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Bundles over Quantum Sphere and Noncommutative Index Theorem

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Abstract. The Noncommutative Index Theorem is used to prove that the Chern numbers of quantum Hopf line bundles over the standard Podleś quantum sphere equal the winding numbers of the representations defining these bundles. This result gives an estimate of the positive cone of the *algebraic* K_0 of the standard quantum sphere.

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0. Introduction

The goal of this paper is to compute the Chern numbers of quantum Hopf line bundles. We do it within the framework of the Hopf–Galois theory of extensions of rings and Chern–Connes pairing between the cyclic cohomology and K-theory. We view the Podleś sphere as the base space of the quantum principal Hopf fibration described algebraically as a Hopf–Galois extension. Noncommutative line bundles over the quantum sphere are constructed as left and right projective bimodules over the coordinate ring of the quantum sphere. They are the bimodules of colinear mappings indexed by the winding number of one-dimensional representations of $U(1)$. The Chern number of the left projective module of such a quantum line bundle is proved to coincide with the winding number of the representation defining the bundle. The Chern number of the corresponding right projective module is shown to be equal to the opposite of its left counterpart, whence it coincides with minus the winding number.

In the following section, we recall a definition of a Hopf–Galois extension and general construction of the bimodule of an associated quantum vector bundle. We also recall the construction of $SL_q(2)$, quantum principal Hopf fibration, quantum Hopf line bundles and the cyclic cohomology classes of the quantum sphere that pair non-trivially with K-theory. In Section 2, we extend the relevant considerations in [8], and compute the pairing between the Chern cyclic cocycle of the quantum sphere and the left projector matrices of quantum Hopf line bundles for an arbitrary winding number. This computation relies on the integrality of the pairing, which is implied by the Noncommutative Index Theorem, and yields the winding number, as expected. Then we argue that the pairing with the right projector matrices equals minus the pairing with the corresponding left matrices. We conclude by noting that the image of the positive cone of the algebraic K_0 of the quantum sphere under the pairing with cyclic cohomology contains $\mathbb{Z}_+ \times \mathbb{Z}$.

To focus attention and access straightforwardly the C^*-algebraic framework, we work over $\mathbb{C}$. We assume that q is a non-zero element in $\mathbb{C}$ that is not a root of 1. We tacitly assume the compact $*$-structure making $SL_q(2)$ into $SU_q(2)$. It is an additional structure whose existence is crucial but a concrete form not directly relevant to the considerations presented herein. For an introduction to quantum groups (comprehensive discussion of $SL_q(2)$ included) we refer to [9]. For a concise and motivating treatment of the Hopf–Galois theory see [17], and for a wider review see [14]. The Chern–Connes pairing and Noncommutative Index Theorem are elaborated upon in [4, 10], and [3] contains a compact account of the matter. A brief note on classical Hopf line bundles within the general context of K-theory and noncommutative geometry can be found in [11, p. 101]. The generalisation to the non-standard Podleś quantum spheres of the Chern–Connes pairing calculated in [8] is carried out in [2].

1. Preliminaries

In this section we recall basic definitions and known results used in the sequel. We begin with a definition of a Hopf–Galois extension. Hopf–Galois extensions describe quantum principal bundles the same way Hopf algebras describe quantum groups. Here a Hopf algebra H plays the role of the algebra of functions on the structure group, and the total space of a bundle is replaced by an H-comodule algebra P.

DEFINITION 1.1. Let H be a Hopf algebra, P be a right H-comodule algebra with multiplication m_P and coaction Δ_R, and $B := P^{coH} := \{p \in P \mid \Delta_R\, p = p \otimes 1\}$ the subalgebra of coinvariants. We say that P is an *H-Galois extension* of B iff the canonical left P-module right H-comodule map $\chi := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \Delta_R) : P \otimes_B P \longrightarrow P \otimes H$ is bijective.

A natural next step is to consider associated quantum vector bundles. More precisely, what we need here is a replacement for the module of sections of an associated vector bundle. In the classical case such sections can be equivalently described as 'functions of type ϱ' from the total space of a principal bundle to a vector space. We follow this construction in the quantum case by considering B-bimodules of colinear maps (linear maps that preserve the comodule structure) $\mathrm{Hom}_\rho(V, P)$ associated with an H-Galois extension $B \subseteq P$ via a corepresentation

$\rho : V \to V \otimes H$ (see [6, 7]). Under certain reasonable assumptions, these bimodules are always left and right finitely generated projective [5, Corollary 2.6]. Thus we remain within the paradigm of the Serre-Swan theorem.

Let us now exemplify the foregoing concepts. Recall first that $A(SL_q(2))$ is a Hopf algebra over $\mathbb{C}$ generated by $1,\ a,\ b,\ c,\ d$ satisfying the following relations:

$$\begin{aligned}
&ab = q^{-1}ba, \qquad cd = q^{-1}dc, \qquad ac = q^{-1}ca, \qquad bd = q^{-1}db,\\
&bc = cb, \qquad ad - da = (q^{-1} - q)bc,\\
&ad - q^{-1}bc = da - qbc = 1\,,
\end{aligned} \tag{2.1}$$

where $q \in \mathbb{C} \setminus \{0\}$. The comultiplication Δ, counit ε, and antipode S of $A(SL_q(2))$ are defined by the following formulas:

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes a & 1 \otimes b \\ 1 \otimes c & 1 \otimes d \end{pmatrix},$$

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

From here we can proceed to the construction of the standard quantum sphere of Podleś and the quantum principal Hopf fibration. The standard quantum sphere is singled out among the principal series of Podleś spheres by the property that it can be constructed as a quantum quotient space [15, p. 200]. In algebraic terms it means that its coordinate ring can be obtained as the subalgebra of coinvariants of a comodule algebra. To carry out this construction, first we need a right coaction on $A(SL_q(2))$ of the commutative and cocommutative Hopf algebra $\mathbb{C}[z, z^{-1}]$ generated by the group-like element z and its inverse. This Hopf algebra can be obtained as the quotient of $A(SL_q(2))$ by the Hopf ideal generated by the off-diagonal generators b and c. Identifying the image of a and d under the Hopf algebra surjection $\pi : A(SL_q(2)) \to \mathbb{C}[z, z^{-1}]$ with z and z^{-1}, respectively, we can describe the right coaction $\Delta_R := (\mathrm{id} \otimes \pi) \circ \Delta$ by the formula

$$\Delta_R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes z & b \otimes z^{-1} \\ c \otimes z & d \otimes z^{-1} \end{pmatrix}.$$

We call the subalgebra of coinvariants defined by this coaction the coordinate ring of the (standard) *quantum sphere*, and denote it by $A(S^2_q)$. Using general tools of the Hopf–Galois theory (e.g., [16, Theorem 1]), it is straightforward to prove that $A(S^2_q) \subseteq A(SL_q(2))$ is a $\mathbb{C}[z, z^{-1}]$-Galois extension. We refer to the quantum principal bundle given by this Hopf–Galois extension as the *quantum principal Hopf fibration*. (An $SO_q(3)$ version of this noncommutative fibration was studied in [1].)

Now we need to define quantum Hopf line bundles associated to the just described Hopf q-fibration and provide their projector matrices.

DEFINITION 1.2. Let $\rho_\mu\colon\ \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}[z, z^{-1}]$, $\rho_\mu(1) = 1 \otimes z^{-\mu}$, $\mu \in \mathbb{Z}$, be a one-dimensional corepresentation of $\mathbb{C}[z, z^{-1}]$. We call the $A(S_q^2)$-bimodule of colinear maps $\mathrm{Hom}_{\rho_\mu}(\mathbb{C}, A(SL_q(2)))$ the (bimodule of) quantum Hopf line bundle of winding number μ.

We deal here with one-dimensional corepresentations, so that we can identify colinear maps with their value at 1. We have

$$\mathrm{Hom}_{\rho_\mu}(\mathbb{C}, A(SL_q(2))) \tilde{=} \{p \in A(SL_q(2)) \mid \Delta_R p = p \otimes z^{-\mu}\} =: P_\mu$$

as $A(S_q^2)$-bimodules. With the help of the PBW basis $a^k b^l c^m,\ b^p c^r d^s,\ k, l, m, p, r, s \in \mathbb{N},\ k > 0$ of $A(SL_q(2))$, one can show that

$$P_\mu = \begin{cases} \sum_{k=0}^{-\mu} A(S_q^2)\, a^{-\mu-k} c^k = \sum_{k=0}^{-\mu} a^{-\mu-k} c^k A(S_q^2) & \text{for } \mu \leqslant 0 \\ \sum_{k=0}^{\mu} A(S_q^2)\, b^k d^{\mu-k} = \sum_{k=0}^{\mu} b^k d^{\mu-k} A(S_q^2) & \text{for } \mu \geqslant 0 \end{cases}$$

and $A(SL_q(2)) = \bigoplus_{\mu\in\mathbb{Z}} P_\mu$ (cf. [12, (1.10)]). Since the goal of this paper is to compute the Chern–Connes pairing [4, p.224] between quantum Hopf line bundles and the cyclic cohomology of the standard quantum sphere, we need explicit formulas for the projector matrices of the former, and generators of the latter. To this end recall first that if $xuv = vu$, then $(u + v)^n = \sum_{k=0}^{n} \binom{n}{k}_x u^k v^{n-k}$, where

$$\binom{n}{k}_x = \frac{(x-1)\cdots(x^n-1)}{(x-1)\cdots(x^k-1)(x-1)\cdots(x^{n-k}-1)}, \quad \binom{n}{0}_x = 1 = \binom{n}{n}_x,$$

are the *x-binomial coefficients* (e.g., see Section IV.2 in [9]). Now, following [8], put

$$(e_\mu)_{kl} = \begin{cases} a^{-\mu-k} c^k \binom{-\mu}{l}_{q^2} (-q)^l b^l d^{-\mu-l} & \text{for } \mu \leqslant 0 \\ b^k d^{\mu-k} \binom{\mu}{l}_{q^2} (-q)^{-l} a^{\mu-l} c^l & \text{for } \mu \geqslant 0. \end{cases} \tag{1.2}$$

Then, for any $\mu \in \mathbb{Z}$, $e_\mu \in M_{|\mu|+1}(A(S_q^2))$, $e_\mu^2 = e_\mu$, and $A(S_q^2)^{|\mu|+1} e_\mu$ is isomorphic to P_μ as a left $A(S_q^2)$-module [8, Proposition 3.2]. Similarly, put

$$(f_\mu)_{lk} = \begin{cases} \binom{-\mu}{l}_{q^2} (-q)^{-l} b^l d^{-\mu-l} a^{-\mu-k} c^k & \text{for } \mu \leqslant 0 \\ \binom{\mu}{l}_{q^2} (-q)^l a^{\mu-l} c^l b^k d^{\mu-k} & \text{for } \mu \geqslant 0. \end{cases} \tag{1.3}$$

Then again, for any $\mu \in \mathbb{Z}$, $f_\mu \in M_{|\mu|+1}(A(S_q^2))$, $f_\mu^2 = f_\mu$, and $f_\mu A(S_q^2)^{|\mu|+1}$ is isomorphic to P_μ as a right $A(S_q^2)$-module [8, Proposition 3.5].

To obtain the desired pairing, we need to evaluate appropriate even cyclic cocycles on the left and right projector matrices provided above. The positive even cyclic cohomology $HC^{2n}(A(S_q^2))$, $n > 0$, is the image of the periodicity operator applied to $HC^0(A(S_q^2))$. In degree zero it is given by two generators (cohomologically non-trivial cyclic cocycles) and the kernel of the periodicity operator [13, p. 174]. Since the pairing is compatible with the action of the periodicity operator, it is completely determined by the aforementioned two cyclic 0-cocycles (traces). (Everything else either pairs with $K_0(A(S_q^2))$ in the same way or trivially.) These traces are explicitly provided in [13]. One of them, denoted by τ^0, is simply the restriction to $A(S_q^2)$ of the counit map of $A(SL_q(2))$. It can be argued that this trace detects the 'rank' of our quantum vector bundles [8, Remark 3.4]. The other trace is given by the following adaptation of [13, (4.4)] to our special case of the standard Podleś quantum sphere:

$$\tau^1((ab)^m\zeta^n) = \begin{cases} (1-q^{2n})^{-1} & \text{for } n > 0,\ m = 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau^1((cd)^m\zeta^n) = \begin{cases} (1-q^{2n})^{-1} & \text{for } n > 0,\ m = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1.4}$$

where $\zeta := -q^{-1}bc$. One can think of this cocycle as the 'Chern cyclic cocycle', and the invariants it computes as the Chern numbers of quantum vector bundles. The fact that it is in degree zero is a quantum effect caused by the non-classical structure of $HC^*(A(S_q^2))$ (see [13]). In the classical case the corresponding cocycle is in degree two, as it comes from the volume form of the two-sphere.

The pairing $\langle[\tau^1],[e_{-1}]\rangle = -1$, $\langle[\tau^1],[1]\rangle = 0$ can be used to conclude the non-cleftness of the quantum principal Hopf fibration [8, Corollary 4.2]. On the other hand, one can directly check that the other cyclic cocycle τ^0 pairs unitally with above projectors:

$$\langle[\tau^0],[e_\mu]\rangle = (\varepsilon \circ \mathrm{Tr})(e_\mu) = 1, \quad \mu \in \mathbb{Z}. \tag{1.5}$$

Since τ^0 and τ^1 come from K-homology [13, p. 175], they pair integrally with $K_0(A(S_q^2))$ (Noncommutative Index Theorem). Thus, taking advantage of the linearity of the pairing, we have a group homomorphism

$$(\tau^0,\tau^1)\colon\ K_0(A(S_q^2)) \ni [p] \mapsto (\langle[\tau^0],[p]\rangle, \langle[\tau^1],[p]\rangle) \in \mathbb{Z}\oplus\mathbb{Z}. \tag{1.6}$$

Furthermore, it follows already from the values of (τ^0,τ^1) on $[1]$ and $[e_{-1}]$ that it is an epimorphism. This epimorphism splits by the freeness of $\mathbb{Z}\oplus\mathbb{Z}$, whence $\mathbb{Z}\oplus\mathbb{Z}$ is a direct summand in $K_0(A(S_q^2))$. We can think of (τ^0,τ^1) as a (possibly incomplete) coordinate system for $K_0(A(S_q^2))$ determining the rank and Chern number, respectively. The point of this paper is that for any $\mu \in \mathbb{Z}$ there exists a rank one projector matrix (quantum line bundle) with its Chern number equal to μ.

2. Chern–Connes Pairing for Quantum Hopf Line Bundles

We are to compute the pairing between the Chern cyclic cocycle τ^1 and both left and right projector matrices of quantum Hopf line bundles P_μ. We refer to the thus obtained invariants as the left and right Chern numbers, respectively. Since τ^1 is a 0-cyclic cocycle, the pairing is given by the formula $\langle[\tau^1],[p]\rangle = (\tau^1\circ \mathrm{Tr})(p)$, where $p \in M_n(A(S_q^2))$, $p^2 = p$, and Tr: $M_n(A(S_q^2)) \to A(S_q^2)$ is the usual matrix trace. We have:

THEOREM 2.1. *The left Chern number and the winding number of any quantum Hopf line bundle coincide:* $(\tau^1\circ \mathrm{Tr})(e_\mu) = \mu,\ \mu\in\mathbb{Z}$.

Proof. We need to consider two cases: $\mu < 0$ and $\mu > 0$. (The case $\mu = 0$ is evident, as $e_0 = 1$ and τ^1 annihilates numbers.)

Case $\mu < 0$*:* To simplify notation put $n = -\mu$. Let us first compute the trace of e_{-n} (see (1.2)) as a polynomial in $\zeta := -q^{-1}bc$:

$$\begin{aligned}\mathrm{Tr}(e_{-n}) &= \sum_{k=0}^{n}\binom{n}{k}_{q^2}(-q)^k a^{n-k}c^k b^k d^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}_{q^2}(-q)^k q^{-2k(n-k)}(bc)^k a^{n-k}d^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}_{q^2} q^{-2k(n-k-1)}\zeta^k\prod_{l=0}^{n-k-1}(1-q^{-2l}\zeta).\end{aligned}$$

The last step follows from the quantum determinant formula $ad = 1 + q^{-1}bc = 1-\zeta$ (see (1.1)) and standard induction. (The expression $\prod_{l=0}^{-1}(\cdots)$ is understood as 1.) To apply τ^1 to $\mathrm{Tr}(e_{-n})$ we need to know more explicitly the coefficients of the above polynomial. For this reason let us recall the definition of *shifted binomials* (cf. [12, p.173]). Let $k[t]$ denote a polynomial ring in one variable and $x\in k$. For natural numbers $0\leqslant l\leqslant \nu$ we define the x-shifted binomial $\left[\begin{smallmatrix}\nu\\ l\end{smallmatrix}\right]_x$ by the equality

$$\sum_{l=0}^{\nu}\begin{bmatrix}\nu\\ l\end{bmatrix}_x t^l := \prod_{l=0}^{\nu-1}(1+x^l t).$$

Now we can use the above calculations and (1.4) to compute the Chern–Connes pairing between $[\tau^1]$ and $[e_{-n}]$:

$$\begin{aligned}\langle[\tau^1],[e_{-n}]\rangle &= (\tau^1\circ\mathrm{Tr})(e_{-n})\\ &= \tau^1\left(\sum_{k=0}^{n}\binom{n}{k}_{q^2} q^{-2k(n-k-1)}\zeta^k\sum_{l=0}^{n-k}\begin{bmatrix}n-k\\ l\end{bmatrix}_{q^{-2}}(-\zeta)^l\right)\\ &= \tau^1\left(\sum_{k=0}^{n}\binom{n}{k}_{q^2} q^{-2k(n-k-1)}\zeta^k\sum_{m=k}^{n}\begin{bmatrix}n-k\\ m-k\end{bmatrix}_{q^{-2}}(-\zeta)^{m-k}\right)\end{aligned}$$

$$= \tau^1\left(\sum_{m=0}^{n}(-\zeta)^m\sum_{k=0}^{m}\binom{n}{k}_{q^2}q^{-2k(n-k-1)}(-1)^k\begin{bmatrix}n-k\\m-k\end{bmatrix}_{q^{-2}}\right)$$
$$= \sum_{m=1}^{n}\frac{(-1)^m}{1-q^{2m}}\sum_{k=0}^{m}\binom{n}{k}_{q^2}q^{-2k(n-k-1)}(-1)^k\begin{bmatrix}n-k\\m-k\end{bmatrix}_{q^{-2}}.$$

The point is to prove that the just obtained number equals $-n$. To do so, let us assume for the time being that $q\in(0,1)$, so that we can use the C^*-algebraic framework. Recall that the 0-cyclic cocycle τ^1 comes from a 1-summable Fredholm module over $A(S_q^2)$ [13, p.175]. Hence, by the Noncommutative Index Theorem, the Chern–Connes pairing between $[\tau^1]$ and any element of $K_0(A(S_q^2))$ is necessarily an integer – the index of an appropriate Fredholm operator. (See, e.g., [4, p. 297], [3, p. 54], [10, Section 12.2.5].) Thus we have

$$(\tau^1\circ \mathrm{Tr})(e_{-n}) = \sum_{m=1}^{n}\frac{(-1)^m}{1-q^{2m}}\sum_{k=0}^{m}\binom{n}{k}_{q^2}q^{-2k(n-k-1)}(-1)^k\begin{bmatrix}n-k\\m-k\end{bmatrix}_{q^{-2}}\in\mathbb{Z}$$

for any $q\in(0,1)$. Observe now that, since $(\tau^1\circ\mathrm{Tr})(e_{-n})$ is a rational function of q, it is continuous, whence constant on $(0,1)$ by the connectedness of $(0,1)$ and the above integrality property. The only rational function on $\mathbb{C}\setminus\{0, \text{roots of } 1\}$ which is constant on the open interval $(0,1)$ is a constant function. Therefore, $(\tau^1\circ\mathrm{Tr})(e_{-n})$ is independent of $q\in(1,\infty)$, and we have $(\tau^1\circ\mathrm{Tr})(e_{-n}) = \lim_{q\to\infty}(\tau^1\circ\mathrm{Tr})(e_{-n})$. Now it suffices to show that

$$\lim_{q\to\infty}(1-q^{2m})^{-1}(-1)^m\sum_{k=0}^{m}\binom{n}{k}_{q^2}q^{-2k(n-k-1)}(-1)^k\begin{bmatrix}n-k\\m-k\end{bmatrix}_{q^{-2}} = -1$$

for any positive integers m and n. To this end we need to analyse the asymptotic behaviour of the fractions

$$F_{n,m,k}(q) := \binom{n}{k}_{q^2}q^{-2k(n-k-1)}(-1)^k\begin{bmatrix}n-k\\m-k\end{bmatrix}_{q^{-2}}$$
$$= \frac{(-1)^k(q^2-1)\cdots(q^{2n}-1)}{q^{2k(n-k-1)}(q^2-1)\cdots(q^{2k}-1)(q^2-1)\cdots(q^{2(n-k)}-1)}\times$$
$$\times\left(\frac{1}{q^{(m-k-1)(m-k)}}+\cdots+\frac{1}{q^{(n-k-1)(n-k)-(n-m-1)(n-m)}}\right).$$

Here the sum in parenthesis equals $\left[\begin{smallmatrix}n-k\\m-k\end{smallmatrix}\right]_{q^{-2}}$ and the powers of q^{-2} are ordered from the smallest to the biggest. The dominating term of $F_{n,m,k}(q)$ if proportional to q to the power of $n(n+1)-k(k+1)-(n-k)(n-k+1)-2k(n-k-1)-(m-k-1)(m-k) = -k^2+k(1+2m)+m-m^2$. Thinking of this expression as a function of k, we see that it is biggest for $k=m$. Hence the dominating term in $\sum_{k=0}^{m}F_{n,m,k}(q)$ is proportional to q^{2m}. All other terms are proportional to q to some power strictly less than $2m$, and will be annihilated by $(1-q^{2m})^{-1}$. Therefore, only $F_{n,m,m}(q)$ contributes to the limit, and we have:

$$\begin{aligned}
&\lim_{q\to\infty}(1-q^{2m})^{-1}(-1)^m\sum_{k=0}^{m}F_{n,m,k}(q)\\
&=\lim_{q\to\infty}\frac{(-1)^m}{1-q^{2m}}\frac{(-1)^m(q^2-1)\cdots(q^{2n}-1)}{q^{2m(n-m-1)}(q^2-1)\cdots(q^{2m}-1)(q^2-1)\cdots(q^{2(n-m)}-1)}\\
&=\lim_{q\to\infty}\frac{-(q^2-1)\cdots(q^{2n}-1)}{(q^{2m}-1)q^{2m(n-m-1)}(q^2-1)\cdots(q^{2m}-1)(q^2-1)\cdots(q^{2(n-m)}-1)}\\
&=-1.
\end{aligned}$$

This proves that $(\tau^1\circ\mathrm{Tr})(e_{-n})=\sum_{m=1}^{n}(-1)=-n$, as needed.

Case $\mu>0$*:* The reasoning is similar to that of the previous case, though the calculation of the limit is more straightforward. Put $n=\mu$. First we compute:

$$\begin{aligned}
\mathrm{Tr}(e_n) &= \sum_{k=0}^{n}\binom{n}{k}_{q^2}(-q)^{-k}b^kd^{n-k}a^{n-k}c^k\\
&=\sum_{k=0}^{n}\binom{n}{k}_{q^2}\zeta^k\prod_{l=0}^{n-k-1}(1-q^{2(l+1)}\zeta)\\
&=\sum_{k=0}^{n}\binom{n}{k}_{q^2}\zeta^k\sum_{l=0}^{n-k}\begin{bmatrix}n-k\\ l\end{bmatrix}_{q^2}(-q^2\zeta)^l\\
&=\sum_{k=0}^{n}\binom{n}{k}_{q^2}\zeta^k\sum_{m=k}^{n}\begin{bmatrix}n-k\\ m-k\end{bmatrix}_{q^2}(-q^2\zeta)^{m-k}\\
&=\sum_{m=0}^{n}\zeta^m\sum_{k=0}^{m}\binom{n}{k}_{q^2}(-1)^{m-k}q^{2(m-k)}\begin{bmatrix}n-k\\ m-k\end{bmatrix}_{q^2}.
\end{aligned}$$

Using the Noncommutative Index Theorem the same way as before, we can again conclude that $(\tau^1\circ\mathrm{Tr})(e_n)\in\mathbb{Z}$ for $q\in(0,1)$. Also by the same argument as before, the integrality of $(\tau^1\circ\mathrm{Tr})(e_n)$ entails that it is independent of q. We can therefore compute it by taking the limit:

$$\begin{aligned}
(\tau^1\circ\mathrm{Tr})(e_n) &= \lim_{q\to 0}(\tau^1\circ\mathrm{Tr})(e_n)\\
&=\lim_{q\to 0}\sum_{m=1}^{n}(1-q^{2m})^{-1}\sum_{k=0}^{m}\binom{n}{k}_{q^2}(-1)^{m-k}q^{2(m-k)}\begin{bmatrix}n-k\\ m-k\end{bmatrix}_{q^2}\\
&=\sum_{m=1}^{n}\sum_{k=0}^{m}(-1)^{m-k}\delta_{k,m}(\delta_{k,m-1}+\delta_{k,m})=n.
\end{aligned}$$

□

Remark 2.2. It follows from a direct calculation that $(\tau^1 \circ \mathrm{Tr})(e_{-2}) = (-1 - q^{-2}) + (-1 + q^{-2}) = -2$, where the first and the second term correspond to $m = 1$ and $m = 2$, respectively, in the general sum. Similarly, $(\tau^1 \circ \mathrm{Tr})(e_2) = (1 + q^2) + (1 - q^2) = 2$.

As for the right projective structure of P_μ, one can infer directly from formulas (1.2), (1.3) and Theorem 2.1 that

$$(\tau^1 \circ \mathrm{Tr})(f_\mu) = (\tau^1 \circ \mathrm{Tr})(e_{-\mu}) = -\mu.$$

Hence we have:

COROLLARY 2.3. *The right Chern number of any quantum Hopf line bundle equals the opposite of its winding number, i.e.,* $(\tau^1 \circ \mathrm{Tr})(f_\mu) = -\mu,\ \mu \in \mathbb{Z}$.

Let us now consider further consequences of Theorem 2.1. Note first that due to the direct sum decomposition $A(SL_q(2)) = \bigoplus_{\mu\in\mathbb{Z}} P_\mu$, one can say that the coordinate ring of $SL_q(2)$ decomposes into *mutually* K_0*-non-equivalent* left and right projective finitely generated $A(S_q^2)$-bimodules. As for the structure of $K_0(A(S_q^2))$, Theorem 2.1 provides us with an estimate of its positive cone. Indeed, combining (1.5) with Theorem 2.1 yields:

COROLLARY 2.4. *The image of the positive cone of* $K_0(A(S_q^2))$ *under* (τ^0, τ^1): $K_0(A(S_q^2)) \longrightarrow \mathbb{Z} \times \mathbb{Z}$ *contains* $\mathbb{Z}_+ \times \mathbb{Z}$ *see (1.6).*

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