x|^j \nu_n(\mathrm{d}n), \qquad n \in \mathbb{N},\ j = 1, 2, 3, 4,
$$

where ν_n is the Lévy measure of L_n.

Consider a sequence (z_n) of real numbers. If

$$
\sum_{n=1}^\infty z_n^2 m_2(n) \lambda_n^{-1} < \infty, \tag{28}
$$

then the real-valued process

$$Y(t) := \sum_{n=1}^{\infty} \int_0^t \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}\widehat{L}_n(s) z_n, \tag{29}$$

is well defined, as the series on the right hand side converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$\sup_{t\leq T} \mathbb{E}\left(Y(t)\right)^2 < \infty, \qquad \forall\, T<\infty.$$

The following lemma will play a crucial role in the proof of Theorems 5.1 and 5.2.

Lemma 7.1. *Let $\varepsilon \in (0,1)$. Assume that (28) holds and that the quantity $A(z,\lambda,\varepsilon)$ equals*

$$\left[\left(\sum_{n=1}^{\infty} z_n^2 m_2(n) \lambda_n^{(\varepsilon-1)/2} \right)^2 + \sum_{n=1}^{\infty} z_n^4 \left(m_2(n)^2 \lambda_n^{\varepsilon-1} + m_4(n) (\lambda_n^{\varepsilon-1} + \lambda_n^{\varepsilon}) \right) \right]$$

is finite. Then for all $0 \leq t-h \leq t \leq t+h < \infty$.

$$\mathbb{E}\left(Y(t+h)-Y(t)\right)^2 \left(Y(t)-Y(t-h)\right)^2 \leq 6A(z,\lambda,\varepsilon) h^{1+\varepsilon}.$$

Proof Set

$$a(t,h) := \mathbb{E}\left(Y(t+h)-Y(t)\right)^2 \left(Y(t)-Y(t-h)\right)^2.$$

Then

$$a(t,h) = \mathbb{E}\left(I_1+I_2+I_3\right)^2 \left(I_4+I_5\right)^2,$$

where

$$\begin{aligned}
I_1 &:= \sum_{n=1}^{\infty} \int_t^{t+h} \mathrm{e}^{-\lambda_n(t+h-s)} \mathrm{d}\widehat{L}_n(s)\, z_n, \\
I_2 &:= \sum_{n=1}^{\infty} \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)} \right) \mathrm{d}\widehat{L}_n(s)\, z_n, \\
I_3 &:= \sum_{n=1}^{\infty} \int_0^{t-h} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)} \right) \mathrm{d}\widehat{L}_n(s)\, z_n, \\
I_4 &:= \sum_{n=1}^{\infty} \int_{t-h}^{t} \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}\widehat{L}_n(s)\, z_n, \\
I_5 &:= \sum_{n=1}^{\infty} \int_0^{t-h} \left(\mathrm{e}^{-\lambda_n(t-s)} - \mathrm{e}^{-\lambda_n(t-h-s)} \right) \mathrm{d}\widehat{L}_n(s)\, z_n.
\end{aligned}$$

We have

$$\mathbb{E}\, I_1^2 (I_4+I_5)^2 = \mathbb{E}\, I_1^2 \, \mathbb{E}\, (I_4+I_5)^2 = \mathbb{E}\, I_1^2 \left(\mathbb{E}\, I_4^2 + \mathbb{E}\, I_5^2 \right),$$

$$\mathbb{E}\, 2 I_1 (I_2+I_3)(I_4+I_5)^2 = 0,$$

and

$$\mathbb{E}\, (I_2+I_3)^2 (I_4+I_5)^2 = \mathbb{E}\, I_2^2 I_4^2 + \mathbb{E}\, I_2^2\, \mathbb{E}\, I_5^2 + 4\, \mathbb{E}\, I_2 I_4\, \mathbb{E}\, I_3 I_5 + \mathbb{E}\, I_3^2 \mathbb{E}\, I_4^2 + \mathbb{E}\, I_3^2 I_5^2.$$

Thus

$$\begin{aligned} a(t,h) \;=\;& \mathbb{E}\, I_1^2 \left(\mathbb{E}\, I_4^2 + \mathbb{E}\, I_5^2 \right) + \mathbb{E}\, I_2^2 I_4^2 + \mathbb{E}\, I_2^2\, \mathbb{E}\, I_5^2 \\ & + 4\, \mathbb{E}\, I_2 I_4\, \mathbb{E}\, I_3 I_5 + \mathbb{E}\, I_3^2 \mathbb{E}\, I_4^2 + \mathbb{E}\, I_3^2 I_5^2. \end{aligned}$$

Given $\delta \in (0,1]$ define

$$C(\delta, z) := \sum_{n=1}^{\infty} z_n^2 m_2(n) \lambda_n^{\delta - 1}, \; D(\delta, z) := \sum_{n=1}^{\infty} z_n^4 \left(m_4(n) + m_2(n)^2 \right) \lambda_n^{2\delta - 2}. \quad (30)$$

Below the last inequality in each estimate follows from the following elementary inequalities

$$\forall\, x > 0, \; \forall\, \delta \in (0,1], \qquad 1 - \mathrm{e}^{-x} \le x^{\delta}, \quad (31)$$

and

$$\sum_{n=1}^{\infty} \frac{u_n}{\lambda_n} \left(1 - \mathrm{e}^{-\lambda_n x} \right) \le x^{\delta} \sum_{n=1}^{\infty} u_n \lambda_n^{\delta-1}, \quad (32)$$

for all $u_n \ge 0$, $\lambda_n > 0$, $x > 0$, and $\delta \in (0,1]$. We have

$$\begin{aligned} \mathbb{E}\, I_1^2 \;&=\; \sum_{n=1}^{\infty} z_n^2 m_2(n) \int_t^{t+h} \mathrm{e}^{-2\lambda_n (t+h-s)} \mathrm{d}s \\ &=\; \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(1 - \mathrm{e}^{-2\lambda_n h} \right) \le C(\delta, z) h^{\delta}, \end{aligned}$$

$$\mathbb{E}\, I_4^2 = \sum_{n=1}^{\infty} z_n^2 m_2(n) \int_{t-h}^{t} \mathrm{e}^{-2\lambda_k (t-s)} \mathrm{d}s = \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(1 - \mathrm{e}^{-2\lambda_n h} \right) \le C(\delta, z) h^{\delta},$$

and

$$\begin{aligned}
\mathbb{E}\, I_2^2 &= \sum_{n=1}^{\infty} z_n^2 m_2(n) \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right)^2 \mathrm{d}s \\
&= \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(1-\mathrm{e}^{-2\lambda_n h}\right)\left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \\
&\leq C(\delta, z) h^{\delta},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\, I_3^2 &= \sum_{n=1}^{\infty} z_n^2 m_2(n) \int_{0}^{t-h} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right)^2 \mathrm{d}s \\
&= \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(e^{-2\lambda_n h}-\mathrm{e}^{-2\lambda_n t}\right)\left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \\
&\leq \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(1-\mathrm{e}^{-\lambda_n h}\right) \leq \frac{1}{2} C(\delta, z) h^{\delta},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\, I_5^2 &= \sum_{n=1}^{\infty} z_n^2 m_2(n) \int_{0}^{t-h} \left(\mathrm{e}^{-\lambda_n(t-s)} - \mathrm{e}^{-\lambda_n(t-h-s)}\right)^2 \mathrm{d}s \\
&= \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_n} \left(1-\mathrm{e}^{-2\lambda_n (t-h)}\right)\left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \\
&\leq \sum_{n=1}^{\infty} \frac{z_n^2 m_2(n)}{2\lambda_k} \left(1-\mathrm{e}^{-\lambda_n h}\right) \leq \frac{1}{2} C(\delta, z) h^{\delta}.
\end{aligned}$$

Clearly

$$\mathbb{E}\, I_2 I_4 \;= \textstyle\sum_{n=1}^{\infty} z_n^2 m_2(n) \int_{t-h}^{t} \mathrm{e}^{-\lambda_n(t-s)} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right) \mathrm{d}s \leq 0,$$

and $\mathbb{E}\, I_3 I_5 \geq 0$.What is left is to find good estimates for the terms $\mathbb{E}\, I_2^2 I_4^2$ and $\mathbb{E}\, I_3^2 I_5^2$. We have $\mathbb{E}\, I_2^2 I_4^2$ equals

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=1}^{\infty} z_k \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s)\right)^2 \left(\sum_{j=1}^{\infty} z_j \int_{t-h}^{t} \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}\widehat{L}_j(s)\right)^2 \\
&\quad = \sum_{k,k',j,j'} \mathbb{E} \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_{k'}(t+h-s)} - \mathrm{e}^{-\lambda_{k'}(t-s)}\right) \mathrm{d}\widehat{L}_{k'}(s) \\
&\times \int_{t-h}^{t} \mathrm{e}^{-\lambda_{j}(t-s)} \mathrm{d}\widehat{L}_{j}(s) \int_{t-h}^{t} \mathrm{e}^{-\lambda_{j'}(t-s)} \mathrm{d}\widehat{L}_{j'}(s) z_k z_{k'} z_j z_{j'} \\
= {} & J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 \quad &:= \quad \sum_{k\neq j} z_k^2 z_j^2 \, \mathbb{E} \left(\int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s) \right)^2 \left(\int_{t-h}^{t} \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}\widehat{L}_j(s) \right)^2 \\
&= \quad \sum_{k\neq j} z_k^2 z_j^2 \, \mathbb{E} \left(\int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s) \right)^2 \mathbb{E} \\
&\qquad\qquad \times \left(\int_{t-h}^{t} \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}\widehat{L}_j(s) \right)^2,
\end{aligned}
$$

$$
\begin{aligned}
J_2 &:= 2 \sum_{k\neq j} z_k^2 z_j^2 \, \mathbb{E} \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s) \int_{t-h}^{t} \mathrm{e}^{-\lambda_k(t-s)} \mathrm{d}\widehat{L}_k(s) \\
&\quad \times \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_j(t+h-s)} - \mathrm{e}^{-\lambda_j(t-s)}\right) \widehat{L}_j(\mathrm{d}s) \int_{t-h}^{t} \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}\widehat{L}_j(s) \\
&= 2 \sum_{k\neq j} z_k^2 z_j^2 \, \mathbb{E} \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s) \int_{t-h}^{t} \mathrm{e}^{-\lambda_k(t-s)} \mathrm{d}\widehat{L}_k(s) \\
&\quad \times \mathbb{E} \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_j(t+h-s)} - \mathrm{e}^{-\lambda_j(t-s)}\right) \mathrm{d}\widehat{L}_j(s) \int_{t-h}^{t} \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}\widehat{L}_j(s),
\end{aligned}
$$

and

$$
J_3 := \sum_{n=1}^{\infty} z_n^4 \, \mathbb{E} \left(\int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right) \mathrm{d}\widehat{L}_n(s) \right)^2 \left(\int_{t-h}^{t} \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}\widehat{L}_n(s) \right)^2 .
$$

We have

$$
\begin{aligned}
J_1 \quad &= \quad \sum_{k\neq j} z_k^2 z_j^2 m_2(k) m_2(j) \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right)^2 \mathrm{d}s \int_{t-h}^{t} \mathrm{e}^{-2\lambda_j(t-s)} \mathrm{d}s \\
&= \quad \sum_{k\neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k \lambda_j} \left(1 - \mathrm{e}^{-\lambda_k h}\right)^2 \left(1 - \mathrm{e}^{-2\lambda_k h}\right) \left(1 - \mathrm{e}^{-2\lambda_j h}\right) \\
&\leq \quad \frac{1}{2} C(\delta, z)^2 h^{2\delta},
\end{aligned}
$$

and

$$
\begin{aligned}
J_2 &= 2\sum_{k\neq j} z_k^2 z_j^2 m_2(k) m_2(j) \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{e}^{-\lambda_k(t-s)} \mathrm{d}s \\
&\quad \times \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_j(t+h-s)} - \mathrm{e}^{-\lambda_j(t-s)}\right) \mathrm{e}^{-\lambda_j(t-s)} \mathrm{d}s \\
&= 2\sum_{k\neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k \lambda_j} \left(1-\mathrm{e}^{-\lambda_k h}\right)\left(1-\mathrm{e}^{-\lambda_j h}\right)\left(1-\mathrm{e}^{-2\lambda_k h}\right)\left(1-\mathrm{e}^{-2\lambda_j h}\right) \\
&\leq C(\delta, z)^2 h^{2\delta}.
\end{aligned}
$$

Finally, by Proposition 2.5, $J_3 = J_{3,1} + J_{3,2} + J_{3,3}$, where

$$
\begin{aligned}
J_{3,1} &= 2\sum_{n=1}^{\infty} z_n^4 m_2(n)^2 \left(\int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right) \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}s\right)^2 \\
&= 2\sum_{n=1}^{\infty} \frac{z_n^4 m_2(n)^2}{4\lambda_n^2} \left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \left(1-\mathrm{e}^{-2\lambda_n h}\right)^2 \\
&\leq D(\delta, z) h^{2\delta},
\end{aligned}
$$

$$
\begin{aligned}
J_{3,2} &= \sum_{n=1}^{\infty} z_n^4 m_4(n) \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right)^2 \mathrm{e}^{-2\lambda_n(t-s)} \mathrm{d}s \\
&= \sum_{n=1}^{\infty} \frac{z_n^4 m_4(n)}{4\lambda_n} \left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \left(1-\mathrm{e}^{-\lambda_n h}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
J_{3,3} &= \sum_{n=1}^{\infty} z_n^4 m_2(n)^2 \int_{t-h}^{t} \left(\mathrm{e}^{-\lambda_n(t+h-s)} - \mathrm{e}^{-\lambda_n(t-s)}\right)^2 \mathrm{d}s \int_{t-h}^{t} \mathrm{e}^{-2\lambda_n(t-s)} \mathrm{d}s \\
&= \sum_{n=1}^{\infty} \frac{z_n^4 m_2(n)^2}{4\lambda_n^2} \left(1-\mathrm{e}^{-\lambda_n h}\right)^2 \left(1-\mathrm{e}^{-2\lambda_n h}\right)^2 \\
&\leq \frac{1}{2} D(\delta, z) h^{2\delta}.
\end{aligned}
$$

Term $J_{3,2}$ will be estimated later. It is the most difficult term to evaluate as in the denominator we have λ_n in the first power.

We proceed to the estimation of $\mathbb{E}\, I_3^2 I_5^2$. We have

$$\begin{aligned}
\mathbb{E}\, I_3^2 I_5^2 &= \mathbb{E}\left(\sum_{k=1}^{\infty} z_k \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s)\right)^2 \\
&\qquad \times \left(\sum_{j=1}^{\infty} z_j \int_0^{t-h} \left(\mathrm{e}^{-\lambda_j(t-s)} - \mathrm{e}^{-\lambda_j(t-h-s)}\right) \mathrm{d}\widehat{L}_j(s)\right)^2 \\
&= U_1 + U_2 + U_3,
\end{aligned}$$

where

$$\begin{aligned}
U_1 &:= \sum_{k \neq j} z_k^2 z_j^2\, \mathbb{E}\left(\int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s)\right)^2 \\
&\quad \times \mathbb{E}\left(\int_0^{t-h} \left(\mathrm{e}^{-\lambda_j(t-s)} - \mathrm{e}^{-\lambda_j(t-h-s)}\right) \mathrm{d}\widehat{L}_j(s)\right)^2,
\end{aligned}$$

$$\begin{aligned}
&U_2 := 2\sum_{k \neq j} z_k^2 z_j^2\, \mathbb{E} \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s) \\
&\times \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t-s)} - \mathrm{e}^{-\lambda_k(t-h-s)}\right) \mathrm{d}\widehat{L}_k(s) \mathbb{E} \int_0^{t-h} \left(\mathrm{e}^{-\lambda_j(t+h-s)} - \mathrm{e}^{-\lambda_j(t-s)}\right) \mathrm{d}\widehat{L}_j(s) \\
&\times \int_0^{t-h} \left(\mathrm{e}^{-\lambda_j(t-s)} - \mathrm{e}^{-\lambda_j(t-h-s)}\right) \mathrm{d}\widehat{L}_j(s),
\end{aligned}$$

and

$$\begin{aligned}
U_3 &:= \sum_{k=1}^{\infty} z_k^4\, \mathbb{E}\left(\int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)} - \mathrm{e}^{-\lambda_k(t-s)}\right) \mathrm{d}\widehat{L}_k(s)\right)^2 \\
&\quad \times \left(\int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t-s)} - \mathrm{e}^{-\lambda_k(t-h-s)}\right) \mathrm{d}\widehat{L}_k(s)\right)^2.
\end{aligned}$$

We have

$$\begin{aligned}
U_1 &= \sum_{k \neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k \lambda_j} \left(\mathrm{e}^{-\lambda_k(t+h)} - \mathrm{e}^{-\lambda_k t}\right)^2 \left(\mathrm{e}^{2\lambda_k(t-h)} - 1\right) \\
&\qquad \times \left(\mathrm{e}^{-\lambda_j t} - \mathrm{e}^{-\lambda_j(t-h)}\right)^2 \left(\mathrm{e}^{2\lambda_j(t-h)} - 1\right)
\end{aligned}$$

$$\leq \sum_{k\neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k\lambda_j} \left(1-\mathrm{e}^{-\lambda_k h}\right)^2 \left(1-e^{-\lambda_j h}\right)^2$$
$$\leq \frac{1}{4} C(\delta,z)^2 h^{2\delta},$$

$$\begin{aligned} U_2 &= 2\sum_{k\neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k\lambda_j} \left(\mathrm{e}^{-\lambda_k(t+h)}-\mathrm{e}^{-\lambda_k t}\right)\left(\mathrm{e}^{-\lambda_k t}-\mathrm{e}^{-\lambda_k(t-h)}\right) \\ &\quad \times \left(\mathrm{e}^{-\lambda_j(t+h)}-\mathrm{e}^{-\lambda_j t}\right)\left(\mathrm{e}^{-\lambda_j t}-\mathrm{e}^{-\lambda_j(t-h)}\right)\left(\mathrm{e}^{2\lambda_k(t-h)}-1\right)\left(\mathrm{e}^{2\lambda_j(t-h)}-1\right) \\ &\leq 2\sum_{k\neq j} \frac{z_k^2 z_j^2 m_2(k) m_2(j)}{4\lambda_k\lambda_j} \left(1-\mathrm{e}^{-\lambda_k h}\right)\left(1-\mathrm{e}^{-\lambda_k h}\right)\left(1-\mathrm{e}^{-\lambda_j h}\right)\left(1-\mathrm{e}^{-\lambda_j h}\right) \\ &\leq \frac{1}{2} C(\delta,z)^2 h^{2\delta}. \end{aligned}$$

Finally, by Proposition 2.5, $U_3 = U_{3,1} + U_{3,2} + U_{3,3}$, where $U_{3,1}$ equals

$$\begin{aligned} &2\sum_{k=1}^{\infty} z_k^4 m_2(k)^2 \left(\int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)}-\mathrm{e}^{-\lambda_k(t-s)}\right)\left(\mathrm{e}^{-\lambda_k(t-s)}-\mathrm{e}^{-\lambda_k(t-h-s)}\right)\mathrm{d}s\right)^2 \\ &\quad = 2\sum_{k=1}^{\infty} \frac{z_k^4 m_2(k)^2}{4\lambda_k^2} \left(\mathrm{e}^{-\lambda_k(t+h)}-\mathrm{e}^{-\lambda_k t}\right)^2\left(\mathrm{e}^{-\lambda_k t}-\mathrm{e}^{-\lambda_k(t-h)}\right)^2\left(\mathrm{e}^{2\lambda_k(t-h)}-1\right)^2 \\ &\quad \leq \frac{1}{2} D(\delta,z) h^{2\delta}, \end{aligned}$$

$$\begin{aligned} U_{3,2} &= \sum_{k=1}^{\infty} z_k^4 m_4(k) \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)}-\mathrm{e}^{-\lambda_k(t-s)}\right)^2\left(\mathrm{e}^{-\lambda_k(t-s)}-\mathrm{e}^{-\lambda_k(t-h-s)}\right)^2\mathrm{d}s \\ &= \sum_{k=1}^{\infty} \frac{z_k^4 m_4(k)}{4\lambda_k} \left(\mathrm{e}^{-\lambda_k(t+h)}-\mathrm{e}^{-\lambda_k t}\right)^2\left(\mathrm{e}^{-\lambda_k t}-\mathrm{e}^{-\lambda_k(t-h)}\right)^2\left(\mathrm{e}^{4\lambda_k(t-h)}-1\right) \\ &\leq \sum_{k=1}^{\infty} \frac{z_k^4 m_4(k)}{4\lambda_k} \left(1-\mathrm{e}^{-\lambda_k h}\right)^2\left(1-\mathrm{e}^{-\lambda_k h}\right)^2, \end{aligned}$$

and $U_{3,3}$ equals

$$\sum_{k=1}^{\infty} z_k^4 m_2(k)^2 \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t+h-s)}-\mathrm{e}^{-\lambda_k(t-s)}\right)^2\mathrm{d}s$$

$$
\begin{aligned}
&\times \int_0^{t-h} \left(\mathrm{e}^{-\lambda_k(t-s)} - \mathrm{e}^{-\lambda_k(t-h-s)}\right)^2 \mathrm{d}s \\
&= \sum_{k=1}^{\infty} \frac{z_k^4 m_2(k)^2}{4\lambda_k^2} \left(\mathrm{e}^{-\lambda_k(t+h)} - \mathrm{e}^{-\lambda_k t}\right)^2 \left(\mathrm{e}^{-\lambda_k t} - \mathrm{e}^{-\lambda_k(t-h)}\right)^2 \left(\mathrm{e}^{2\lambda_k(t-h)} - 1\right)^2 \\
&\le \frac{1}{4} D(\delta, z) h^{2\delta}.
\end{aligned}
$$

So far we have not estimated the terms $U_{3,2}$ and $J_{3,2}$. However we note that

$$
R := \sum_{n=1}^{\infty} \frac{z_n^4 m_4(n)}{2\lambda_n} \left(1 - \mathrm{e}^{-\lambda_n h}\right) \left(1 - \mathrm{e}^{-4\lambda_n h}\right),
$$

dominates $U_{3,2} + J_{3,2}$. Summing up, we obtain the following estimate

$$
\begin{aligned}
&\forall\, \delta \in (0,1] \text{ and } \forall\, 0 \le t-h \le t \le t+h \le T, \\
&\mathbb{E}\left(Y(t+h) - Y(t)\right)^2 \left(Y(t) - Y(t-h)\right)^2 \\
&\le 6 \left[\left(\sum_{n=1}^{\infty} z_n^2 m_2(n) \lambda_n^{\delta-1} \right)^2 + \sum_{n=1}^{\infty} z_n^4 \left(m_4(n) + m_2(n)^2\right) \lambda_n^{2\delta-2} \right] h^{2\delta} + R.
\end{aligned}
$$

Put $\delta = (\varepsilon+1)/2$. In order to estimate R note that

$$
\begin{aligned}
R &= \sum_{n=1}^{\infty} \frac{z_n^4 m_4(n) \lambda_n^{2\delta}}{2\lambda_n} \frac{\left(1 - \mathrm{e}^{-\lambda_n h}\right)}{\lambda_n^{\delta}} \frac{\left(1 - \mathrm{e}^{-4\lambda_n h}\right)}{\lambda_n^{\delta}} \\
&\le \frac{1}{2} \left(\sum_{n=1}^{\infty} z_n^4 m_4(n) \lambda_n^{\varepsilon} \right) \sup_{\lambda > 0} \frac{\left(1 - \mathrm{e}^{-\lambda h}\right)}{\lambda^{\delta}} \frac{\left(1 - \mathrm{e}^{-4\lambda h}\right)}{\lambda^{\delta}}.
\end{aligned}
$$

Since for all $x, y > 0$ and $\lambda > 0$,

$$
\begin{aligned}
\lambda^{-1-\varepsilon} \left(1 - \mathrm{e}^{-\lambda x}\right) \left(1 - \mathrm{e}^{-4\lambda y}\right) &\le 4^{(1+\varepsilon)/2} \lambda^{-1-\varepsilon} \lambda^{2(1+\varepsilon)/2} (xy)^{(1+\varepsilon)/2} \\
&= 2^{1+\varepsilon} (xy)^{(1+\varepsilon)/2} \\
&\le 2^{(1+\varepsilon)/2} |x+y|^{1+\varepsilon},
\end{aligned}
$$

we have

$$
R \le \sum_{n=1}^{\infty} z_n^4 m_4(n) \lambda_n^{\varepsilon} h^{1+\varepsilon},
$$

which gives the desired estimate. □

8. Proof of Theorem 5.1

Remark 8.1. In the proof we cannot apply directly Lemma 7.1 putting $\lambda = \gamma$. Indeed the quantity $A(z,\gamma,\varepsilon)$ appearing in the formulation of Lemma 7.1 dominates

$$\sum_{n=1}^{\infty} z_n^4 m_4(n) \gamma_n^{\varepsilon}.$$

To apply the Chentsov criterion we need to find an $\varepsilon > 0$ such that

$$\sup_{|z|_{l^2}\le 1} A(z,\gamma,\varepsilon) < \infty.$$

Thus in particular we would need

$$\sup_{|z|_{l^2}\le 1} \sum_{n=1}^{\infty} z_n^4 m_4(n) \gamma_n^{\varepsilon} < \infty.$$

This condition is never satisfied if the sequences $\{m_4(n)\}$ and $\{m_2(n)\}$ are constant and by consequence $\lim_{n\to\infty} \gamma_n = \infty$, which corresponds to the case where (Z_n) are identically distributed.

Recall that the processes L_k were defined at the beginning of the previous section. Let $z \in l^2$. Set

$$\lambda_n := \gamma_n \wedge |z_n|^{-2/\varepsilon}, \qquad n \in \mathbb{N}. \tag{33}$$

Then

$$\sum_{n=1}^{\infty} |z_n| \left| \int_0^t \mathrm{e}^{-\gamma_n(t-s)} \mathrm{d}Z_n(s) \right| \le \sum_{n=1}^{\infty} |z_n| \int_0^t \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}L_n(s) = Y(z)(t) + r(z)(t),$$

where

$$\begin{aligned}
Y(z)(t) &:= \sum_{n=1}^{\infty} |z_n| \int_0^t \mathrm{e}^{-\lambda_n(t-s)} \mathrm{d}\widehat{L}_n(s), \\
r(z)(t) &:= \sum_{n=1}^{\infty} |z_n| \int_0^t \mathrm{e}^{-\lambda_n(t-s)} m_1(n) \mathrm{d}s \le \sum_{n=1}^{\infty} |z_n| m_1(n) \lambda_n^{-1} \\
&\le \sum_{n=1}^{\infty} \left(|z_n|^{1+2/\varepsilon} m_1(n) + |z_n| m_1(n) \gamma_n^{-1} \right) \\
&\le |z|_{l^2}^{1+2/\varepsilon} \sup_n m_1(n) + |z|_{l^2} \left(\sum_{n=1}^{\infty} m_1(n)^2 \gamma_n^{-2} \right)^{1/2}.
\end{aligned}$$

Therefore, thanks to (21) and (22),

$$\sup_{z\in l^2:|z|_{l^2}\le R}\ \sup_{t\in[0,T]} r(z)(t)<\infty, \qquad \forall\, R<\infty. \tag{34}$$

Obviously (28) holds. Let us denote by M the supremum appearing in (22). Then the quantity $A(z,\lambda,\varepsilon)$ appearing in Lemma 7.1, is dominated by

$$\left(|z|^2_{l^2}M + M\sum_{n=1}^{\infty} z_n^2|z_n|^{(1-\varepsilon)/\varepsilon}\right)^2 \\ +(M+M^2)\sum_{n=1}^{\infty}\left(z_n^4+z_n^4|z_n|^{-2}+2z_n^4|z_n|^{\frac{2(1-\varepsilon)}{\varepsilon}}\right).$$

Therefore

$$\sup_{z\in l^2:|z|_{l^2\le R}} A(z,\lambda,\varepsilon)<\infty, \qquad \forall\, R<\infty.$$

By Lemma 7.1, and Corollary 2.2, for any $q\in[1,4)$, and $R>0$,

$$\sup_{z\in l^2:|z|_{l^2}\le R}\mathbb{E}\sup_{t,s\in[0,T]}|Y(z)(t)-Y(z)(s)|^q\le C_1\mathbb{E}|Y(z)(T)|^q+C_2, \tag{35}$$

where C_1 and C_2 are constant. By the Bichteler–Jacod inequality for Poisson integrals in infinite dimensions (see [18]) it follows that there is a constant C depending only on T, such that

$$\mathbb{E}|Y(z)(T)|^4\le C\sum_{n=1}^{\infty} m_4(n)z_n^4\lambda_n^{-1}+C\left(\sum_{n=1}^{\infty} m_2(n)z_n^2\lambda_n^{-1}\right)^2. \tag{36}$$

Therefore,

$$\sup_{z\in l^2:|z|_{l^2}\le R}\mathbb{E}|Y(z)(T)|^4<\infty. \tag{37}$$

Combining (34)–(37) we obtain (23). The càdlàg property follows from the càdlàg property of all processes X_k as the series converges uniformly in $t\in[0,T]$; see the arguments below the formulation of the theorem. □

9. Proof of Theorem 5.2

As in the previous section, (λ_n) is given by (33). Using the arguments from the previous section we see that the proof will be complete as soon as we show that there is a sequence of strictly positive numbers (β_n), increasing to $+\infty$, such that for any $R>0$,

$$\sup_{z\in l^2_{1/\beta}:|z|_{l^2_{1/\beta}}\le R}\left\{A(z,\lambda,\varepsilon)+\sum_{n=1}^{\infty}m_4(n)z_n^4\lambda_n^{-1}+\left(\sum_{n=1}^{\infty}m_2(n)z_n^2\lambda_n^{-1}\right)^2\right\}<\infty$$

and

$$\sup_{z\in l^2_{1/\beta}:|z|_{l^2_{1/\beta}}\le R}\left\{\sum_{n=1}^{\infty}\left(|z_n|^{1+2/\varepsilon}m_1(n)+|z_n|m_1(n)\gamma_n^{-1}+z_n^2m_2(n)\lambda_n^{-1}\right)\right\}<\infty.$$

Let us denote by $M(R,\beta)$ the second supremum above. For any sequence (β_n), we have

$$\begin{aligned}M(R,\beta)&\le R^{1+\frac{2}{\varepsilon}}\sup_n\beta_n^{\frac12+\frac1\varepsilon}m_1(n)\\&+R\left(\sum_{n=1}^{\infty}\beta_n m_1(n)^2\gamma_n^{-2}\right)^{1/2}+R^2\sup_n\beta_n m_2(n)\gamma_n+R^{2+2\varepsilon}\sup_n\beta_n^{1+\varepsilon}m_2(n).\end{aligned}$$

Next

$$\begin{aligned}A(z,\lambda,\varepsilon)&\le R^4\sup_n\beta_n^2m_2(n)^2\gamma_n^{\varepsilon-1}+R^{4+2\frac{1-\varepsilon}{\varepsilon}}\sup_n\beta_n^{2-\frac{\varepsilon-1}{\varepsilon}}m_2(n)^2\\&+R^4\sup_n\left(\beta_n^2m_4(n)\gamma_n^{\varepsilon-1}+\beta_n^2m_2(n)^2\gamma_n^{\varepsilon-1}\right)+R^2\sup_n\beta_n m_2(n)^2\\&+R^{4+2\frac{1-\varepsilon}{\varepsilon}}\sup_n\beta_n^{2-\frac{\varepsilon-1}{2}}\left(m_2(n)^2+m_4(n)\right),\end{aligned}$$

and finally

$$\begin{aligned}&\sum_{n=1}^{\infty}m_4(n)z_n^4\lambda_n^{-1}+\left(\sum_{n=1}^{\infty}m_2(n)z_n^2\lambda_n^{-1}\right)^2\\&\le R^4\sup_n\beta_n^2m_4(n)\gamma_n^{-1}+R^4\sup_n\beta_n^2m_2(n)^2\gamma_n^{-2}\\&+R^{4+\frac{2}{\varepsilon}}\sup_n\beta_n^{\frac1\varepsilon+2}m_4(n)+R^{4+\frac4\varepsilon}\sup_n\beta_n^{2+\frac2\varepsilon}m_2(n)^2.\end{aligned}$$

By direct calculations and assumption (24) we arrive at the statement of the theorem. □

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