

## Projective Module Description of the $q$ -Monopole

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**Abstract:** The Dirac  $q$ -monopole connection is used to compute projector matrices of quantum Hopf line bundles for arbitrary winding number. The Chern–Connes pairing of cyclic cohomology and  $K$ -theory is computed for the winding number  $-1$ . The non-triviality of this pairing is used to conclude that the quantum principal Hopf fibration is non-cleft. Among general results, we provide a left-right symmetric characterization of the canonical strong connections on quantum principal homogeneous spaces with an injective antipode. We also provide for arbitrary strong connections on algebraic quantum principal bundles (Hopf–Galois extensions) their associated covariant derivatives on projective modules.

### Introduction

The goal of this paper is to provide a better understanding of the relationship between the quantum-group and  $K$ -theory approach to the noncommutative-geometry gauge theory. The latter approach is based on the classical Serre-Swan theorem that allows one to think of vector bundles as projective modules. The former comes from the concept of a Hopf–Galois extension which describes a quantum principal bundle the same way Hopf algebras describe quantum groups. Here a Hopf algebra  $H$  plays the role of the algebra of functions on the structure group, and the total space of a bundle is replaced by an  $H$ -comodule algebra  $P$ . We rely on the Hopf–Galois theory to derive our noncommutative-geometric constructions. On the other hand, it is the machinery of noncommutative geometry that allows us to obtain a Galois-theoretic result: We employ the Chern–Connes pairing to prove the non-cleftness of the Hopf–Galois extension of the algebraic quantum principal Hopf fibration.

We begin in Sect. 1 with some preliminaries about Hopf–Galois extensions, connections and connection 1-forms on algebraic quantum principal bundles, and connections on projective modules. In Sect. 2 we extend the existing theory with some general results

about strong connections, their covariant derivatives on projective modules, and bicovariant splittings of canonical Hopf algebra surjections. We also discuss how to obtain projector matrices from splittings of the multiplication map. In Sect. 3 we first define (the space of sections of) a quantum Hopf line bundle as a bimodule associated to the quantum principal Hopf fibration via a one-dimensional corepresentation of the Hopf algebra  $k[z, z^{-1}]$ . Then we use a canonical strong connection on the quantum principal Hopf fibration (Dirac  $q$ -monopole) to compute, for any one-dimensional corepresentation, left and right projector matrices of the thus defined quantum Hopf line bundles. This computation is the main part of our paper and provides the projective-module characterization of the  $q$ -monopole. Further results relating to the Chern–Connes pairing are in Sect. 4. We end with the Appendix where we show that the only invertible elements of the coordinate ring of  $SL_q(2)$  are non-zero numbers, and use it as an alternative way to conclude the non-cleftness of the quantum Hopf fibration.

To focus attention and take advantage of the cyclic cohomology results in [MNW91], we work over a ground field  $k$  of characteristic zero, and assume that  $q$  is a non-zero element in  $k$  that is not a root of 1. We use the Sweedler notation  $\Delta h = h_{(1)} \otimes h_{(2)}$  (summation understood) and its derivatives. The antipode of the Hopf algebra is a linear map  $S : H \rightarrow H$ , and the counit is an algebra map  $\varepsilon : H \rightarrow k$  obeying certain properties. The convolution product of two linear maps from a coalgebra to an algebra is denoted in the following way:  $(f * g)(c) := f(c_{(1)})g(c_{(2)})$ . We use interchangeably the words “colinear” and “covariant” with respect to linear maps that preserve the comodule structure. For an introduction to noncommutative geometry, quantum groups, Hopf–Galois extensions and quantum-group gauge theory we refer to [C-A94, L-G97], [M-S95], [S-HJ94] and [BM93, BM98] respectively.

### 1. Preliminaries

We begin by recalling basic definitions and known results.

**Definition 1.1.** *Let  $\mathcal{E}$  be a left  $B$ -module, and  $(\Omega(B), d)$  a differential algebra on  $B$ . A linear map  $\nabla : \Omega^*(B) \otimes_B \mathcal{E} \rightarrow \Omega^{*+1}(B) \otimes_B \mathcal{E}$  is called a connection (covariant derivative) on  $\mathcal{E}$  iff  $\forall \xi \in \mathcal{E}, \lambda \in \Omega(B) : \nabla(\lambda \otimes_B \xi) = \lambda(\nabla \xi) + d\lambda \otimes_B \xi$ .*

In the case of the universal differential algebra the existence of a connection is *equivalent* to the projectivity of  $\mathcal{E}$  [CQ95, Corollary 8.2]; [L-G97, Proposition 8.2.3]. If  $\mathcal{E}$  is projective then a connection exists for any differential algebra because it can be obtained from the universal differential algebra and the canonical surjection onto a given differential algebra [C-A94, p. 555].

**Definition 1.2.** *Let  $H$  be a Hopf algebra,  $P$  be a right  $H$ -comodule algebra with multiplication  $m_P$  and coaction  $\Delta_R$ , and  $B := P^{coH} := \{p \in P \mid \Delta_R p = p \otimes 1\}$  the subalgebra of coinvariants. We say that  $P$  is a (right)  $H$ -Galois extension of  $B$  iff the canonical left  $P$ -module right  $H$ -comodule map*

$$\chi := (m_P \otimes id) \circ (id \otimes_B \Delta_R) : P \otimes_B P \longrightarrow P \otimes H$$

*is bijective. We say that  $P$  is a faithfully flat  $H$ -Galois extension of  $B$  iff  $P$  is faithfully flat as a right and left  $B$ -module.*

For a comprehensive review of the concept of faithful flatness see [B-N72].

**Definition 1.3.** An  $H$ -Galois extension is called **cleft** iff there exists a unital convolution invertible linear map  $\Phi : H \rightarrow P$  satisfying  $\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta$ . We call  $\Phi$  a *cleaving map* of  $P$ .

Note that, in general,  $\Phi$  is *not* uniquely determined by its defining conditions. Observe also that the unitality assumption for the cleaving map is unnecessary in the sense that any right colinear convolution invertible mapping can be normalised to be unital. Indeed, let  $\tilde{\Phi}$  be such a mapping, and  $\tilde{\Phi}(1) := b$ . By the colinearity, we have that  $b \in B$ , and the convolution invertibility entails that  $b$  is invertible. Also,  $b^{-1} \otimes 1 = b^{-1} \Delta_R(bb^{-1}) = b^{-1}b \Delta_R(b^{-1}) = \Delta_R(b^{-1})$ . It is straightforward to check that  $\Phi := b^{-1}\tilde{\Phi}$  is right colinear, convolution invertible and unital. Let us also remark that a cleaving map is necessarily injective:

$$\begin{aligned} & (m_P \circ (m_P \otimes id) \circ (id \otimes \Phi^{-1} \otimes id) \circ (id \otimes \Delta) \circ \Delta_R \circ \Phi)(h) \\ &= \Phi(h_{(1)})\Phi^{-1}(h_{(2)})h_{(3)} = h, \quad \forall h \in H. \end{aligned}$$

To fix convention, let us recall that the universal differential calculus (grade one of the universal differential algebra) can be defined as the kernel of the multiplication map  $\Omega^1 B := \text{Ker}(B \otimes B \xrightarrow{m} B)$  with the differential  $db := 1 \otimes b - b \otimes 1$  (e.g., see [L-G97, Sect. 7.1]). (We abuse the notation and use the same letter  $d$  to signify both the universal and general differential.) The following are the universal-differential-calculus versions of more general definitions in [BM93,H-PM96]:

**Definition 1.4 ([BM93]).** Let  $B \subseteq P$  be an  $H$ -Galois extension. Denote by  $\Omega^1 P$  the universal differential calculus on  $P$ . A left  $P$ -module projection  $\Pi$  on  $\Omega^1 P$  is called a **connection** on a quantum principal bundle iff

1.  $\text{Ker } \Pi = P(\Omega^1 B)P$  (horizontal forms),
2.  $\Delta_{\mathcal{R}} \circ \Pi = (\Pi \otimes id) \circ \Delta_{\mathcal{R}}$  (right covariance).

Here  $\Delta_{\mathcal{R}}$  is the right coaction on differential forms given by the formula  $\Delta_{\mathcal{R}}(ada') := a_{(0)}da'_{(0)} \otimes a_{(1)}a'_{(1)}$ , where  $\Delta_{\mathcal{R}a} := a_{(0)} \otimes a_{(1)}$  (summation understood). Coaction on higher order forms is defined in the same manner.

**Definition 1.5 ([BM93]).** Let  $P, H, B$  and  $\Omega^1 P$  be as above. A  $k$ -homomorphism  $\omega : H \rightarrow \Omega^1 P$  such that  $\omega(1) = 0$  is called a **connection form** iff it satisfies the following properties:

1.  $(m_P \otimes id) \circ (id \otimes \Delta_R) \circ \omega = 1 \otimes (id - \varepsilon)$  (fundamental vector field condition),
2.  $\Delta_{\mathcal{R}} \circ \omega = (\omega \otimes id) \circ ad_R$ ,  $ad_R(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$  (right adjoint covariance).

For every Hopf-Galois extension there is a one-to-one correspondence between connections and connection forms (see [M-S97, Proposition 2.1]). In particular, the connection  $\Pi^\omega$  associated to a connection form  $\omega$  is given by the formula:

$$\Pi^\omega(dp) = p_{(0)}\omega(p_{(1)}). \quad (1.1)$$

$\Pi^\omega$  is a left  $P$ -module homomorphism, so that it suffices to know its values on exact forms.

**Definition 1.6 ([H-PM96]).** Let  $\Pi$  be a connection in the sense of Definition 1.4. It is called **strong** iff  $(id - \Pi)(dP) \subseteq (\Omega^1 B)P$ . We say that a connection form is **strong** iff its associated connection is strong.

A natural next step is to consider associated quantum vector bundles. More precisely, what we need here is a replacement of the module of sections of an associated vector bundle. In the classical case such sections can be equivalently described as “functions of type  $\varrho$ ” from the total space of a principal bundle to a vector space. We follow this construction in the quantum case by considering  $B$ -bimodules of colinear maps  $\text{Hom}_\rho(V, P)$  associated with an  $H$ -Galois extension  $B \subseteq P$  via a corepresentation  $\rho : V \rightarrow V \otimes H$  (see [D-M96]). For our later purpose, we need the following reformulation of [BM93, Prop. A.7]:

**Lemma 1.7.** *Let  $B \subseteq P$  be a cleft  $H$ -Galois extension and  $\rho : V \rightarrow V \otimes H$  a right corepresentation of  $H$  on  $V$ . Then the space of colinear maps  $\text{Hom}_\rho(V, P)$  is isomorphic as a left  $B$ -module to the free module  $\text{Hom}(V, B)$ .*

## 2. Strong Connections on Associated Projective Modules

First we study a general setting for translating strong connections on algebraic quantum principal bundles to connections on projective modules. The associated bimodule of colinear maps is finitely generated projective as a left module over the subalgebra of coinvariants under rather unrestrictive assumptions. However, we do not assume the projectivity of this module in the following two propositions, as it is needed only later to ensure the existence of a connection. Also, although we work only with the universal differential algebra in the sequel, we do not assume here that the differential algebra is universal. It suffices that it is right-covariant, i.e., the right coaction is well-defined on differential forms, and right-covariant and right-flat in the second proposition. On the other hand, we do not aim here at the utmost generality but try to keep our noncommutative-geometric motivation evident.

**Proposition 2.1.** *Let  $H$  be a Hopf algebra with a bijective antipode,  $P$  a faithfully flat  $H$ -Galois extension of  $B$ , and  $V \xrightarrow{\rho} V \otimes H$  ( $\dim V < \infty$ ) a coaction. Denote by  $\text{Hom}_\rho(V, P)$  the  $B$ -bimodule of colinear homomorphisms from  $V$  to  $P$ , and choose a right-covariant differential algebra  $\Omega(P)$ . Then the following map*

$$\check{\ell} : \Omega(B) \otimes_B \text{Hom}_\rho(V, P) \longrightarrow \text{Hom}_\rho(V, \Omega(B)P), \quad (\check{\ell}(\lambda \otimes_B \varphi))(v) = \lambda\varphi(v),$$

*is an isomorphism of graded left  $\Omega(B)$ -modules.*

*Proof.* It suffices to show that  $\check{\ell}$  has an inverse. By choosing a linear basis  $\{\lambda_\mu\}$  of  $\Omega(B)$ , for any  $\varphi \in \text{Hom}_\rho(V, \Omega(B)P)$  we can write  $\varphi(v) = \sum_\mu \lambda_\mu \varphi^\mu(v)$ . The point is now to show that we can always choose each  $\varphi^\mu$  to be an element of  $\text{Hom}_\rho(V, P)$ . It can be done by assuming flatness of  $\Omega(B)$  (see Proposition 2.3), or by employing our assumptions on the Hopf-Galois extension.

**Lemma 2.2.** *Under the assumptions of Proposition 2.1, for any  $\varphi \in \text{Hom}_\rho(V, \Omega(B)P)$  there exist colinear homomorphisms  $\tilde{\varphi}^\mu \in \text{Hom}_\rho(V, P)$  such that  $\varphi(v) = \sum_\mu \lambda_\mu \tilde{\varphi}^\mu(v)$ ,  $\forall v \in V$ .*

*Proof.* By assumption, we have

$$\begin{aligned} ((\Delta_{\mathcal{R}} \circ \varphi) \otimes id)(v_{(0)} \otimes v_{(1)}) &= (((\varphi \otimes id) \circ \rho) \otimes id)(v_{(0)} \otimes v_{(1)}), \text{ i.e.,} \\ \sum_\mu \lambda_\mu \varphi^\mu(v_{(0)})_{(0)} \otimes \varphi^\mu(v_{(0)})_{(1)} \otimes v_{(1)} &= \sum_\mu \lambda_\mu \varphi^\mu(v_{(0)}) \otimes v_{(1)} \otimes v_{(2)}. \end{aligned}$$

Taking advantage of the faithful flatness of  $P$ , Theorem I in [S-HJ90] and (1.6) in [D-Y85] (Remark 3.3 in [S-HJ90]), we know that there exists a unital colinear map  $j : H \rightarrow P$ . Applying

$$m_{\Omega(P)} \circ (id \otimes (j \circ m)) \circ (id \otimes S \otimes id),$$

where  $m_{\Omega(P)}$  and  $m$  are appropriate multiplication maps, to both sides of the above equality, we get

$$\sum_{\mu} \lambda_{\mu} \varphi^{\mu}(v_{(0)})_{(0)} j(S(\varphi^{\mu}(v_{(0)})_{(1)})v_{(1)}) = \sum_{\mu} \lambda_{\mu} \varphi^{\mu}(v_{(0)}) j(S(v_{(1)})v_{(2)}),$$

Hence, by the unitality of  $j$ , we obtain

$$\varphi(v) = \sum_{\mu} \lambda_{\mu} \varphi^{\mu}(v_{(0)})_{(0)} j(S(\varphi^{\mu}(v_{(0)})_{(1)})v_{(1)}).$$

On the other hand, using the colinearity of  $j$  it is straightforward to verify that each of the maps  $v \xrightarrow{\tilde{\varphi}^{\mu}} \varphi^{\mu}(v_{(0)})_{(0)} j(S(\varphi^{\mu}(v_{(0)})_{(1)})v_{(1)})$  is colinear.  $\square$

The next step is to take advantage of the existence of the translation map  $H \xrightarrow{\tau} P \otimes_B P$ ,  $\tau(h) := \chi^{-1}(1 \otimes h)$  (see Definition 1.2), and define an auxiliary isomorphism

$$f : \Omega(B)P \rightarrow \Omega(B) \otimes_B P, \quad f := (m \otimes_B id) \circ (id \otimes \tau) \circ \Delta_{\mathcal{R}}.$$

From the definition of the translation map it follows that

$$\begin{aligned} f(\lambda p) &= \lambda p_{(0)} \tau(p_{(1)}) = \lambda p_{(0)} \chi^{-1}(1 \otimes p_{(1)}) \\ &= \lambda \chi^{-1}(p_{(0)} \otimes p_{(1)}) = \lambda \chi^{-1}(\chi(1 \otimes_B p)) = \lambda \otimes_B p. \end{aligned}$$

(Note that  $f$  is the inverse of the multiplication map.) Moreover, let  $I$  be the restriction to  $\text{Hom}_{\rho}(V, \Omega(B)P)$  of the canonical isomorphism from  $\text{Hom}(V, \Omega(B)P)$  to  $\Omega(B)P \otimes V^*$ . Then we have a well-defined map

$$\begin{aligned} \hat{\ell} &:= (id \otimes_B I^{-1}) \circ (f \otimes id) \circ I : \text{Hom}_{\rho}(V, \Omega(B)P) \longrightarrow \Omega(B)P \otimes_B \text{Hom}_{\rho}(V, P), \\ \hat{\ell}(\varphi) &= ((id \otimes_B I^{-1}) \circ (f \otimes id)) \left( \sum_i \sum_{\mu} \lambda_{\mu} \tilde{\varphi}^{\mu}(e_i) \otimes e^i \right) = \sum_{\mu} \lambda_{\mu} \otimes_B \tilde{\varphi}^{\mu}, \end{aligned}$$

where  $\{e_i\}$  is a basis of  $V$ ,  $\{e^i\}$  its dual, and (by the above lemma) we choose  $\tilde{\varphi}^{\mu} \in \text{Hom}_{\rho}(V, P)$  such that  $\varphi(v) = \sum_{\mu} \lambda_{\mu} \tilde{\varphi}^{\mu}(v)$ . It is straightforward to check that  $\hat{\ell} = \check{\ell}^{-1}$ , as desired.  $\square$

**Proposition 2.3.** *Let  $H$  be a Hopf algebra and  $P \supseteq B$  an  $H$ -Galois extension. Let  $\check{\ell}$  be the map defined in Proposition 2.1. Then if  $\Omega(B)$  is flat as a right  $B$ -module,  $\check{\ell}$  is an isomorphism of graded left  $\Omega(B)$ -modules.*

*Proof.* Let  $\check{\rho} : \text{Hom}(V, \Omega(B)P) \rightarrow \text{Hom}(V, \Omega(B)P \otimes H)$  be a left  $\Omega(B)$ -linear homomorphism defined by the formula  $\check{\rho}(\varphi)(v) = \varphi(v_{(0)}) \otimes v_{(1)} - \varphi(v)_{(0)} \otimes \varphi(v)_{(1)}$ , and let  $\tilde{\rho}$  denote its restriction to  $\text{Hom}(V, P)$ . Evidently, we have  $\text{Ker } \check{\rho} = \text{Hom}_\rho(V, \Omega(B)P)$  and  $\text{Ker } \tilde{\rho} = \text{Hom}_\rho(V, P)$ . Moreover, since  $\Omega(B)$  is flat as a right  $B$ -module, we have the following commutative diagram with exact rows of left  $\Omega(B)$ -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega(B) \otimes_B \text{Hom}_\rho(V, P) & \longrightarrow & \Omega(B) \otimes_B \text{Hom}(V, P) & \xrightarrow{id \otimes \tilde{\rho}} & \Omega(B) \otimes_B \text{Hom}(V, P \otimes H) \\ & & \downarrow \check{\ell} & & \downarrow \ell & & \downarrow \tilde{\ell} \\ 0 & \longrightarrow & \text{Hom}_\rho(V, \Omega(B)P) & \longrightarrow & \text{Hom}(V, \Omega(B)P) & \xrightarrow{\check{\rho}} & \text{Hom}(V, \Omega(B)P \otimes H). \end{array} \quad (2.1)$$

Here  $\ell$  is defined by the formula  $\ell(\sum_\mu \lambda_\mu \otimes_B \varphi^\mu)(v) = \sum_\mu \lambda_\mu \varphi^\mu(v)$ , and  $\tilde{\ell}$  is given the same way. With the help of the translation map  $H \xrightarrow{\tau} P \otimes_B P$ , reasoning as in the proof of the preceding proposition, one can show that  $\ell$  and  $\tilde{\ell}$  are isomorphisms. By standard diagram chasing (or completing the left hand side of (2.1) with zeros and invoking the Five Isomorphism Lemma), one can conclude from the diagram (2.1) that  $\check{\ell}$  is also an isomorphism.  $\square$

If  $\omega$  is a strong connection form, then

$$(id - \Pi^\omega) \circ d \circ \text{Hom}_\rho(V, P) \subseteq \text{Hom}_\rho(V, \Omega^1(B)P).$$

Assuming also that the conditions allowing us to utilise one of the above propositions are fulfilled, we can define the *covariant derivative* associated to  $\omega$  in the following way:

$$\nabla^\omega : \text{Hom}_\rho(V, P) \longrightarrow \Omega^1(B) \otimes_B \text{Hom}_\rho(V, P), \quad \nabla^\omega \xi := \check{\ell}^{-1}((id - \Pi^\omega) \circ d \circ \xi). \quad (2.2)$$

One can check that  $\nabla^\omega$  satisfies the Leibniz rule  $\nabla^\omega(b\xi) = b\nabla^\omega \xi + db \otimes_B \xi$ . Hence  $\nabla^\omega$  can be extended (by the Leibniz rule) to an endomorphism of  $\Omega(B) \otimes_B \text{Hom}_\rho(V, P)$  which is of degree 1 with respect to the grading of  $\Omega(B)$ .

Our second group of results concerns the canonical connection on a quantum principal homogeneous space (principal homogenous  $H$ -Galois extension), which is the general construction behind the Dirac  $q$ -monopole. A principal homogenous  $H$ -Galois extension  $B \subseteq P$  is a Hopf-Galois extension obtained from a surjective Hopf algebra map  $\pi : P \rightarrow H$  which defines the right comodule structure by the formula  $\Delta_R := (id \otimes \pi) \circ \Delta$ . We know from the proof of [BM93, Proposition 5.3] that if  $B \subseteq P$  is a principal homogenous  $H$ -Galois extension, and  $i : H \rightarrow P$  is a linear unital map such that  $\pi \circ i = id$  (splitting of  $\pi$ ) and

$$(id \otimes \pi) \circ ad_R \circ i = (i \otimes id) \circ ad_R, \quad (2.3)$$

then  $\omega := (S * d) \circ i$  is a connection form in the sense of Definition 1.5. (Note that since  $i$  is a splitting of a Hopf algebra map, it is counital:  $\varepsilon_H = \varepsilon_H \circ \pi \circ i = \varepsilon_P \circ i$ .) We call the thus constructed connection the *canonical connection (form)* associated to splitting  $i$ . (In what follows, we skip writing ‘‘form’’ for the sake of brevity.) The next step is towards a left-right symmetric characterization of strong canonical connections.

**Proposition 2.4.** *The canonical connection associated to splitting  $i : H \rightarrow P$  satisfying the above conditions is strong if and only if the splitting  $i$  obeys in addition the right covariance condition*

$$(i \otimes id) \circ \Delta = \Delta_R \circ i.$$

*Proof.* First we need to reduce the strongness condition for the canonical connection to a simpler form:

**Lemma 2.5.** *The canonical connection  $\omega$  associated to  $i : H \rightarrow P$  is strong if and only if*

$$i(h_{(2)})_{(2)} \otimes h_{(1)} \overline{S\pi(i(h_{(2)})_{(1)})} = i(h) \otimes 1, \quad \forall h \in H. \quad (2.4)$$

*Proof.* To simplify the notation, let us put  $\pi(p) = \bar{p}$ . Also, let  $\Pi^\omega$  denote the connection associated to  $\omega$ , i.e.,  $\Pi^\omega(dp) = p_{(1)}\omega(\overline{p_{(2)}})$ . (We take advantage of the fact that  $\Delta_R = (id \otimes \pi) \circ \Delta$ , see (1.1).) Using the Leibniz rule we obtain:

$$\begin{aligned} & (id - \Pi^\omega)(dp) \\ &= d(p_{(1)} S(i(\overline{p_{(2)}})_{(1)})) i(\overline{p_{(2)}})_{(2)} - p_{(1)} S(i(\overline{p_{(2)}})_{(1)}) d(i(\overline{p_{(2)}})_{(2)}) \\ &= d(p_{(1)} S(i(\overline{p_{(2)}})_{(1)})) i(\overline{p_{(2)}})_{(2)} \\ &= 1 \otimes p - p_{(1)} S(i(\overline{p_{(2)}})_{(1)}) \otimes i(\overline{p_{(2)}})_{(2)}. \end{aligned}$$

On the other hand, applying  $\Delta_R \otimes id$  to  $p_{(1)} S(i(\overline{p_{(2)}})_{(1)}) \otimes i(\overline{p_{(2)}})_{(2)}$  yields

$$p_{(1)} S(i(\overline{p_{(3)}})_{(2)}) \otimes \overline{p_{(2)}} S(i(\overline{p_{(3)}})_{(1)}) \otimes i(\overline{p_{(3)}})_{(3)}.$$

Remembering that  $(\Omega^1 B)P \subseteq B \otimes P$ , we conclude that the strongness condition (see Definition 1.6, cf. [M-S97, (11)]) of the canonical connection is equivalent to

$$p_{(1)} S(i(\overline{p_{(3)}})_{(2)}) \otimes \overline{p_{(2)}} S(i(\overline{p_{(3)}})_{(1)}) \otimes i(\overline{p_{(3)}})_{(3)} = p_{(1)} S(i(\overline{p_{(2)}})_{(1)}) \otimes 1 \otimes i(\overline{p_{(2)}})_{(2)}.$$

The above equation is of the form  $(id * f_1)(p) = (id * f_2)(p)$ . Since the antipode  $S$  is the convolution inverse of  $id$ , it is equivalent to  $f_1(p) = f_2(p)$ . Therefore we can cancel the  $p_{(1)}$  product from both sides. Also, since  $\pi$  is surjective and a coalgebra map, we can replace  $\pi(p)$  by a general element  $h \in H$ . Thus we arrive at

$$S(i(h_{(2)})_{(2)}) \otimes h_{(1)} \overline{S(i(h_{(2)})_{(1)})} \otimes i(h_{(2)})_{(3)} = S(i(h)_{(1)}) \otimes 1 \otimes i(h)_{(2)}.$$

Moreover, for any Hopf algebra the map  $(S \otimes id) \circ \Delta$  is injective (apply  $\varepsilon \otimes id$ ). Consequently, the strongness is equivalent to the condition

$$i(h_{(2)})_{(2)} \otimes h_{(1)} \overline{S(i(h_{(2)})_{(1)})} = i(h) \otimes 1, \quad \forall h \in H,$$

as claimed.  $\square$

Note now that we can write the adjoint covariance of  $i$ , in an explicit manner, as

$$i(h)_{(2)} \otimes S(i(h)_{(1)}) \overline{i(h)_{(3)}} = i(h_{(2)}) \otimes (Sh_{(1)})h_{(3)}, \quad \forall h \in H. \quad (2.5)$$

In this case

$$\begin{aligned} & i(h_{(1)}) \otimes h_{(2)} \\ &= i(h_{(3)}) \otimes h_{(1)} S(h_{(2)})h_{(4)} \\ &= (1 \otimes h_{(1)})((i \otimes id) \circ ad_R)(h_{(2)}) \\ &= i(h_{(2)})_{(2)} \otimes h_{(1)} \overline{S(i(h_{(2)})_{(1)})} \overline{i(h_{(2)})_{(3)}}. \end{aligned}$$

Assume that  $\omega$  is strong. Hence, by the above lemma, the strongness condition implies that  $i(h_{(1)}) \otimes h_{(2)} = i(h)_{(1)} \otimes \overline{i(h)}_{(2)}$  as required. Conversely, using the right covariance of  $i$  for the first step and (2.5) for the second, we compute the left hand side of (2.4) as

$$\begin{aligned} & i(h_{(2)})_{(2)} \otimes h_{(1)} \overline{S(i(h_{(2)}_{(1)}))} \overline{i(h_{(2)}_{(3)})} \overline{S(i(h_{(2)}_{(4)}))} \\ &= i(h_{(2)})_{(2)} \otimes h_{(1)} \overline{S(i(h_{(2)}_{(1)}))} \overline{i(h_{(2)}_{(3)})} Sh_{(3)} \\ &= i(h_{(3)}) \otimes h_{(1)} S(h_{(2)}) h_{(4)} S(h_{(5)}) \\ &= i(h) \otimes 1. \end{aligned}$$

Hence the canonical connection is strong by Lemma 2.5.  $\square$

**Corollary 2.6.** *Assume that antipode  $S$  is injective. Then strong canonical connections are in 1-1 correspondence with linear unital splittings of  $\pi$  obeying the two conditions*

$$(i \otimes id) \circ \Delta = \Delta_R \circ i, \quad (id \otimes i) \circ \Delta = \Delta_L \circ i,$$

where  $\Delta_R = (id \otimes \pi) \circ \Delta$ ,  $\Delta_L = (\pi \otimes id) \circ \Delta$ .

*Proof.* Assume first that the canonical connection associated to  $i$  is strong. Then, by the preceding proposition,  $i$  is right covariant and (2.5) holds. Hence

$$\begin{aligned} & i(h_{(1)})_{(2)} \otimes \overline{S(i(h_{(1)}_{(1)}))} h_{(2)} \\ &= i(h)_{(1)(2)} \otimes \overline{S(i(h)_{(1)(1)})} \overline{i(h)}_{(2)} \\ &= i(h)_{(2)} \otimes \overline{S(i(h)_{(1)})} \overline{i(h)}_{(3)} \\ &= i(h_{(2)}) \otimes (Sh_{(1)})h_{(3)}. \end{aligned}$$

Reasoning as in the proof of Lemma 2.5, we can cancel  $h_{(2)}$  and  $h_{(3)}$  from the two sides. Then cancelling  $S$  from both sides (we assume  $S$  to be injective), we have  $i(h)_{(2)} \otimes \overline{i(h)}_{(1)} = i(h_{(2)}) \otimes h_{(1)}$ , which is the left covariance condition.

Conversely, if the left and right covariance conditions hold then

$$\begin{aligned} & i(h_{(2)}) \otimes (Sh_{(1)})h_{(3)} \\ &= i(h_{(2)(1)}) \otimes (Sh_{(1)})h_{(2)(2)} \\ &= i(h_{(2)})_{(1)} \otimes (Sh_{(1)})\overline{i(h_{(2)})_{(2)}} \\ &= i(h)_{(2)(1)} \otimes \overline{S(i(h)_{(1)})} \overline{i(h)}_{(2)(2)}, \end{aligned}$$

which is the same as (2.5). Invoking again the preceding proposition, we can conclude that the canonical connection associated to  $i$  is strong as required.  $\square$

*Remark 2.7.* Let  $\pi : P \rightarrow H$  be a Hopf algebra surjection. If a linear map  $i : H \rightarrow P$  is counital and left or right colinear, then  $i$  is a splitting of  $\pi$ , i.e.,  $\pi \circ i = id$ . Indeed, if  $i$  is right colinear ( $i(h)_{(1)} \otimes \pi(i(h)_{(2)}) = i(h_{(1)}) \otimes h_{(2)}$ ), we have:

$$\begin{aligned} (\pi \circ i)(h) &= \varepsilon((\pi \circ i)(h)_{(1)}) (\pi \circ i)(h)_{(2)} \\ &= \varepsilon(\pi(i(h)_{(1)})) \pi(i(h)_{(2)}) = \varepsilon(\pi(i(h_{(1)}))) h_{(2)} = h. \end{aligned}$$

The left-sided case is analogous.

We end this section by showing how to obtain a projector matrix (explicit embedding of a projective module in a free module) from the canonical strong connection. It is known [DGH] that strong connection forms on  $P$  are *equivalent* to unital left  $B$ -linear right  $H$ -colinear splittings of the multiplication map  $m : B \otimes P \rightarrow P$ . Explicitly, if  $\omega$  is a strong connection form, then

$$s : P \longrightarrow B \otimes P, \quad s(p) = p \otimes 1 + p_{(0)}\omega(p_{(1)}) \quad (2.6)$$

gives the desired splitting. (Solving this equation for  $\omega$  one gets  $\omega(h) = h^{[1]}s(h^{[2]}) - 1 \otimes \varepsilon(h)$ , where  $h^{[1]} \otimes_B h^{[2]} = \chi^{-1}(1 \otimes h)$ , summation understood, see Definition 1.2.) In particular, for the canonical strong connection associated to a bicovariant splitting  $i$  (i.e.,  $\omega = (S * d) \circ i$ ), we have:

$$s(p) = p_{(1)}Si(\overline{p_{(2)}})_{(1)} \otimes i(\overline{p_{(2)}})_{(2)}. \quad (2.7)$$

Note that a splitting of the multiplication map is almost the same as a projector matrix, for it is an embedding of  $P$  in the free  $B$ -module  $B \otimes P$ . (We will use formula (2.7) in the next section to compute projector matrices of quantum Hopf line bundles from the Dirac  $q$ -monopole connection.) To turn (2.7) into a concrete recipe for producing finite size projector matrices of finitely generated projective modules, let us note the following general lemma:

**Lemma 2.8.** *Let  $A$  be an algebra and  $M$  a projective left  $A$ -module generated by linearly independent generators  $g_1, \dots, g_n$ . Also, let  $\{\tilde{g}_\mu\}_{\mu \in I}$  be a completion of  $\{g_1, \dots, g_n\}$  to a linear basis of  $M$ ,  $f_2$  be a left  $A$ -linear splitting of the multiplication map  $m : A \otimes M \rightarrow M$  given by the formula  $f_2(g_k) = \sum_{l=1}^n a_{kl} \otimes g_l + \sum_{\mu \in I} a_{k\mu} \otimes \tilde{g}_\mu$ , and  $c_{\mu l} \in A$  a choice of coefficients such that  $\tilde{g}_\mu = \sum_{l=1}^n c_{\mu l} g_l$ . Then  $e_{kl} = a_{kl} + \sum_{\mu \in I} a_{k\mu} c_{\mu l}$  defines a projector matrix of  $M$ , i.e.,  $e \in M_n(A)$ ,  $e^2 = e$  and  $A^n e$  and  $M$  are isomorphic as left  $A$ -modules.*

*Proof.* Note first that we do not lose any generality by assuming  $g_1, \dots, g_n$  to be linearly independent (we can always remove generators that are linear combinations of other generators), and that a splitting of the multiplication map always exists by the projectivity assumption (cf. [CQ95, Sect. 8]). Let  $N$  be the kernel of the surjection  $f_1 : A^n \rightarrow M = A^n/N$ ,  $f_1(e_k) = g_k$ ,  $k \in \{1, \dots, n\}$ , where  $\{e_k\}_{k \in \{1, \dots, n\}}$  is the standard basis of  $A^n$ , i.e.,  $e_k$  is the row with zeros everywhere except for the  $k^{\text{th}}$  place where there is 1. We have the following commutative diagram of left  $A$ -module homomorphisms whose rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A \otimes N & \longrightarrow & A \otimes A^n & \xrightleftharpoons{id \otimes f_1} & A \otimes M & \longrightarrow & 0 \\ & & & & \downarrow f_4 & & f_3 \uparrow & & \\ 0 & \longrightarrow & N & \longrightarrow & A^n & \xrightarrow{f_1} & M & \longrightarrow & 0. \end{array} \quad (2.8)$$

Here  $f_2$  is a splitting of the multiplication map ( $m \circ f_2 = id$ ),  $f_3$  a splitting of  $id \otimes f_1$  (which exists because  $A \otimes M$  is free), and  $f_4$  is the multiplication map on  $A \otimes A^n$ . From the commutativity of the diagram we can infer that  $f_4 \circ f_3 \circ f_2$  is a splitting of  $f_1$ :

$$f_1 \circ f_4 \circ f_3 \circ f_2 = m \circ (id \otimes f_1) \circ f_3 \circ f_2 = id.$$

Hence  $f_e := f_4 \circ f_3 \circ f_2 \circ f_1$  is an idempotent ( $f_e^2 = f_e$ ) and  $f_e(A^n)$  is isomorphic to  $M$ , as needed. To compute a matrix of  $f_e$ , we choose a splitting  $f_3$  so that  $f_3(1 \otimes g_k) = 1 \otimes e_k$ ,  $f_3(1 \otimes \tilde{g}_\mu) = 1 \otimes \sum_{l=1}^n c_{\mu l} e_l$ ,  $\sum_{l=1}^n c_{\mu l} g_l = \tilde{g}_\mu$ ,  $k \in \{1, \dots, n\}$ ,  $\mu \in I$ . Then

$$\begin{aligned} f_e(e_k) &= (f_4 \circ f_3 \circ f_2)(g_k) \\ &= (f_4 \circ f_3)\left(\sum_{l=1}^n a_{kl} \otimes g_l + \sum_{\mu \in I} a_{k\mu} \otimes \tilde{g}_\mu\right) \\ &= f_4\left(\sum_{l=1}^n a_{kl} \otimes e_l + \sum_{\mu \in I} a_{k\mu} \otimes \sum_{l=1}^n c_{\mu l} e_l\right) \\ &= \sum_{l=1}^n (a_{kl} + \sum_{\mu \in I} a_{k\mu} c_{\mu l}) e_l. \end{aligned}$$

This means that  $(a_{kl} + \sum_{\mu \in I} a_{k\mu} c_{\mu l})_{k,l \in \{1, \dots, n\}}$  is a projector matrix of  $M$ , as claimed.  $\square$

Observe that if  $a_{k\mu} = 0$  for all  $k$  and  $\mu$ , the matrix elements of  $e$  are simply  $a_{kl}$ , and can be directly read off from the formula for splitting  $f_2$  written in terms of the module generators  $g_1, \dots, g_n$ . By a completely analogous reasoning, the same kind of lemma is true for right modules.

### 3. Projective Module Form of the Dirac $q$ -Monopole

Recall that  $A(SL_q(2))$  is a Hopf algebra over a field  $k$  generated by  $1, \alpha, \beta, \gamma, \delta$ , satisfying the following relations:

$$\begin{aligned} \alpha\beta &= q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \beta\gamma = \gamma\beta, \quad \gamma\delta = q^{-1}\delta\gamma, \\ \alpha\delta - \delta\alpha &= (q^{-1} - q)\beta\gamma, \quad \alpha\delta - q^{-1}\beta\gamma = \delta\alpha - q\beta\gamma = 1, \end{aligned} \quad (3.1)$$

where  $q \in k \setminus \{0\}$ . The comultiplication  $\Delta$ , counit  $\varepsilon$ , and antipode  $S$  of  $A(SL_q(2))$  are defined by the following formulas:

$$\begin{aligned} \Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \alpha \otimes 1 & \beta \otimes 1 \\ \gamma \otimes 1 & \delta \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes \alpha & 1 \otimes \beta \\ 1 \otimes \gamma & 1 \otimes \delta \end{pmatrix}, \\ \varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q\beta \\ -q^{-1}\gamma & \alpha \end{pmatrix}. \end{aligned}$$

Now we need to recall the construction of the standard quantum sphere of Podleś and the quantum principal Hopf fibration. The standard quantum sphere is singled out among the principal series of Podleś quantum spheres by the property that it can be constructed as a quantum quotient space [P-P87]. In algebraic terms it means that its coordinate ring can be obtained as the subalgebra of coinvariants of a comodule algebra. To carry out this construction, first we need the right coaction on  $A(SL_q(2))$  of the commutative and cocommutative Hopf algebra  $k[z, z^{-1}]$  generated by the grouplike element  $z$  and its inverse. This Hopf algebra can be obtained as the quotient of  $A(SL_q(2))$  by the Hopf ideal generated by the off-diagonal generators  $\beta$  and  $\gamma$ . Identifying the image of

$\alpha$  and  $\delta$  under the Hopf algebra surjection  $\pi : A(SL_q(2)) \rightarrow k[z, z^{-1}]$  with  $z$  and  $z^{-1}$  respectively, we can describe the right coaction  $\Delta_R := (id \otimes \pi) \circ \Delta$  by the formula:

$$\Delta_R \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes z & \beta \otimes z^{-1} \\ \gamma \otimes z & \delta \otimes z^{-1} \end{pmatrix}.$$

We call the subalgebra of coinvariants defined by this coaction the coordinate ring of the (standard) *quantum sphere*, and denote it by  $A(S_q^2)$ . Since

$$k[z, z^{-1}] = A(SL_q(2)) / (A(S_q^2) \cap \text{Ker}\varepsilon) A(SL_q(2))$$

by Remark 3.4, we know from the general argument that  $A(S_q^2) \subseteq A(SL_q(2))$  is a principal homogenous  $k[z, z^{-1}]$ -Galois extension. (If  $P$  is a Hopf algebra,  $I$  a Hopf ideal,  $B$  the subalgebra of coinvariants under the coaction  $\Delta_R = (id \otimes \pi) \circ \Delta$ ,  $P \xrightarrow{\pi} P/I$ , and  $I = (B \cap \text{Ker}\varepsilon)P$ , then we can define the inverse of the canonical map by  $\chi^{-1}(p' \otimes \pi(p)) = p'Sp_{(1)} \otimes_B p_{(2)}$ .) We refer to the quantum principal bundle given by this Hopf-Galois extension as the *quantum principal Hopf fibration*. (An  $SO_q(3)$  version of this quantum fibration was studied introduced in [BM93].)

The main point of this section is to compute projector matrices of quantum Hopf line bundles associated to the just described Hopf  $q$ -fibration.

**Definition 3.1.** Let  $\rho_n : k[z, z^{-1}] \rightarrow k \otimes k[z, z^{-1}]$ ,  $\rho_n(1) = 1 \otimes z^{-n}$ ,  $n \in \mathbb{Z}$ , be a one-dimensional corepresentation of  $k[z, z^{-1}]$ . We call the  $A(S_q^2)$ -bimodule of colinear maps  $\text{Hom}_{\rho_n}(k, A(SL_q(2)))$  the (bimodule of) **quantum Hopf line bundle of winding number  $n$** .

Since we deal here with one-dimensional corepresentations, we identify colinear maps with their value at 1. We have

$$\text{Hom}_{\rho_n}(k, A(SL_q(2))) \cong \{p \in A(SL_q(2)) \mid \Delta_R p = p \otimes z^{-n}\} =: P_n$$

as  $A(S_q^2)$ -bimodules. With the help of the PBW basis  $\alpha^k \beta^l \gamma^m$ ,  $\beta^p \gamma^r \delta^s$ ,  $k, l, m, p, r, s \in \mathbb{N}_0$ ,  $k > 0$  of  $A(SL_q(2))$ , one can show that

$$P_n = \begin{cases} \sum_{k=0}^{-n} A(S_q^2) \alpha^{-n-k} \gamma^k = \sum_{k=0}^{-n} \alpha^{-n-k} \gamma^k A(S_q^2) & \text{for } n \leq 0 \\ \sum_{k=0}^n A(S_q^2) \beta^k \delta^{n-k} = \sum_{k=0}^n \beta^k \delta^{n-k} A(S_q^2) & \text{for } n \geq 0, \end{cases}$$

and  $A(SL_q(2)) = \bigoplus_{n \in \mathbb{Z}} P_n$  (cf. [MMNNU91, (1.10)]).

Next, similarly to [BM93, BM98], we consider the canonical connection induced by the bicovariant splitting  $i(z^n) = \alpha^n$ ,  $i(z^{-n}) = \delta^n$ . By Corollary 2.6 it induces a strong connection. We call this connection the (Dirac) *q-monopole*. Now, formula (2.7) gives us a splitting  $s : A(SL_q(2)) \rightarrow A(S_q^2) \otimes A(SL_q(2))$ , and we can claim:

**Proposition 3.2.** Put

$$(e_n)_{kl} = \begin{cases} \alpha^{-n-k} \gamma^k \binom{-n}{l}_{q^2} (-q)^l \beta^l \delta^{-n-l} & \text{for } n \leq 0 \\ \beta^k \delta^{n-k} \binom{n}{l}_{q^2} (-q)^{-l} \alpha^{n-l} \gamma^l & \text{for } n \geq 0. \end{cases}$$

Then, for any  $n \in \mathbb{Z}$ ,  $e_n \in M_{|n|+1}(A(S_q^2))$ ,  $e_n^2 = e_n$ , and  $A(S_q^2)^{|n|+1} e_n$  is isomorphic to  $P_n$  as a left  $A(S_q^2)$ -module.

*Proof.* Recall first that if  $qxy = yx$ , then  $(x + y)^n = \sum_{k=0}^n \binom{n}{k}_q x^k y^{n-k}$ , where

$$\binom{n}{k}_q = \frac{(q-1)\dots(q^n-1)}{(q-1)\dots(q^k-1)(q-1)\dots(q^{n-k}-1)}$$

are the  $q$ -binomial coefficients. (See, e.g., [M-S95, p.85].) Taking advantage of formula (2.7) in the  $q$ -monopole case, we compute:

$$\begin{aligned} s(\alpha^{m-k}\gamma^k) &= \alpha^{m-k}\gamma^k Si(z^m)_{(1)} \otimes i(z^m)_{(2)} \\ &= \sum_{l=0}^m \alpha^{m-k}\gamma^k \binom{m}{l}_{q^2} S(\alpha^{m-l}\beta^l) \otimes \alpha^{m-l}\gamma^l \\ &= \sum_{l=0}^m \alpha^{m-k}\gamma^k \binom{m}{l}_{q^2} (-q)^l \beta^l \delta^{m-l} \otimes \alpha^{m-l}\gamma^l. \end{aligned}$$

Similarly,  $s(\beta^k\delta^{n-k}) = \sum_{l=0}^n \beta^k\delta^{n-k} \binom{n}{l}_{q^2} (-q)^{-l} \alpha^{n-l}\gamma^l \otimes \beta^l\delta^{n-l}$ . Thus we have verified that  $s$  preserves the direct sum decomposition of  $A(SL_q(2))$ , i.e.,  $s(P_n) \subseteq A(S_q^2) \otimes P_n$ ,  $n \in \mathbb{Z}$ . Hence, by restriction, we have a splitting of the left multiplication map for each  $P_n$ . The claim of the proposition follows directly from Lemma 2.8 and the above formulas for  $s$ .  $\square$

*Remark 3.3.* Observe that for  $n \geq 0$  we can write  $e_n = uv^T$ , where

$$u^T = (\delta^n, \dots, \beta^k\delta^{n-k}, \dots, \beta^n) \quad \text{and} \quad v^T = (S(\delta^n), \dots, \binom{n}{k}_{q^2} S(\gamma^k\delta^{n-k}), \dots, S(\gamma^n)).$$

Since

$$v^T u = \sum_{k=0}^n \binom{n}{k}_{q^2} S(\gamma^k\delta^{n-k})\beta^k\delta^{n-k} = S((\delta^n)_{(1)})(\delta^n)_{(2)} = \varepsilon(\delta^n) = 1,$$

we can directly see that  $e_n^2 = e_n$ . The case  $n \leq 0$  is similar.

*Remark 3.4.* We can define the fibre of a quantum vector bundle over a classical point (understood as a number-valued algebra homomorphism) as the localization of the module of “sections” of this bundle at the kernel of this homomorphism. The standard Podleś quantum sphere that we consider here has one classical point given by the restriction of the counit map  $\varepsilon$ . Let us consider the quantum Hopf line bundles as left  $A(S_q^2)$ -modules  $P_n$ . We can then regard the localization  $P_n/A(S_q^2)^+ P_n$ ,  $A(S_q^2)^+ := \text{Ker}\varepsilon \cap A(S_q^2)$ , as the fibre vector space of  $P_n$  over the point given by  $A(S_q^2)^+$ . (Note that  $P_n/A(S_q^2)^+ P_n$  is automatically a vector space over  $A(S_q^2)/A(S_q^2)^+ = k$ .) Since  $\varepsilon(A(S_q^2)^+ P_n) = 0$ ,  $\varepsilon$  induces a linear map  $\tilde{\varepsilon} : P_n/A(S_q^2)^+ P_n \rightarrow k$  given by the formula  $\tilde{\varepsilon}(p/A(S_q^2)^+ P_n) = \varepsilon(p)$ . Assume now that  $n \geq 0$ . Arbitrary  $p \in P_n$  can be written as  $p = \sum_{l=0}^n b_l \beta^l \delta^{n-l}$ ,  $b_l \in A(S_q^2)$ . Hence  $\tilde{\varepsilon}(p/A(S_q^2)^+ P_n) = \varepsilon(b_0)$ , and we can conclude

that  $\tilde{\varepsilon}$  is surjective. Note now that  $\beta = (-q^{-1}\beta\gamma)\beta + (q\alpha\beta)\delta$ , and consequently, for  $l > 0$ ,  $\beta^l\delta^{n-l} = (-q^{-1}\beta\gamma)\beta^l\delta^{n-l} + (q\alpha\beta)\delta\beta^{l-1}\delta^{n-l} \in A(S_q^2)^+P_n$ . It follows that

$$\begin{aligned} \left(\sum_{l=0}^n b_l\beta^l\delta^{n-l}\right)/A(S_q^2)^+P_n &= b_0\delta^n/A(S_q^2)^+P_n \\ &= \varepsilon(b_0)\delta^n/A(S_q^2)^+P_n + (b_0 - \varepsilon(b_0))\delta^n/A(S_q^2)^+P_n \\ &= \varepsilon(b_0)\delta^n/A(S_q^2)^+P_n. \end{aligned}$$

This entails the injectivity of  $\tilde{\varepsilon}$ . Thus  $\tilde{\varepsilon}$  is an isomorphism, and we can infer that the fibre  $P_n/A(S_q^2)^+P_n$  is a one-dimensional vector space, exactly as expected for a line bundle. The reasoning for  $n \leq 0$  is analogous, and relies on the identity  $\gamma = (-q\beta\gamma)\gamma + (q^{-1}\delta\gamma)\alpha$ . This agrees with the fact that  $A(SL_q(2)) = \bigoplus_{n \in \mathbb{Z}} P_n$  and  $A(SL_q(2))/A(S_q^2)^+A(SL_q(2)) = k[z, z^{-1}] = \bigoplus_{n \in \mathbb{Z}} kz^n$ . The latter equality can be directly seen as follows: Since  $\beta$  and  $\gamma$   $q$ -commute with all monomials, the two-sided ideal  $\langle \beta, \gamma \rangle = \beta A(SL_q(2)) + \gamma A(SL_q(2))$ . Thus, as  $\beta, \gamma \in A(S_q^2)^+A(SL_q(2))$  by the above formulas, we have  $\langle \beta, \gamma \rangle \subseteq A(S_q^2)^+A(SL_q(2))$ . On the other hand, since  $A(S_q^2)^+$  is the ideal in  $A(S_q^2)$  generated by  $\alpha\beta, \beta\gamma, \gamma\delta$ , we also have  $A(S_q^2)^+A(SL_q(2)) \subseteq \langle \beta, \gamma \rangle$ . Hence  $k[z, z^{-1}] = A(SL_q(2))/\langle \beta, \gamma \rangle = A(SL_q(2))/A(S_q^2)^+A(SL_q(2))$ .

To compute projector matrices of the quantum Hopf line bundles thought of as right  $A(S_q^2)$ -modules, we need a right-sided version of formula (2.7). A natural first candidate appears to be:

$$\tilde{s}(p) = i(\overline{p(1)})_{(1)} \otimes S(i(\overline{p(1)})_{(2)})p_{(2)}. \quad (3.2)$$

It is evidently a splitting of the multiplication map  $m : A(SL_q(2)) \otimes A(SL_q(2)) \rightarrow A(SL_q(2))$ . Only now it is right linear under left coinvariants. By left coinvariants we understand here  $\tilde{A}(S_q^2) := \{p \in A(SL_q(2)) \mid \Delta_L p = 1 \otimes p\}$ , where  $\Delta_L = (\pi \otimes id) \circ \Delta$ . On generators, we have explicitly:

$$\Delta_L \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z \otimes \alpha & z \otimes \beta \\ z^{-1} \otimes \gamma & z^{-1} \otimes \delta \end{pmatrix}.$$

Using the PBW basis  $\alpha^k\beta^l\gamma^m, \beta^p\gamma^r\delta^s$ ,  $k, l, m, p, r, s \in \mathbb{N}_0$ ,  $k > 0$  of  $A(SL_q(2))$ , one can show that  $\tilde{A}(S_q^2)$  is a unital subalgebra of  $A(SL_q(2))$  generated by  $\alpha\gamma, \beta\delta, \beta\gamma$ . We want to prove now that the image of  $\tilde{s}$  lies in  $A(SL_q(2)) \otimes \tilde{A}(S_q^2)$ . To this end we note that the right covariance of  $i$  implies the formula  $i(h)_{(1)} \otimes \overline{i(h)_{(3)}} \otimes i(h)_{(2)} = i(h_{(1)})_{(1)} \otimes h_{(2)} \otimes i(h_{(1)})_{(2)}$ . With the above formula at hand, one can verify that  $((id \otimes \Delta_L) \circ \tilde{s})(p) = i(\overline{p(1)})_{(1)} \otimes 1 \otimes S(i(\overline{p(1)})_{(2)})p_{(2)}$ , as needed. Thus we can conclude that  $\tilde{s}$  is a right  $\tilde{A}(S_q^2)$ -linear splitting of the multiplication map  $A(SL_q(2)) \otimes \tilde{A}(S_q^2) \rightarrow A(SL_q(2))$ . However,  $\tilde{A}(S_q^2)$  and  $A(S_q^2)$  are different subalgebras of  $A(SL_q(2))$ , and we want to find projector matrices for  $P_n$  thought of as right  $A(S_q^2)$ -modules. To our aid comes the transpose automorphism of  $A(SL_q(2))$  defined on generators by

$$T \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

One can check directly that  $T$  is well defined. In particular, when we work over  $\mathbb{C}$ ,  $A(SL_q(2))$  has a natural  $*$ -algebra structure for  $q$  real, namely

$$* \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\gamma \\ -q\beta & \alpha \end{pmatrix},$$

and we can simply define  $T = * \circ S$ . This automorphism gives an isomorphism between  $A(S_q^2)$  and  $\tilde{A}(S_q^2)$ . We have  $T(A(S_q^2)) = \tilde{A}(S_q^2)$  and  $T(\tilde{A}(S_q^2)) = A(S_q^2)$ . (Note that  $T^2 = id$ .) It is straightforward to verify that  $\check{s} := (T \otimes T) \circ \tilde{s} \circ T$  is a right  $A(S_q^2)$ -linear splitting of the right multiplication map  $m : A(SL_q(2)) \otimes A(S_q^2) \rightarrow A(SL_q(2))$ . We can now proceed as in the left-sided case to prove:

**Proposition 3.5.** *Put*

$$(f_n)_{lk} = \begin{cases} \binom{-n}{l}_{q^2} (-q)^{-l} \beta^l \delta^{-n-l} \alpha^{-n-k} \gamma^k & \text{for } n \leq 0 \\ \binom{n}{l}_{q^2} (-q)^l \alpha^{n-l} \gamma^l \beta^k \delta^{n-k} & \text{for } n \geq 0. \end{cases}$$

Then, for any  $n \in \mathbb{Z}$ ,  $f_n \in M_{|n|+1}(A(S_q^2))$ ,  $f_n^2 = f_n$ , and  $f_n A(S_q^2)^{|n|+1}$  is isomorphic to  $P_n$  as a right  $A(S_q^2)$ -module.

*Proof.* We have:

$$\begin{aligned} \check{s}(\alpha^{m-k} \gamma^k) &= (T \otimes T)(\tilde{s}(\alpha^{m-k} \beta^k)) \\ &= (T \otimes T)(i(z^m)_{(1)} \otimes S(i(z^m)_{(2)}) \alpha^{m-k} \beta^k) \\ &= (T \otimes T) \left( \sum_{l=0}^m \binom{m}{l}_{q^2} \alpha^{m-l} \beta^l \otimes S(\alpha^{m-l} \gamma^l) \alpha^{m-k} \beta^k \right) \\ &= (T \otimes T) \left( \sum_{l=0}^m \alpha^{m-l} \beta^l \otimes \binom{m}{l}_{q^2} (-q)^{-l} \gamma^l \delta^{m-l} \alpha^{m-k} \beta^k \right) \\ &= \sum_{l=0}^m \alpha^{m-l} \gamma^l \otimes \binom{m}{l}_{q^2} (-q)^{-l} \beta^l \delta^{m-l} \alpha^{m-k} \gamma^k. \end{aligned}$$

Similarly,  $\check{s}(\beta^k \delta^{n-k}) = \sum_{l=0}^n \beta^l \delta^{n-l} \otimes \binom{n}{l}_{q^2} (-q)^l \alpha^{n-l} \gamma^l \beta^k \delta^{n-k}$ . Hence  $\check{s}(P_n) \subseteq P_n \otimes$

$A(S_q^2)$ ,  $n \in \mathbb{Z}$ . By restriction of  $\check{s}$ , we have a splitting of the right multiplication map for each  $P_n$ . The claim of the proposition follows from the right-sided version of Lemma 2.8 and the above formulas for  $\check{s}$ .  $\square$

Finally, let us observe that, identifying  $\text{Hom}_{\rho_n}(k, A(SL_q(2)))$  with  $P_n$ , we can view the covariant derivative

$$\nabla_n^\omega : \text{Hom}_{\rho_n}(k, A(SL_q(2))) \rightarrow \Omega^1 A(S_q^2) \otimes_{A(S_q^2)} \text{Hom}_{\rho_n}(k, A(SL_q(2)))$$

associated to the  $q$ -monopole by (2.2), as the Grassmannian connection associated to the splitting  $s_n := s|_{P_n}$ . More precisely, let  $\psi : \text{Hom}_{\rho_n}(k, A(SL_q(2))) \rightarrow P_n$ ,  $\psi(\xi) = \xi(1)$  be the identification isomorphism mentioned above. The Grassmannian connection associated to the splitting  $s_n : P_n \rightarrow A(S_q^2) \otimes P_n$  is by definition the connection

$\tilde{\nabla}_n^s : P_n \rightarrow \Omega^1 A(S_q^2) \otimes P_n$  given by the formula  $\tilde{\nabla}_n^s p = \sum_i db_i \otimes_{A(S_q^2)} p_i$ , where  $\sum_i b_i \otimes p_i := s(p)$ . (See [CQ95, (54)] or [L-G97, (8.27)] for the right-sided version.) We want to show that

$$\nabla_n^\omega = (id \otimes_{A(S_q^2)} \psi^{-1}) \circ \tilde{\nabla}_n^s \circ \psi, \quad n \in \mathbb{Z},$$

or equivalently that

$$\forall \xi \in \text{Hom}_{\rho_n}(k, A(SL_q(2))), \quad n \in \mathbb{Z} :$$

$$(\check{\ell}(\nabla_n^\omega \xi))(1) = ((\check{\ell} \circ (id \otimes_{A(S_q^2)} \psi^{-1}) \circ \tilde{\nabla}_n^s \circ \psi)(\xi))(1).$$

(See Proposition 2.1 and (2.2).) Notice that we can use here either Proposition 2.1 or Proposition 2.3 to guarantee that  $\nabla_n^\omega$ ,  $n \in \mathbb{Z}$ , makes sense. Indeed, since  $k[z, z^{-1}]$  admits the Haar functional ( $h_H : k[z, z^{-1}] \rightarrow k$ ,  $h_H(z^n) = \delta_{0n}$ ), we can construct a unital right colinear mapping  $j : k[z, z^{-1}] \rightarrow A(SL_q(2))$ ,  $j := \eta \circ h_H$ , where  $\eta : k \rightarrow A(SL_q(2))$  is the unit map, so that  $A(SL_q(2))$  is injective as a right  $k[z, z^{-1}]$ -comodule. Thus, as the antipode of  $k[z, z^{-1}]$  is bijective,  $A(SL_q(2))$  is left and right faithfully flat over  $A(S_q^2)$  by [S-HJ90, Theorem I], and Proposition 2.1 applies. (In fact, we used the existence of a unital right colinear mapping to prove Proposition 2.1.) Also,  $\Omega^1 A(S_q^2)$  is isomorphic with  $A(S_q^2)/k \otimes A(S_q^2)$  as a right  $A(S_q^2)$ -module via  $db \mapsto b/k \otimes 1$ , so that it is free, whence flat. Therefore Proposition 2.3 applies as well. Now, we put  $s(\xi(1)) = b_i \otimes \xi(1)_i$ ,  $\xi_i(1) = \xi(1)_i$ , and taking advantage of  $m \circ s_n = id$ , (2.6), (1.1) and (2.2) compute:

$$\begin{aligned} & ((\check{\ell} \circ (id \otimes_{A(S_q^2)} \psi^{-1}) \circ \tilde{\nabla}_n^s \circ \psi)(\xi))(1) \\ &= ((\check{\ell} \circ (id \otimes_{A(S_q^2)} \psi^{-1}))(\sum_i db_i \otimes_{A(S_q^2)} \xi(1)_i))(1) \\ &= \sum_i (db_i) \xi(1)_i \\ &= 1 \otimes (m \circ s_n)(\xi(1)) - s_n(\xi(1)) \\ &= 1 \otimes \xi(1) - \xi(1) \otimes 1 - \xi(1)_{(0)} \omega(\xi(1)_{(1)}) \\ &= d\xi(1) - \Pi^\omega(d\xi(1)) \\ &= (\check{\ell}(\nabla_n^\omega \xi))(1). \end{aligned}$$

This is exactly as one should expect, since we have constructed the splitting  $s : A(SL_q(2)) \rightarrow A(S_q^2) \otimes A(SL_q(2))$  from the connection form  $\omega$  by formula (2.6).

#### 4. Chern–Connes Pairing for the $n = -1$ Bimodule

The aim of this section is to compute the left and right Chern numbers of the left and right finitely generated projective bimodule  $P_{-1}$  describing the quantum Hopf line bundle of winding number  $-1$ . This computation is a simple example of the Chern–Connes pairing between  $K$ -theory and cyclic cohomology [C-A94, L-JL97].

To obtain the desired Chern numbers we need to evaluate (to pair) the appropriate even cyclic cocycle with the left and right projector matrix respectively. Since the positive even cyclic cohomology  $HC^{2n}(A(S_q^2))$ ,  $n > 0$ , is the image of the periodicity operator applied to  $HC^0(A(S_q^2))$ , and the pairing is compatible with the action of the periodicity

operator, the even cyclic cocycle computing Chern numbers is necessarily of degree zero, i.e., a trace. This trace is explicitly provided in [MNW91, (4.4)]. Adapting [MNW91, (4.4)] to our special case of the standard Podleś quantum sphere, we obtain:

$$\begin{aligned} \tau^1((\alpha\beta)^m \zeta^n) &= \begin{cases} (1 - q^{2n})^{-1} & \text{for } n > 0, m = 0, \\ 0 & \text{otherwise,} \end{cases} \\ \tau^1((\gamma\delta)^m \zeta^n) &= \begin{cases} (1 - q^{2n})^{-1} & \text{for } n > 0, m = 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

where  $\zeta := -q^{-1}\beta\gamma$ .

The fact that the ‘‘Chern cyclic cocycle’’ is in degree zero is a quantum effect caused by the non-classical structure of  $HC^*(A(S_q^2))$  (see [MNW91]). In the classical case the corresponding cocycle is in degree two, as it comes from the volume form of the two-sphere.

Since  $\tau^1$  is a 0-cyclic cocycle, the pairing is given by the formula  $\langle [\tau^1], [p] \rangle = (\tau^1 \circ Tr)(p)$ , where  $p \in M_n(A(S_q^2))$ ,  $p^2 = p$ , and  $Tr : M_n(A(S_q^2)) \rightarrow A(S_q^2)$  is the usual matrix trace. The following proposition establishes the pairing between the cyclic cohomology class  $[\tau^1]$  and the  $K_0$ -classes  $[e_{-1}]$  and  $[f_{-1}]$  of the left and right projector matrix of bimodule  $P_{-1}$  respectively:

**Proposition 4.1.** *Let  $\tau^1 : A(S_q^2) \rightarrow k$  be the trace (4.1), and  $e_{-1}, f_{-1}$  the projectors given in Propositions 3.2 and 3.5. Then  $(\tau^1 \circ Tr)(e_{-1}) = -1$  and  $(\tau^1 \circ Tr)(f_{-1}) = 1$ .*

*Proof.* Taking advantage of (3.1) and (4.1), we get:

$$(\tau^1 \circ Tr) \begin{pmatrix} \alpha\delta & -\beta\alpha \\ \gamma\delta & -q\beta\gamma \end{pmatrix} = \tau^1(1 + (q^{-1} - q)\beta\gamma) = \tau^1((q^2 - 1)\zeta) = -1.$$

Similarly,

$$(\tau^1 \circ Tr) \begin{pmatrix} \delta\alpha & \delta\gamma \\ -\alpha\beta & -q^{-1}\beta\gamma \end{pmatrix} = 1,$$

as claimed.  $\square$

This computation is in agreement with the classical situation. Only there the sign change of the Chern number when switching (by transpose) from the left to right projector matrix is due to the anticommutativity of the standard differential forms on manifolds. Here the sign change relies on the noncommutativity of the algebra.

Since every free module can be represented in  $K_0$  by the identity matrix, we obtain that the pairing of the cyclic cohomology class  $[\tau^1]$  with the  $K_0$ -class of any free  $A(S_q^2)$ -module always vanishes:

$$\langle [\tau^1], [I] \rangle = \tau^1(n) = 0, \quad n \in \mathbb{N}.$$

Now, combining Proposition 4.1 with Lemma 1.7 yields:

**Corollary 4.2.** *The Hopf–Galois extension of the quantum principal Hopf fibration is not cleft.*

## Appendix

In this appendix we provide a direct proof of non-cleftness of the quantum principal Hopf fibration which is possible in the purely algebraic setting. This complements our  $K$ -theoretic proof. Thus, suppose that there exists a cleaving map  $\Phi : k[z, z^{-1}] \rightarrow A(SL_q(2))$ . The existence of the convolution inverse  $\Phi^{-1}$  entails  $\Phi(z)\Phi^{-1}(z) = \varepsilon(z)$ , whence  $\Phi(z)$  must be invertible in  $A(SL_q(2))$ . The polynomial  $\Phi(z)$  cannot be constant because then  $\Phi(z)$  and  $\Phi(1) = 1$  would be linearly dependent, which contradicts the injectivity of  $\Phi$  (see Sect. 1). Therefore to prove the non-cleftness it suffices to show that all invertible elements of  $A(SL_q(2))$  are non-zero numbers.

One can do it using the direct sum decomposition  $A(SL_q(2)) = \bigoplus_{m,n \in \mathbb{Z}} A[m, n]$ , where

$$A[m, n] = \{p \in A(SL_q(2)) \mid \pi(p_{(1)}) \otimes p_{(2)} = z^m \otimes p, p_{(1)} \otimes \pi(p_{(2)}) = p \otimes z^n\}$$

(see [MMNNU91, (1.10)].) To be consistent with [MMNNU91], let us put now  $k = \mathbb{C}$ . (See, however, the bottom of p.360 in [MMNNU91].) We know from [MMNNU91, p.363] that we can write any element of  $A(SL_q(2))$  as a sum  $\sum_{m,n} p_{m,n}(\zeta)e_{m,n}$  or  $\sum_{k,l} e_{k,l}r_{k,l}(\zeta)$ , where  $\zeta := -q^{-1}\beta\gamma$ ,  $p_{m,n}, r_{k,l} \in \mathbb{C}[\zeta]$ ,  $e_{m,n} \in A[m, n]$ . Assume now that  $\sum_{m,n} p_{m,n}(\zeta)e_{m,n} \sum_{k,l} e_{k,l}r_{k,l}(\zeta) = 1$ . Since both sums are finite, there exist indices  $m_+ := \max\{m \in \mathbb{Z} \mid p_{m,n} \neq 0\}$ ,  $n_+ := \max\{n \in \mathbb{Z} \mid p_{m_+,n} \neq 0\}$ ,  $m_- := \min\{m \in \mathbb{Z} \mid p_{m,n} \neq 0\}$ ,  $n_- := \min\{n \in \mathbb{Z} \mid p_{m_-,n} \neq 0\}$ , and similarly  $k_+, k_-, l_+, l_-$ . We have

$$\begin{aligned} A[0, 0] \ni e_{0,0} = 1 &= \sum_{m,n} p_{m,n}(\zeta)e_{m,n} \sum_{k,l} e_{k,l}r_{k,l}(\zeta) \\ &= \sum_{m,n,k,l} p_{m,n}(\zeta)s_{m,n,k,l}(\zeta)\tilde{r}_{k,l}(\zeta)e_{m+k,n+l}. \end{aligned} \quad (4.2)$$

Here  $s_{m,n,k,l}(\zeta)e_{m+k,n+l} := e_{m,n}e_{k,l}$  (see [MMNNU91, p.363]), and  $\tilde{r}_{k,l}(\zeta)$  is obtained from  $r_{k,l}(\zeta)$  by commuting it over  $e_{m+k,n+l}$ , i.e.,  $e_{m+k,n+l}r_{k,l}(\zeta) = \tilde{r}_{k,l}(\zeta)e_{m+k,n+l}$ . It follows from the commutation relations (3.1) that the coefficients of  $\tilde{r}_{k,l}$  are  $q$  to some powers times the corresponding coefficients of  $r_{k,l}$ . In particular,  $r_{k,l} = 0 \Leftrightarrow \tilde{r}_{k,l} = 0$ . Since  $p_{m_+,n_+}(\zeta)e_{m_+,n_+}$ ,  $e_{k_+,l_+}r_{k_+,l_+}(\zeta)$  and  $p_{m_-,n_-}(\zeta)e_{m_-,n_-}$ ,  $e_{k_-,l_-}r_{k_-,l_-}(\zeta)$  are the only terms that can contribute to the direct summand  $A[m_+ + k_+, n_+ + l_+]$  and  $A[m_- + k_-, n_- + l_-]$  respectively, we can conclude from Eq. (4.2) that either  $m_+ + k_+, n_+ + l_+, m_- + k_-, n_- + l_-$  are all zero, or else  $p_{m_\pm, n_\pm}(\zeta)s_{m_\pm, n_\pm, k_\pm, l_\pm}(\zeta)\tilde{r}_{k_\pm, l_\pm}(\zeta)e_{m_\pm + k_\pm, n_\pm + l_\pm} = 0$ . From [MMNNU91, p.363] we know, however, that  $e_{m_\pm + k_\pm, n_\pm + l_\pm}$  is a (left and right) basis of  $A[m_\pm + k_\pm, n_\pm + l_\pm]$  over  $\mathbb{C}[\zeta]$ .

Also, using formulas  $\alpha^j \delta^j = \prod_{i=1}^j (1 - q^{-2(i-1)}\zeta)$ ,  $d^j a^j = \prod_{i=1}^j (1 - q^{2i}\zeta)$  one can check that  $e_{m,n}e_{k,l} \neq 0$ , whence  $s_{m_\pm, n_\pm, k_\pm, l_\pm} \neq 0$ . Thus, as there are no zero divisors in  $\mathbb{C}[\zeta]$  and  $r_{k,l} = 0 \Leftrightarrow \tilde{r}_{k,l} = 0$ , we can conclude that  $p_{m_\pm, n_\pm} = 0$  or  $r_{k_\pm, l_\pm} = 0$ . This, however, contradicts the definition of  $m_\pm, n_\pm, k_\pm, l_\pm$ . Therefore  $m_\pm = -k_\pm$  and  $n_\pm = -l_\pm$ . Consequently, as  $m_- \leq m_+$  and  $k_- \leq k_+$ , we have  $m_- = m_+ = -k_+ = -k_-$ . Hence also  $n_- = n_+ = -l_+ = -l_-$ . Put  $m_0 = m_- = m_+$  and  $n_0 = n_- = n_+$ . It follows now that  $\sum_{m,n} p_{m,n}(\zeta)e_{m,n} = p_{m_0, n_0}(\zeta)e_{m_0, n_0}$  and  $\sum_{k,l} e_{k,l}r_{k,l}(\zeta) = e_{-m_0, -n_0}r_{-m_0, -n_0}(\zeta)$ . This way (4.2) reduces to  $p_{m_0, n_0}(\zeta)s_{m_0, n_0, -m_0, -n_0}(\zeta)\tilde{r}_{-m_0, -n_0}(\zeta) = 1$ . Hence all three of the above polynomials must be non-zero constants. Using again [MMNNU91, p.363] and remembering that

$\alpha^j \delta^j$  and  $\delta^j \alpha^j$  are polynomials in  $\zeta$  of degree  $j$ , we can infer that  $m_0 = 0 = n_0$ . (Otherwise  $s_{m_0, n_0, -m_0, -n_0}$  is not of degree 0.) Consequently  $\sum_{m,n} p_{m,n}(\zeta) e_{m,n} = p_{0,0}(\zeta)$ ,  $\sum_{k,l} e_{k,l} r_{k,l}(\zeta) = \tilde{r}_{0,0}(\zeta) = r_{0,0}(\zeta)$ , and  $p_{0,0}$ ,  $r_{0,0}$  are invertible constant polynomials, as needed.

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