Algebras and Representation Theory (2006) 9: 121–146
DOI: 10.1007/s10468-005-3080-y

A Locally Trivial Quantum Hopf Fibration

PIOTR M. HAJAC[1,2], RAINER MATTHES[2,3] and WOJCIECH SZYMANSKI[4]
[1]*Mathematisches Institut, Universität München, Theresienstr. 39, Munich, 80333, Germany and Instytut Matematyczny, Polska Akademia Nauk, ul. Sniadeckich 8, Warsaw, 00-950 Poland. url: http://www.fuw.edu.pl/ pmh*
[2]*Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, ul. Hoza 74, Warsaw, 00-682 Poland.*
[3]*Fachbereich Physik der TU Clausthal, Leibnizstr. 10, D-38678 Clausthal-Zellerfeld, Germany. e-mail: matthes@fuw.edu.pl, matthes@itp.uni-leipzig.de*
[4]*School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia. e-mail: Wojciech.Szymanski@newcastle.edu.au*

(Received: September 2002; accepted: December 2002)

Presented by S. L. Woronowicz

Abstract. The irreducible $*$-representations of the polynomial algebra $\mathcal{O}(S^3_{pq})$ of the quantum 3-sphere introduced by Calow and Matthes are classified. The K-groups of its universal C^*-algebra are shown to coincide with their classical counterparts. The U(1)-action on $\mathcal{O}(S^3_{pq})$ corresponding for $p = 1 = q$ to the classical Hopf fibration is proven to be Galois (free). The thus obtained locally trivial Hopf–Galois extension is shown to be equivariantly projective (admitting a strong connection) and non-cleft. The latter is proven by determining an appropriate pairing of cyclic cohomology and K-theory.

Mathematics Subject Classifications (2000): 16W30, 46L87.

Key words: Hopf–Galois extensions, K-theory, representations of C^*-algebras.

Introduction

Splitting and gluing topological spaces along 2-spheres or 2-tori are standard procedures in the study of 3-dimensional manifolds. Fibering such manifolds is another important tool revealing their geometry. In the case of S^3, we have the well-known Heegaard splittings and Hopf fibration. The former present S^3 as two copies of a solid torus glued along their boundaries, and the latter as a non-trivial principal $\mathrm{U}(1)$-bundle over S^2.

In [M-K91a], K. Matsumoto applied the idea of a Heegaard splitting to construct a noncommutative 3-sphere S^3_θ out of two quantum solid tori. Then the U(1)-action on S^3_θ was defined and the quotient space $S^3_\theta/\mathrm{U}(1)$ proven to coincide with S^2 [M-K91b]. Thus a noncommutative Hopf fibration was constructed. Here we study along these lines a different but analogously constructed example of a quantum 3-sphere [CM02].

Throughout the paper we use the jargon of Noncommutative Geometry referring to quantum spaces as objects dual to noncommutative algebras in the sense of the Gelfand correspondence between spaces and function algebras. The unadorned tensor product means the completed tensor product when placed between C^*-algebras (this is not ambiguous as all C^*-algebras we consider are nuclear), and the algebraic tensor product over $\mathbb{C}$ otherwise. The algebras are assumed to be associative and over $\mathbb{C}$. They are also unital unless the contrary is obvious from the context. C_0 (locally compact Hausdorff space) means the (non-unital) algebra of vanishing-at-infinity continuous functions on this space. By $\mathcal{O}$ (quantum space) we denote the polynomial algebra of a quantum space, and by C (quantum space) the corresponding C^*-algebra. By classical points we understand 1-dimensional $*$-representations. In this paper, the C^*-completion (C^*-closure) of a $*$-algebra always means the completion with respect to the supremum norm over all $*$-representations in bounded operators.

First, we recall the necessary facts and definitions. This includes the construction of the quantum 3-sphere S^3_{pq} introduced in [CM02], which is also obtained by gluing two quantum solid tori. Here, however, the noncommutativity comes from the quantum disc [KL93] rather than the quantum torus (see [R-MA90] and references therein). Also, contrary to S^3_θ, the sphere S^3_{pq} was constructed in the spirit of *locally trivial principal bundles*. Indeed, it can be easily noted that as both the base and the fibre of the Matsumoto noncommutative Hopf fibration are classical, the noncommutativity of S^3_θ rules out its local triviality. On the other hand, S^3_{pq} is by construction a locally trivial quantum $\mathrm{U}(1)$-space.

We begin the main part of our paper by classifying the unitary classes of irreducible $*$-representations of the polynomial algebra $\mathcal{O}(S^3_{pq})$, finding a basis of $\mathcal{O}(S^3_{pq})$, and defining the C^*-algebra $C(S^3_{pq})$. Then we prove that $K_i(C(S^3_{pq})) \cong \mathbb{Z} \cong K^i(S^3)$, $i \in \{0, 1\}$. Next, we discuss the fact that $C(S^3_{pq})$ is a graph C^*-algebra for a certain 2-graph. On the algebraic side, we show that S^3_{pq} is a quantum *principal* $\mathrm{U}(1)$-bundle in the sense of Hopf–Galois theory. More precisely, by constructing a strong connection, we prove that $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ is an *equivariantly projective* $\mathcal{O}(\mathrm{U}(1))$-Galois extension. From a formula for the strong connection, we determine the idempotent matrices of all quantum line bundles associated to the noncommutative Hopf fibration $S^3_{pq} \to S^2_{pq}$. Finally, we pair a trace on $\mathcal{O}(S^2_{pq})$ with the idempotent corresponding to the identity representation of $\mathrm{U}(1)$. Since this pairing turns out to be non-trivial, we conclude that the extension is non-cleft. Thus we have an example of a locally trivial equivariantly projective noncommutative Hopf–Galois extension which not only is not trivial, but also is *not* a cross-product (cleft) construction.

Recently, there was an outburst of new examples of noncommutative 3 and 4-spheres. For a comparison study of these and older constructions we refer to [D-L03]. Let us only mention that there exist at least two more classes of non-classical Hopf fibrations, notably the quantum Hopf fibrations coming from $\mathrm{SU}_q(2)$ and the super Hopf fibration. (See [BM00] and references therein for the former, and [DGH01] and its references for the latter.) Although $C(S^3_{pq})$ is not isomorphic to $C(\mathrm{SU}_q(2))$ (different sets of classical points), the C^*-algebra of the generic Podles quantum sphere coincides with the C^*-algebra of the base space of our locally trivial quantum Hopf fibration, i.e., $C(S^2_{pq}) \cong C(S^2_{\mu c})$, $p, q, \mu \in (0, 1)$, $c > 0$ [CM00, Proposition 21]. This holds despite the fact that the polynomial $*$-subalgebras $\mathcal{O}(S^2_{pq})$ and $\mathcal{O}(S^2_{\mu c})$ are non-isomorphic, as shown in Proposition 1.4.

1. Preliminaries

1.1. LOCALLY TRIVIAL H-EXTENSIONS

The idea of a locally trivial H-extension can be traced back to [BM93, P-MJ94, D-M96, BK96, CM02]. The prerequisite idea of the gluing of quantum spaces can be traced much further. In terms of C^*-algebras, the gluing corresponds to the pullback construction, which is essential, e.g., for the Busby invariant. We refer to [P-GK99] for lots of generalities on pullbacks of C^*-algebras. Here, let us only recall the needed definitions and fix the terminology.

DEFINITION 1.1 ([BK96, CM00]). A *covering* of an algebra B is a family $\{J_i\}_{i\in I}$ of ideals with zero intersection. Let $\pi_i\colon B \to B_i := B/J_i$, $\pi^i_j\colon B_i \to B_{ij} := B/(J_i + J_j)$ be the quotient maps. A covering $\{J_i\}_{i\in I}$ is called *complete* iff the homomorphism

$$B \ni b \mapsto (\pi_i(b))_{i\in I} \in B_c := \left\{(b_i)_{i\in I} \in \prod_{i\in I} B_i \;\middle|\; \forall i, j \in I : \pi^i_j(b_i) = \pi^j_i(b_j)\right\}$$

is surjective. (It is automatically injective.)

Note that finite coverings consisting of closed ideals in C^*-algebras and two-element coverings are always complete. See [CM00] for more information about completeness of coverings, in particular for an example of a non-complete covering. Let B and P be algebras and H be a Hopf algebra. Assume that B is a subalgebra of P via an injective homomorphism $\iota\colon B \to P$, which we also write as $B \subseteq P$. The inclusion $B \subseteq P$ is called an *H-extension* if P is a right H-comodule algebra via a right coaction $\Delta_R\colon P \to P \otimes H$ such that B coincides with the subalgebra of coinvariants, i.e., $B = P^{coH} := \{p \in P \mid \Delta_R(p) = p \otimes 1\}$.

DEFINITION 1.2 ([CM02]). A *locally trivial H-extension*⋆ is an H-extension $B \subseteq P$ together with the following (local) data:

(i) a complete finite covering $\{J_i\}_{i\in I}$ of B;
(ii) surjective homomorphisms $\chi_i\colon P \to B_i \otimes H$ (local trivializations) such that

⋆ In [CM02] locally trivial H-extensions were called locally trivial quantum principal fibre bundles.

(a) $\chi_i \circ \iota = \pi_i \otimes 1$ (B-linearity),
(b) $(\chi_i \otimes \mathrm{id}) \circ \Delta_R = (\mathrm{id} \otimes \Delta) \circ \chi_i$ (H-colinearity),
(c) $\{\mathrm{Ker}\, \chi_i\}_{i \in I}$ is a complete covering of P.

Locally trivial H-extensions can be reconstructed from the local data. First, it follows from the proof of [CM02, Proposition 2] that there exist homomorphisms $\varphi_{ij}\colon B_{ij} \otimes H \to B_{ij} \otimes H$ (to be thought of as the change of local trivializations) that are uniquely determined by the formula

$$\varphi_{ij} \circ (\pi_i^j \otimes \mathrm{id}) \circ \chi_j = (\pi_j^i \otimes \mathrm{id}) \circ \chi_i. \tag{1.1}$$

As argued in [CM02] (see there formulas (5) and (6) and the remark after the proof of Proposition 2), these maps are isomorphisms satisfying

$$(\mathrm{id} \otimes \Delta) \circ \varphi_{ij} = (\varphi_{ij} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \quad (H\text{-colinearity}), \tag{1.2}$$

$$\varphi_{ij}(\pi_{ij}(b'b) \otimes h) = \pi_{ij}(b')\varphi_{ij}(\pi_{ij}(b) \otimes h) \quad (B\text{-linearity}). \tag{1.3}$$

Next, in analogy with the classical situation, one defines transition functions $\tau_{ji}\colon H \to B_{ij}$ by

$$\tau_{ji}(h) = (\mathrm{id} \otimes \varepsilon)(\varphi_{ij}(1 \otimes h)). \tag{1.4}$$

Due to the aforementioned colinearity and linearity, one can equivalently rewrite this expression as

$$\varphi_{ij}(b \otimes h) = b\tau_{ji}(h_{(1)}) \otimes h_{(2)}. \tag{1.5}$$

It can be shown (see [BK96] or [CM02, Proposition 4]) that the transition functions of a locally trivial H-extension are homomorphisms with the following properties:

$$\tau_{ii} = \varepsilon, \tag{1.6}$$

$$\tau_{ji} \circ S = \tau_{ij}, \tag{1.7}$$

$$\tau_{ij}(H) \subseteq Z(B_{ij}) \quad (\text{centre of } B_{ij}), \tag{1.8}$$

$$\pi_k^{ij} \circ \tau_{ij} = m_{B_{ijk}} \circ ((\pi_j^{ik} \circ \tau_{ik}) \otimes (\pi_i^{jk} \circ \tau_{kj})) \circ \Delta. \tag{1.9}$$

Here $\pi_k^{ij}\colon B_{ij} \to B_{ijk} := B/(J_i + J_j + J_k)$ is the quotient map and $m_{B_{ijk}}$ is the multiplication in B_{ijk}.

Conversely, let us consider an algebra B with a complete finite covering $\{J_i\}_{i \in I}$ and a Hopf algebra H. Assume that we have a family of homomorphisms $\tau_{ji}\colon H \to B_{ij}$ satisfying (1.6)–(1.9). Define φ_{ij} by the formula (1.5) and put

$$\widetilde{P} = \left\{ (f_i)_{i \in I} \in \textstyle\prod_{i \in I}(B_i \otimes H) \,\middle|\, (\pi_j^i \otimes \mathrm{id})(f_i) = \varphi_{ij}((\pi_i^j \otimes \mathrm{id})(f_j)) \right\}. \tag{1.10}$$

One can verify that the formulas

$$\widetilde{\Delta}_R((f_i)_{i \in I}) = ((\mathrm{id} \otimes \Delta)(f_i))_{i \in I}, \tag{1.11}$$

$$\widetilde{\chi}_j((f_i)_{i \in I}) = f_j, \tag{1.12}$$

$$\widetilde{\iota}(b) = (\pi_i(b) \otimes 1)_{i \in I}, \tag{1.13}$$

turn $\widetilde{P}$ into a locally trivial H-extension of B. Moreover, we have:

PROPOSITION 1.3 ([CM02]). *Let P be the locally trivial H-extension of B corresponding to a covering $\{J_i\}_{i\in I}$ and $\tau_{ji}\colon H \to B_{ij}$ its transition functions. Let $\widetilde{P}$ be the locally trivial H-extension constructed from the τ_{ji}'s. Then the formula $p \mapsto (\chi_i(p))_{i\in I}$ defines an isomorphism of locally trivial H-extensions P and $\widetilde{P}$.*

1.2. CONSTRUCTION OF S^3_{pq}

Our starting point is the coordinate algebra $\mathcal{O}(D_p)$ of the quantum disc, which is defined as the universal unital $*$-algebra generated by x fulfilling the relation

$$x^*x - pxx^* = 1 - p, \quad 0 < p < 1. \tag{1.14}$$

This is a one-parameter sub-family of the two-parameter family of quantum discs defined in [KL93]. Using all bounded representations of $\mathcal{O}(D_p)$, one can define its C^*-closure $C(D_p)$. It can be shown that the C^*-algebra $C(D_p)$ is isomorphic with the Toeplitz algebra $\mathcal{T}$ (e.g., see [CM00, Proposition 15]), and that $\|x\| = 1$ (see [KL93, Proposition IV.1(I)]). Let us also mention that there are unbounded representations of the relation (1.14). They are given, e.g., in [KS97, Section 5.2.6].

Next, we glue two quantum discs to get a quantum S^2. Let $\mathcal{O}(S^1)$ be the universal $*$-algebra generated by the unitary u. Then we have a natural epimorphism $\pi_p\colon \mathcal{O}(D_p) \to \mathcal{O}(S^1)$ given by $\pi_p(x) = u$. (This corresponds to embedding S^1 into D_p as its boundary.) Now, we can define $\mathcal{O}(S^2_{pq})$ in the following manner [CM00]:

$$\mathcal{O}(S^2_{pq}) := \{(f, g) \in \mathcal{O}(D_p) \oplus \mathcal{O}(D_q) \mid \pi_p(f) = \pi_q(g)\}. \tag{1.15}$$

The algebra $\mathcal{O}(S^2_{pq})$ has a complete covering $\{\mathrm{Ker}\, pr_1, \mathrm{Ker}\, pr_2\}$, where pr_1 and pr_2 are the restrictions to $\mathcal{O}(S^2_{pq})$ of the projections on $\mathcal{O}(D_p)$ and $\mathcal{O}(D_q)$, respectively (see [CM00, Proposition 8]). Furthermore, one has canonical isomorphisms $\mathcal{O}(S^2_{pq})/\mathrm{Ker}\, pr_1 \cong \mathcal{O}(D_p)$ and $\mathcal{O}(S^2_{pq})/\mathrm{Ker}\, pr_2 \cong \mathcal{O}(D_q)$ and $\mathcal{O}(S^2_{pq})/(\mathrm{Ker}\, pr_1 + \mathrm{Ker}\, pr_2) \cong \mathcal{O}(S^1)$. As was shown in [CM00, Proposition 17], $\mathcal{O}(S^2_{pq})$ can be identified with the universal $*$-algebra generated by f_0 and f_1 satisfying the relations

$$f_0 = f_0^*, \tag{1.16}$$

$$f_1^* f_1 - q f_1 f_1^* = (p - q) f_0 + 1 - p, \tag{1.17}$$

$$f_0 f_1 - p f_1 f_0 = (1 - p) f_1, \tag{1.18}$$

$$(1 - f_0)(f_1 f_1^* - f_0) = 0. \tag{1.19}$$

The isomorphism is given by $f_1 \mapsto (x, y)$, $f_0 \mapsto (xx^*, 1)$. Here x denotes the generator of the $*$-algebra $\mathcal{O}(D_p)$, and y that of $\mathcal{O}(D_q)$. In terms of these generators, the irreducible $*$-representations of $\mathcal{O}(S^2_{pq})$ can be given as follows [CM00, Proposition 19]:

$$\rho_\theta(f_0) = 1, \qquad \rho_\theta(f_1) = \mathrm{e}^{i\theta}, \quad \theta \in [0, 2\pi) \quad \text{(classical points)}, \tag{1.20}$$

$$\rho_1(f_0)e_k = (1 - p^k)e_k, \qquad \rho_1(f_1)e_k = \sqrt{1 - p^{k+1}}\, e_{k+1}, \quad k \geqslant 0; \tag{1.21}$$

$$\rho_2(f_0)e_k = e_k, \qquad \rho_2(f_1)e_k = \sqrt{1 - q^{k+1}}\, e_{k+1}, \quad k \geqslant 0. \tag{1.22}$$

Here $\{e_k\}_{k\geqslant 0}$ is an orthonormal basis of a separable Hilbert space.

In the classical case $p = q = 1$, the relations (1.16)–(1.19) reduce to commutativity and the geometrical relations (1.16) and (1.19). Adding by hand the conditions $|f_0| \leqslant 1$, $|f_1| \leqslant 1$ (which are automatic in the noncommutative case [CM00, Proposition 19]), one obtains as the corresponding geometric space a closed cone. The irreducible representations given above allow one to make an analogous picture also in the noncommutative case. The sum of the squares of the hermitian generators $f_+ = \frac{1}{2}(f_1 + f_1^*)$ and $f_- = \frac{i}{2}(f_1 - f_1^*)$ is diagonal in the representations, and one can imagine a discretized version of the above cone, with the edge being the circle of classical points and the remainder of the cone being formed by "non-classical circles", accumulating at this edge (cf. [CM00, pp. 387–388]).

It follows from the relations (1.16)–(1.19) that $\|\rho(f_i)\| \leqslant 1$ in every bounded representation of $\mathcal{O}(S^2_{pq})$. Therefore, one can form a C^*-algebra $C(S^2_{pq})$ using bounded representations. As noticed already at the end of the introduction, this C^*-algebra is isomorphic to the C^*-algebra $C(S^2_{\mu c})$ of the Podles spheres for $c > 0$. Such an isomorphism does not exist on the level of polynomial $*$-algebras. Indeed, there are two infinite-dimensional irreducible representations π_+ and π_- of $C(S^2_{\mu c})$ [P-P87, Proposition 4] whose restrictions to $\mathcal{O}(S^2_{\mu c})$ are faithful. (Their faithfulness can be proved by direct arguments using a vector space basis of $\mathcal{O}(S^2_{\mu c})$ provided in [P-P89, Section 3].) On the other hand, the representations ρ_1 and ρ_2 are not faithful ($\rho_1(f_1 f_1^* - f_0) = 0$ and $\rho_2(1 - f_0) = 0$) and (up to unitary equivalence), these are the only infinite-dimensional irreducible representations of $\mathcal{O}(S^2_{pq})$. Thus, we have

PROPOSITION 1.4. *For any* $p, q, \mu \in (0, 1)$, $c > 0$, *there is no* $*$*-algebra isomorphism between* $\mathcal{O}(S^2_{pq})$ *and* $\mathcal{O}(S^2_{\mu c})$.

We are now ready for the definition of $\mathcal{O}(S^3_{pq})$. We consider $\mathcal{O}(\mathrm{U}(1))$ as the $*$-algebra $\mathcal{O}(S^1)$ generated by the unitary u: $\mathrm{e}^{i\varphi} \mapsto \mathrm{e}^{i\varphi}$ and equipped with the Hopf algebra structure given by $\Delta(u) = u \otimes u$, $\varepsilon(u) = 1$, $S(u) = u^*$. We view $\mathcal{O}(S^2_{pq})$ as an algebra with the complete covering $\{\mathrm{Ker}\, pr_1, \mathrm{Ker}\, pr_2\}$. The homomorphisms $\tau_{12} := \mathrm{id}$: $\mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(S^1)$, $\tau_{21} := S$, $\tau_{11} := \varepsilon =: \tau_{22}$, evidently fulfill the axioms (1.6)–(1.9). Therefore, we can proceed along the lines of Subsection 1.1, and define the following locally trivial $\mathcal{O}(\mathrm{U}(1))$-extension:

DEFINITION 1.5 ([CM02, p. 152]). Let τ_{ji}: $\mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(S^2_{pq})_{ij}$ be the homomorphisms given above. We define $\mathcal{O}(S^3_{pq})$ as the locally trivial $\mathcal{O}(\mathrm{U}(1))$-extension of $\mathcal{O}(S^2_{pq})$ given by the τ_{ji}'s via (1.5) and (1.10). Explicitly, the algebra $\mathcal{O}(S^3_{pq})$ is

$$\{(a_1, a_2) \in (\mathcal{O}(D_p) \otimes \mathcal{O}(\mathrm{U}(1)) \oplus (\mathcal{O}(D_q) \otimes \mathcal{O}(\mathrm{U}(1)) \mid (\pi_p \otimes \mathrm{id})(a_1) = \varphi_{12}((\pi_q \otimes \mathrm{id})(a_2))\},$$

where π_p: $\mathcal{O}(D_p) \to \mathcal{O}(S^1)$, $\pi_p(x) = u$, π_q: $\mathcal{O}(D_q) \to \mathcal{O}(S^1)$, $\pi_q(y) = u$.

Note that we glue two quantum solid tori $D_p \times \mathrm{U}(1)$ and $\mathrm{U}(1) \times D_q$ along their classical boundaries, which are T^2. The subspace of the classical points of the resulting S^3_{pq} is precisely the locus of the gluing (see Section 2). In terms of generators and relations, $\mathcal{O}(S^3_{pq})$ can be characterised in the following way:

LEMMA 1.6 ([CM02, Proposition 21]). *The algebra $\mathcal{O}(S^3_{pq})$ is isomorphic to the universal unital $*$-algebra generated by a and b satisfying the relations*

$$a^*a - qaa^* = 1 - q, \tag{1.23}$$

$$b^*b - pbb^* = 1 - p, \tag{1.24}$$

$$ab = ba, \qquad a^*b = ba^*, \tag{1.25}$$

$$(1 - aa^*)(1 - bb^*) = 0. \tag{1.26}$$

The isomorphism is given by $(1 \otimes u, y \otimes u) \mapsto a$ and $(x \otimes u^*, 1 \otimes u^*) \mapsto b$. Using this identification and the description of $\mathcal{O}(S^2_{pq})$ in terms of the generators f_0 and f_1 ((1.16)–(1.19)), we can write the structural $*$-homomorphisms of the locally trivial $\mathcal{O}(\mathrm{U}(1))$-extension $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ in the form

$$\Delta_R(a) = a \otimes u, \quad \Delta_R(b) = b \otimes u^*, \tag{1.27}$$

$$\chi_p(a) = 1 \otimes u, \quad \chi_q(a) = y \otimes u, \quad \chi_p(b) = x \otimes u^*, \quad \chi_q(b) = 1 \otimes u^*, \tag{1.28}$$

$$\iota(f_1) = ba, \quad \iota(f_0) = bb^*. \tag{1.29}$$

As shown in Section 2, we can define the C^*-algebra of $\mathcal{O}(S^3_{pq})$ as the universal unital C^*-algebra generated by a and b satisfying the relations (1.23)–(1.26) and $\|a\| = 1 = \|b\|$. The last condition follows from the remaining ones for $p, q \in (0, 1)$ for the same reason it is automatically true for the generator of $\mathcal{O}(D_p)$ (see [KL93, Proposition IV.1(I)]), but for $p = 1 = q$ we need to put it by hand. In the classical case ($p = 1 = q$), this C^*-algebra coincides with $C(S^3)$ [M-K91a, p. 334]. By the universality of the C^*-algebra $C(S^3_{pq})$, the right coaction Δ_R extends to $C(S^3_{pq})$ and is equivalent to a U(1)-action on $C(S^3_{pq})$ (use [W-NE93, Proposition T.5.21]). The latter reduces for $p = 1 = q$ to the U(1)-action on S^3 yielding the Hopf fibration. (Our convention for the action differs from the convention in [M-K91a].) To understand more precisely the classical case (see [N-GL97, Section 0.3] for related details), let us prove the following:

PROPOSITION 1.7. *Define $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid (1 - |z_1|^2)(1 - |z_2|^2) = 0,$ $|z_1|, |z_2| \leqslant 1\}$ and $S^3 = \{(c_1, c_2) \in \mathbb{C}^2 \mid |c_1|^2 + |c_2|^2 = 1\}$. The group* U(1) *acts on X and S^3 via $(z_1, z_2) \cdot e^{i\varphi} = (z_1 e^{i\varphi}, z_2 e^{-i\varphi})$ and $(c_1, c_2) \cdot e^{i\varphi} = (c_1 e^{i\varphi}, c_2 e^{i\varphi})$, respectively, and X and S^3 are homeomorphic as* U(1)*-spaces.*

Proof. The U(1)-action on $\mathbb{C}^2$ clearly restricts to both X and S^3. Note first that we can equivalently write the equation $(1 - |z_1|^2)(1 - |z_2|^2) = 0$ in the form $|z_1|^2 + |z_2|^2 = 1 + |z_1|^2|z_2|^2$. This suggests that we can define a map $X \xrightarrow{f} S^3$ by the formula (cf. [MT92, p. 38] for the case of S^3_θ):

$$f((z_1, z_2)) = (1 + |z_1|^2|z_2|^2)^{-\frac{1}{2}}(z_1, \overline{z_2}). \tag{1.30}$$

Indeed, f is a continuous U(1)-map into S^3. To find the inverse of f, we look for a map of the form $(c_1, c_2) \mapsto \alpha(c_1, \overline{c_2})$.* A direct computation provides us with the formula:

$$g((c_1, c_2)) = \frac{\sqrt{2}(c_1, \overline{c_2})}{\sqrt{1 + \big|\,|c_1|^2 - |c_2|^2\,\big|}} =: (g_1, g_2). \tag{1.31}$$

Taking advantage of $|c_1|^2 + |c_2|^2 = 1$, we compute

$$|g_1|^2 + |g_2|^2 = \frac{2}{1 + \big|\,|c_1|^2 - |c_2|^2\,\big|} \tag{1.32}$$

and

$$\begin{aligned} 1 + |g_1|^2|g_2|^2 &= \frac{(1 + \big|\,|c_1|^2 - |c_2|^2\,\big|)^2 + 4|c_1|^2|c_2|^2)}{(1 + \big|\,|c_1|^2 - |c_2|^2\,\big|)^2} \\ &= \frac{2}{1 + \big|\,|c_1|^2 - |c_2|^2\,\big|}. \end{aligned} \tag{1.33}$$

Hence $(1-|g_1|^2)(1-|g_2|^2) = 0$. Furthermore, as $\big|\,|c_1|^2-|c_2|^2\,\big| = |\,2|c_1|^2-1\,| = |\,2|c_2|^2 - 1\,|$, we obtain

$$|g_i| \leqslant 1 \Leftrightarrow 2|c_i|^2 \leqslant 1 + |\,2|c_i|^2 - 1\,|. \tag{1.34}$$

We have two cases. For $2|c_i|^2 \geqslant 1$, the latter inequality reads $0 \leqslant 0$. Otherwise, i.e., for $2|c_i|^2 < 1$, it is the same as $2|c_i|^2 \leqslant 1$. Thus we have a continuous map $S^3 \xrightarrow{g} X$. As both f and g are evidently U(1)-maps, it only remains to prove that they are mutually inverse. Remembering (1.33), we have

$$(f \circ g)((c_1, c_2)) = \frac{\sqrt{2}(c_1, c_2)}{\sqrt{1 + \big|\,|c_1|^2 - |c_2|^2\,\big|}} \frac{\sqrt{1 + \big|\,|c_1|^2 - |c_2|^2\,\big|}}{\sqrt{2}} = (c_1, c_2). \tag{1.35}$$

For the other identity, note first that, due to $(1 - |z_1|^2)(1 - |z_2|^2) = 0$, we have $|z_j|^2 \leqslant |z_i|^2 = 1$. To avoid confusion, let us fix $|z_2|^2 \leqslant |z_1|^2 = 1$. (The other case behaves in the same way.) Now we can compute

$$\begin{aligned} (g \circ f)((z_1, z_2)) &= \frac{\sqrt{2}(z_1, z_2)}{\sqrt{1 + |z_1|^2|z_2|^2}\sqrt{1 + \left|\frac{|z_1|^2-|z_2|^2}{1+|z_1|^2|z_2|^2}\right|}} \\ &= \frac{\sqrt{2}(z_1, z_2)}{\sqrt{1 + |z_1|^2|z_2|^2 + \big|\,|z_1|^2 - |z_2|^2\,\big|}} \\ &= \frac{\sqrt{2}(z_1, z_2)}{\sqrt{|z_1|^2 + |z_2|^2 + \big|\,|z_1|^2 - |z_2|^2\,\big|}} \\ &= (z_1, z_2). \end{aligned} \tag{1.36}$$

This ends the proof. □

* We are grateful to Andrzej Sitarz for his involvement here.

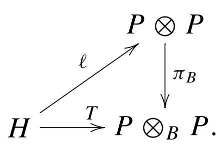
1.3. HOPF–GALOIS EXTENSIONS AND ASSOCIATED MODULES

We refer to [M-S93] for generalities concerning Hopf–Galois theory. Recall that an H-extension $B \subseteq P$ is called *Hopf–Galois* iff the canonical map

$$\mathrm{can}\colon\ P \otimes_B P \longrightarrow P \otimes H, \quad p \otimes p' \longmapsto p\Delta_R(p'), \tag{1.37}$$

is bijective. Note that can is surjective whenever, for any generator h of H, the element $1 \otimes h$ is in the image of can (cf. [S-P00, pp. 106–107]). Indeed, if the tensors $\sum_i h_i \otimes \tilde{h}_i$ and $\sum_j g_j \otimes \tilde{g}_j \in P \otimes P$ are such that $\mathrm{can}(\sum_i h_i \otimes_B \tilde{h}_i) = 1 \otimes h$ and $\mathrm{can}(\sum_j g_j \otimes_B \tilde{g}_j) = 1 \otimes g$, then $\sum_{ij} g_j h_i \otimes \tilde{h}_i \tilde{g}_j \in P \otimes P$ has the property $\mathrm{can}(\sum_{ij} g_j h_i \otimes_B \tilde{h}_i \tilde{g}_j) = 1 \otimes hg$. Hence, for any monomial $w \in H$, the element $1 \otimes w$ is in the image of can, and its surjectivity follows from its left P-linearity.

The restricted inverse of can, $T := \mathrm{can}^{-1} \circ (1 \otimes \mathrm{id})$, is called the *translation map*. We are interested in unital bicolinear liftings of the translation map T because, when the antipode S of H is bijective, they can be interpreted as strong connections on algebraic quantum principal bundles [BH04, Lemma 2.3]. More precisely, we take the canonical surjection π_B and demand that the following diagram be commutative:



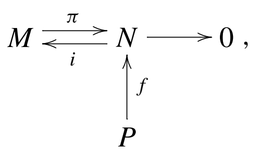
Then we equip $P \otimes P$ with an H-bicomodule structure via the maps

$$\Delta_L^{\otimes} := \big((S^{-1} \otimes \mathrm{id}) \circ (\mathrm{flip}) \circ \Delta_R\big) \otimes \mathrm{id} \quad \text{and} \quad \Delta_R^{\otimes} := \mathrm{id} \otimes \Delta_R, \tag{1.38}$$

and require that

$$\Delta_L^{\otimes} \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta \quad \text{and} \quad \Delta_R^{\otimes} \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta. \tag{1.39}$$

Finally, we ask that $\ell(1) = 1 \otimes 1$. It follows from [BH04, Lemmas 2.2 and 2.3] that the existence of such a lifting is equivalent to P being an $(H^*)^{op}$-equivariantly projective left B-module. (Here $(H^*)^{op}$ means the convolution algebra of functionals on H taken with the opposite multiplication.) We call such Hopf–Galois extensions equivariantly projective. In general (cf. [CE56, p. 197]), we say that a (B, A)-bimodule P is an *A-equivariantly projective left B-module* iff for every diagram with the exact row



where M, N are (B, A)-bimodules, π and f are (B, A)-bimodule maps and i is a right A-module splitting of π, there exists a (B, A)-bimodule map g rendering the following diagram commutative

$$\begin{array}{ccccc} M & \underset{i}{\overset{\pi}{\rightleftarrows}} & N & \longrightarrow & 0\,. \\ & \nwarrow^{g} & \uparrow{\scriptstyle f} & & \\ & & P & & \end{array}$$

If $B \subseteq P$ is a Hopf–Galois H-extension and $\rho\colon\ V \to V \otimes H$ is a coaction, then we can define the associated left B-module $\mathrm{Hom}_\rho(V, P)$ of all colinear maps from V to P. Such modules play the geometric role of the modules of sections of associated vector bundles. Therefore, to be in line with the Serre–Swan theorem and K-theory, it is desirable to have them finitely generated projective. It turns out that this is always the case for $\dim V < \infty$ and equivariantly projective Hopf–Galois extensions with bijective antipodes (see [DGH01, Corollary 2.6], cf. [BH04, Theorem 3.1] for a more general context).

Assume that ρ is a 1-dimensional corepresentation. Then it is given by a group-like g, $\rho(1) = 1 \otimes g$. Assume further that $B \subseteq P$ is an H-Galois extension admitting a strong connection ℓ (equivariant projectivity). Put $\ell(g) = \sum_k l_k(g) \otimes p_k$, where $\{p_k\}_k$ is a basis of P. Then it follows from [BH04, Theorem 3.1] that $E_{jk} := p_j l_k(g) \in B$, the matrix $E := (E_{jk})$ is idempotent ($E^2 = E$), and we have an isomorphism of left B-modules $\mathrm{Hom}_\rho(\mathbb{C}, P) \cong B^n E$. Here n is the number of nonzero elements in the sum $\sum_k l_k(g) \otimes p_k$.

2. Representations and the C^*-algebra of $\mathcal{O}(S^3_{pq})$

To begin with, we classify the bounded irreducible $*$-representations of $\mathcal{O}(S^3_{pq})$. We do it much as in the case of the quantum real projective space $\mathbb{R}P^2_q$ [HMS03a, Theorem 4.5]. As a result, we obtain two S^1-families of infinite-dimensional representations and a T^2-family of one-dimensional representations (classical points). The latter proves that S^3_{pq} differs from other quantum 3-spheres (see [D-L03, Section 3] for details).

THEOREM 2.1. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_k\}_{k \geqslant 0}$. Any irreducible $*$-representation of $\mathcal{O}(S^3_{pq})$ in bounded operators on a Hilbert space is unitarily equivalent to one of the following:*

$$\rho_{1\theta}(a)e_k = \mathrm{e}^{i\theta} e_k, \qquad \rho_{1\theta}(b)e_k = \sqrt{1 - p^{k+1}}\, e_{k+1}, \quad \theta \in [0, 2\pi); \tag{2.1}$$

$$\rho_{2\theta}(a)e_k = \sqrt{1 - q^{k+1}}\, e_{k+1}, \qquad \rho_{2\theta}(b)e_k = \mathrm{e}^{i\theta} e_k, \quad \theta \in [0, 2\pi); \tag{2.2}$$

$$\rho_{\theta_1\theta_2}(a) = \mathrm{e}^{i\theta_1}, \qquad \rho_{\theta_1\theta_2}(b) = \mathrm{e}^{i\theta_2}, \quad \theta_1, \theta_2 \in [0, 2\pi). \tag{2.3}$$

Proof. Let ρ be a $*$-representation of $\mathcal{O}(S^3_{pq})$ in bounded operators on a Hilbert space $\widetilde{\mathcal{H}}$. Then it follows immediately from the relations (1.23)–(1.26) that

$\mathrm{Ker}(1 - \rho(aa^*))$ and $\mathrm{Ker}(1 - \rho(bb^*))$ are invariant subspaces. Hence $\mathcal{H}_0 := \mathrm{Ker}(1-\rho(aa^*)) \cap \mathrm{Ker}(1-\rho(bb^*))$ is invariant. Let $\varphi \neq 0$ be a vector in the orthogonal complement of the closure of the sum $\mathrm{Ker}(1-\rho(aa^*))+\mathrm{Ker}(1-\rho(bb^*))$. Then, due to the invariance of this complement, $(1 - \rho(aa^*))(1 - \rho(bb^*))\varphi \neq 0$, which contradicts (1.26). Therefore this complement must be zero, i.e., $\widetilde{\mathcal{H}}$ automatically coincides with the closure of $\mathrm{Ker}(1 - \rho(aa^*)) + \mathrm{Ker}(1 - \rho(bb^*))$. Thus we have the direct sum decomposition

$$\widetilde{\mathcal{H}} = \mathcal{H}' \oplus \mathcal{H}'' \oplus \mathcal{H}_0 \tag{2.4}$$

into ρ-invariant subspaces. Here $\mathcal{H}' := \mathrm{Ker}(1-\rho(aa^*)) \ominus \mathcal{H}_0$ and likewise $\mathcal{H}'' := \mathrm{Ker}(1 - \rho(bb^*)) \ominus \mathcal{H}_0$ are appropriate orthogonal complements. Our strategy is to look for irreducible representations on these three subspaces separately. We abuse notation by using ρ to denote also its restrictions.

For the restriction of ρ to $\mathcal{H}_0$ we have $\rho(a)\rho(a^*) = 1 = \rho(b)\rho(b^*)$. It follows from the disc-like relations (1.23) and (1.24) that also $\rho(a^*)\rho(a) = 1 = \rho(b^*)\rho(b)$, so that $\rho(a)$ and $\rho(b)$ are unitary. Since $\rho(a)$ and $\rho(b)$ commute, we arrive at the third family of representations.

Consider now the restriction of ρ to $\mathcal{H}'$. On this invariant subspace, we have $\rho(aa^*) = 1$, and it follows from (1.23) that $\rho(a)$ is unitary. The operator $1-\rho(bb^*)$ is injective on $\mathcal{H}'$. Our aim is to determine the spectrum of $1 - \rho(bb^*) \in B(\mathcal{H}')$. First we show that $\mathrm{spec}(1 - \rho(bb^*)) \subseteq [0, 1]$. From (1.24) we can conclude that $\|\rho(b^*b - pbb^*)\| = 1 - p$. Hence $\|\rho(b^*b)\| - p\|\rho(bb^*)\| \leqslant 1 - p$, which gives

$$\|\rho(b)\|^2 = \|\rho(b^*b)\| = \|\rho(bb^*)\| \leqslant 1. \tag{2.5}$$

This means $0 \leqslant \rho(bb^*) \leqslant 1$, and therefore also $0 \leqslant 1 - \rho(bb^*) \leqslant 1$, which yields the desired inclusion for the spectrum.

If $0 \in \mathrm{spec}(1 - \rho(bb^*))$, it cannot be an eigenvalue, since this would contradict the injectivity of $1 - \rho(bb^*)$. Therefore, 0 cannot be isolated in the spectrum. If 1 were the only element of the spectrum, this would mean $1 - \rho(bb^*) = 1$ on $\mathcal{H}'$, i.e., $\rho(b) = 0$, contradicting (1.24). Hence we conclude that there exists $\lambda \in (0, 1) \cap \mathrm{spec}(1 - \rho(bb^*))$. By [KR97, Lemma 3.2.13], there exists a sequence of unit vectors $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}'$ such that

$$\lim_{n\mapsto\infty} \|\rho(1 - bb^*)\varphi_n - \lambda\varphi_n\| = 0. \tag{2.6}$$

To prove that also $p^{-1}\lambda$ must be in the spectrum, we first consider the estimate

$$\begin{aligned} &\|\rho((1 - bb^*)b^*)\varphi_n - p^{-1}\lambda\rho(b^*)\varphi_n\| \\ &\quad = \|p^{-1}\rho(b^*(1 - bb^*))\varphi_n - p^{-1}\lambda\rho(b^*)\varphi_n\| \\ &\quad \leqslant p^{-1}\|\rho(b^*)\|\|\rho(1 - bb^*)\varphi_n - \lambda\varphi_n\|. \end{aligned} \tag{2.7}$$

Next, to show that $\|\rho(b^*)\varphi_n\| \geqslant C$ for some $C > 0$ and all sufficiently big n, we compute

$$\begin{aligned} \|\rho(bb^*)\varphi_n\| &= \|(1 - \lambda - \rho(1 - bb^* - \lambda))\varphi_n\| \\ &\geqslant |1 - \lambda| - \|\rho(1 - bb^* - \lambda)\varphi_n\|. \end{aligned} \tag{2.8}$$

Using $\|\rho(b)\|\|\rho(b^*)\varphi_n\| \geqslant \|\rho(bb^*)\varphi_n\|$, we conclude that

$$\|\rho(b^*)\varphi_n\| \geqslant \frac{1}{\|\rho(b)\|}(|1-\lambda| - \|\rho(1-bb^*-\lambda)\varphi_n\|). \tag{2.9}$$

Due to (2.6), the term within the parentheses approaches $1-\lambda > 0$ for $n \to \infty$, so that there is $N \in \mathbb{N}$ and $C > 0$ such that $\|\rho(b^*)\varphi_n\| \geqslant C$ for $n > N$. Consequently, we can form a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of unit vectors consisting of $\eta_n := \frac{\rho(b^*)\varphi_n}{\|\rho(b^*)\varphi_n\|}$ for $n > N$ and arbitrarily chosen unit vectors η_n for $n \leqslant N$. It is now immediate from (2.6) and the estimate (2.7) that

$$\lim_{n\mapsto\infty} \|\rho(1-bb^*)\eta_n - p^{-1}\lambda\eta_n\| = 0. \tag{2.10}$$

Employing again [KR97, Lemma 3.2.13], we have that $p^{-1}\lambda \in \mathrm{spec}(\rho(1-bb^*))$.

We can iterate this reasoning until $p^{-k}\lambda = 1$. This has to be true for some k, as otherwise we would contradict $\mathrm{spec}(\rho(1-bb^*)) \subseteq [0, 1]$. It follows that

$$\{1, p, \ldots, p^k\} \subseteq \mathrm{spec}(\rho(1-bb^*)) \subseteq \{1, p, p^2, \ldots\} \cup \{0\}. \tag{2.11}$$

We now show that the latter inclusion is an equality. Let ξ_k be a (non-zero) eigenvector corresponding to the eigenvalue p^k, i.e., $\rho(1-bb^*)\xi^k = p^k\xi_k$. Then, using (1.24), we have

$$\rho(1-bb^*)\rho(b)\xi_k = p\rho(b)\rho(1-bb^*)\xi_k = p^{k+1}\rho(b)\xi_k. \tag{2.12}$$

Using the same relation $(1-bb^*)b = pb(1-bb^*)$, we obtain

$$\begin{aligned}\|\rho(b)\xi_k\|^2 &= \langle\rho(b)\xi_k \mid \rho(b)\xi_k\rangle\\ &= \langle\rho(b^*b)\xi_k \mid \xi_k\rangle\\ &= \langle(p\rho(bb^*) + 1 - p)\xi_k \mid \xi_k\rangle\\ &= p\|\rho(b^*)\xi_k\|^2 + (1-p)\|\xi_k\|^2 > 0.\end{aligned} \tag{2.13}$$

Hence, $\rho(b)\xi_k$ is a non-zero eigenvector to the eigenvalue p^{k+1}. This proves that

$$\mathrm{spec}(\rho(1-bb^*)) = \{1, p, p^2, \ldots\} \cup \{0\}. \tag{2.14}$$

We are ready now to construct a set of orthonormal vectors. Since 1 is isolated in $\mathrm{spec}(\rho(1-bb^*))$, there exists a normalised eigenvector ξ given by $\rho(1-bb^*)\xi = \xi$. Making again use of $(1-bb^*)b = pb(1-bb^*)$, we obtain

$$\begin{aligned}\|\rho(b^{k+1})\xi\|^2 &= \langle\rho(b^*bb^k)\xi \mid \rho(b^k)\xi\rangle\\ &= \langle\rho((p(bb^*-1)+1)b^k)\xi \mid \rho(b^k)\xi\rangle\\ &= \langle\rho(b^k)\xi \mid \rho(b^k)\xi\rangle - p^{k+1} < \rho(b^k(1-bb^*))\xi \mid \rho(b^k)\xi\rangle\\ &= (1-p^{k+1})\langle\rho(b^k)\xi \mid \rho(b^k)\xi\rangle\\ &= (1-p^{k+1})\|\rho(b^k)\xi\|^2.\end{aligned} \tag{2.15}$$

Thus $e_k := \frac{\rho(b^k)\xi}{\|\rho(b^k)\xi\|}$ are normalised eigenvectors of $\rho(1-bb^*)$ corresponding to the eigenvalue p^k. Note also that the vectors e_k are orthogonal as they are eigenvectors to different eigenvalues of the selfadjoint operator $1 - \rho(bb^*)$. Furthermore, it follows from the foregoing computation that

$$\rho(b)e_k = \frac{\rho(b^{k+1})\xi}{\|\rho(b^k)\xi\|} = \frac{\rho(b^{k+1})\xi}{\|\rho(b^{k+1})\xi\|}\frac{\|\rho(b^{k+1})\xi\|}{\|\rho(b^k)\xi\|} = \sqrt{1-p^{k+1}}\, e_{k+1}. \quad (2.16)$$

On the other hand,

$$\|\rho(b^*)\xi\|^2 = \langle\rho(b^*)\xi \mid \rho(b^*)\xi\rangle = \langle\rho(bb^*)\xi \mid \xi\rangle = 0, \quad (2.17)$$

so that $\rho(b^*)\xi = 0$. Computing as in (2.15) and (2.16), we obtain

$$\rho(b^*)e_k = \frac{\rho(b^*b^k)\xi}{\|\rho(b^k)\xi\|} = \sqrt{1-p^k}\, e_{k-1}, \quad k > 0. \quad (2.18)$$

Finally, it follows from $\mathcal{H}' \subseteq \mathrm{Ker}(\rho(1-aa^*))$ and (1.23) that $\rho(a)$ is unitary on $\mathcal{H}'$. Since it also commutes with $\rho(b)$ and $\rho(b^*)$, it belongs to the centre of the representation. Hence, as the centre is always trivial in an irreducible representation, $\rho(a)$ has to be a multiple of the identity operator. Therefore, the closed span of the orthonormal vectors $\{e_k\}_{k\geqslant 0}$ is invariant under the whole algebra. Consequently, any irreducible $*$-representation on $\mathcal{H}'$ is unitarily equivalent to one of the first family of representations. The second family is derived in the same way exchanging the roles of a and b, and p and q. □

Since, due to the disc-like relations (1.23)–(1.24), we have $\|\rho(a)\| \leqslant 1$ and $\|\rho(b)\| \leqslant 1$ for any bounded $*$-representation of $\mathcal{O}(S^3_{pq})$ [KL93, Proposition IV.1(I)], we can define the C^*-algebra $C(S^3_{pq})$ of $\mathcal{O}(S^3_{pq})$ as follows:

DEFINITION 2.2. The C^*-algebra $C(S^3_{pq})$ is the enveloping C^*-algebra of $\mathcal{O}(S^3_{pq})$ for the sup-norm over all *bounded* $*$-representations.

Observe that it follows from Theorem 2.1 that $\|\rho(a)\| = 1 = \|\rho(b)\|$ for any irreducible representation, so that $\|a\| = 1 = \|b\|$. On the other hand, using the unbounded representations of the quantum disc, which can be obtained from the unbounded representations of the oscillator algebra given in [KS97, Section 5.2.6], one obtains immediately two families of irreducible unbounded $*$-representations $\rho_{1\theta\gamma}$ and $\rho_{2\theta\gamma}$ of $\mathcal{O}(S^3_{pq})$. On a Hilbert space with an orthonormal basis $\{e_\mu\}_{\mu\in\mathbb{Z}}$, they are given by the formulas

$$\rho_{1\theta\gamma}(a)e_\mu = \sqrt{1+q^{\mu+1}\gamma}\, e_{\mu+1}, \quad \rho_{1\theta\gamma}(a^*)e_\mu = \sqrt{1+q^{\mu}\gamma}\, e_{\mu-1}, \quad (2.19)$$

$$\rho_{1\theta\gamma}(b)e_\mu = \mathrm{e}^{i\theta}e_\mu, \quad \rho_{1\theta\gamma}(b^*)e_\mu = \mathrm{e}^{-i\theta}e_\mu, \quad (2.20)$$

$$\gamma \in (q,1],\ \theta \in [0,2\pi),$$

and analogous expressions for $\rho_{2\theta\gamma}$, with a and b exchanged and q replaced by p.

To end this section, let us determine a vector space basis for $\mathcal{O}(S^3_{pq})$.

THEOREM 2.3. *Let $\mu, \nu \in \mathbb{Z}$. Put $a_\mu = a^\mu$ if $\mu \geqslant 0$ and $a_\mu = a^{*|\mu|}$ if $\mu < 0$. Define b_ν in the same manner. Then the elements*

$$\left\{a_\mu(1-aa^*)^m(1-bb^*)^n b_\nu \mid \mu,\nu \in \mathbb{Z},\ m,n \in \mathbb{N},\ mn=0\right\} \tag{2.21}$$

form a vector space basis of $\mathcal{O}(S^3_{pq})$. Furthermore, $\mathcal{O}(S^3_{pq})$ is faithfully embedded into $C(S^3_{pq})$.

Proof. At first we show that the elements (2.21) linearly span $\mathcal{O}(S^3_{pq})$. To this end, consider a word W in $\{a, a^*, b, b^*\}$. Using relations (1.25) we can rewrite W as W_aW_b, where W_a is a word in $\{a, a^*\}$ and W_b is a word in $\{b, b^*\}$. Proceeding by induction on the length of W_a and using the relation

$$(1-aa^*)a = qa(1-aa^*), \tag{2.22}$$

one can show that W_a is a linear combination of polynomials of the form $a_\mu(1-aa^*)^m$. Likewise, using

$$(1-bb^*)b = pb(1-bb^*), \tag{2.23}$$

we can prove that W_b is a linear combination of polynomials of the form $(1-bb^*)^n b_\nu$. Finally, since $a_\mu(1-aa^*)^m(1-bb^*)^n b_\nu = 0$ unless $mn = 0$ (see (1.26)), we can write any monomial W as a linear combination of elements of (2.21). Hence they span $\mathcal{O}(S^3_{pq})$.

Now, let us assume that x is a finite linear combination of elements of (2.21), i.e.,

$$x = \sum_{\mu,\nu\in\mathbb{Z}} \sum_{\substack{m,n\in\mathbb{N},\\ mn=0}} x_{\mu,m,n,\nu} a_\mu(1-aa^*)^m(1-bb^*)^n b_\nu. \tag{2.24}$$

Since x is annihilated in all representations if and only if its image is zero in the enveloping C^*-algebra, to complete the proof of the theorem, it suffices to show the implication

$$\begin{gathered}\forall \text{ bounded } *\text{-representations } \rho:\ \rho(x) = 0\\ \Downarrow\\ \forall\, \mu,\nu \in \mathbb{Z},\ m,n \in \mathbb{N},\ mn=0:\ x_{\mu,m,n,\nu} = 0.\end{gathered} \tag{2.25}$$

As only finitely many of the coefficients $x_{\mu,m,n,\nu}$ are non-zero, there exists $k \in \mathbb{N}$ such that $x_{\mu,m,n,\nu} = 0$ for all $\nu < -k$. Let us choose such a $k \in \mathbb{N}$ and assume that $\rho_{1\theta}(x) = 0$. Then it follows from (2.1) that

$$\rho_{1\theta}(x)e_k = \sum_{\substack{\mu,\nu\in\mathbb{Z},\\ \nu\geqslant -k}} \sum_{n\in\mathbb{N}} x_{\mu,0,n,\nu} \mathrm{e}^{i\mu\theta}(p^{k+\nu})^n \Lambda_{\nu k} e_{k+\nu} = 0, \tag{2.26}$$

where

$$\Lambda_{\nu k} = \begin{cases} \sqrt{(1-p^k)\cdots(1-p^{k+\nu+1})}, & -k \leqslant \nu < 0, \\ 1, & \nu = 0, \\ \sqrt{(1-p^{k+1})\cdots(1-p^{k+\nu})}, & \nu > 0. \end{cases} \tag{2.27}$$

Since the vectors e_i are linearly independent and the $\Lambda_{\nu k}$'s are always nonzero, for any $\nu \in \mathbb{Z}$ we have

$$\sum_{\mu\in\mathbb{Z}} \sum_{n\in\mathbb{N}} x_{\mu,0,n,\nu} \mathrm{e}^{i\mu\theta} (p^{k+\nu})^n = 0. \tag{2.28}$$

Next, as this equation is valid for all $\theta \in [0, 2\pi)$, by the uniqueness of the Fourier coefficients, we can conclude that

$$\sum_{n\in\mathbb{N}} x_{\mu,0,n,\nu} (p^{k+\nu})^n = 0 \tag{2.29}$$

for any $\mu, \nu \in \mathbb{Z}$. Since $p \notin \{-1, 0, 1\}$ and there are infinitely many k's with the property $x_{\mu,m,n,\nu} = 0$ for $\nu < -k$, the above polynomial in $p^{k+\nu}$ vanishes at infinitely many points, so that all its coefficients must be zero. Thus we have shown that $x_{\mu,0,n,\nu} = 0$ for all $\mu, \nu \in \mathbb{Z}$, $n \in \mathbb{N}$. The vanishing of the remaining coefficients $x_{\mu,m,0,\nu}$ can be proved using $\rho_{2\theta}$ instead of $\rho_{1\theta}$. □

3. *K*-theory of $C(S^3_{pq})$

The purpose of this section is to show that the topological K-groups of S^3_{pq} coincide with the K-groups of the classical 3-sphere.

THEOREM 3.1. *The K-groups of the C*-algebra* $C(S^3_{pq})$ *are* $K_0(C(S^3_{pq})) \cong \mathbb{Z} \cong K_1(C(S^3_{pq}))$.

Proof. Let $\mathcal{T}$ denote the Toeplitz algebra, and s its generating proper isometry (the unilateral shift). The ideal of $\mathcal{T}$ generated by $1 - ss^*$ is isomorphic with the C^*-algebra $\mathcal{K}$ of compact operators on a separable Hilbert space. We denote by π the canonical surjection from $\mathcal{T} \otimes \mathcal{T}$ onto $(\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K})$.

LEMMA 3.2. *The* C^**-algebras* $C(S^3_{pq})$ *and* $(\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K})$ *are isomorphic.*

Proof. The universality of $C(S^3_{pq})$ for the relations (1.23)–(1.26) and the representation formulas (2.1)–(2.2) imply that there exists a C^*-algebra homomorphism

$\alpha\colon\ C(S^3_{pq}) \to (\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K})$ such that

$$\begin{aligned}\alpha(a) &= \pi(\rho_{2\theta}(a) \otimes 1)\\ &= \pi\left(\sum_{n=0}^{\infty}\big(\sqrt{1-q^{n+1}} - \sqrt{1-q^n}\big)s^{n+1}s^{*n} \otimes 1\right),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\alpha(b) &= \pi(1 \otimes \rho_{1\theta}(b))\\ &= \pi\left(\sum_{n=0}^{\infty}\big(\sqrt{1-p^{n+1}} - \sqrt{1-p^n}\big)1 \otimes s^{n+1}s^{*n}\right).\end{aligned} \tag{3.2}$$

(We abuse notation by denoting with the same symbol representations of $\mathcal{O}(S^3_{pq})$ and $C(S^3_{pq})$.)

Let us now construct the inverse of α. By (1.23), we have $a^*a = 1-q+qaa^* \geqslant 1-q$, whence a^*a is invertible. Therefore, so is $|a| = \sqrt{a^*a}$. Likewise, (1.24) implies that $|b|$ is invertible. As both $a|a|^{-1}$ and $b|b|^{-1}$ are isometries, we can define a C^*-algebra homomorphism $\tilde{\beta}\colon\ \mathcal{T} \otimes \mathcal{T} \to C(S^3_{pq})$ by

$$\tilde{\beta}(s \otimes 1) = a|a|^{-1}, \qquad \tilde{\beta}(1 \otimes s) = b|b|^{-1}. \tag{3.3}$$

It follows from (1.26) that

$$\begin{aligned}&(1-a|a|^{-2}a^*)(1-b|b|^{-2}b^*)\\ &\quad= (1-a|a|^{-2}a^*)(1-aa^*)(1-bb^*)(1-b|b|^{-2}b^*)\\ &\quad= 0.\end{aligned} \tag{3.4}$$

Thus $\tilde{\beta}((1-ss^*) \otimes (1-ss^*)) = (1-a|a|^{-2}a^*)(1-b|b|^{-2}b^*) = 0$. Since the smallest ideal of $\mathcal{T} \otimes \mathcal{T}$ containing $(1-ss^*)\otimes(1-ss^*)$ coincides with $\mathcal{K}\otimes\mathcal{K}$, we have $\tilde{\beta}(\mathcal{K} \otimes \mathcal{K}) = \{0\}$, and consequently $\tilde{\beta}$ induces a C^*-algebra homomorphism $\beta\colon\ (\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K}) \to C(S^3_{pq})$.

It is straightforward to verify on the generators that $\alpha \circ \beta = \mathrm{id}$:

$$(\alpha \circ \beta)(\pi(s \otimes 1)) = \alpha(a|a|^{-1}) = \pi(\rho_{2\theta}(a|a|^{-1}) \otimes 1) = \pi(s \otimes 1). \tag{3.5}$$

(The case $1 \otimes s$ is analogous.) For the identity $\beta \circ \alpha = \mathrm{id}$, note that $\rho_{2\theta}$ and $\rho_{1\theta}$ are injective on the C^*-subalgebras C_a and C_b generated by a and b, respectively. Indeed, since $C(D_r)$ is the universal C^*-algebra for the relation $z^*z - rzz^* = 1-r$, $r \in (0, 1)$, we have natural C^*-algebra epimorphisms $\pi_a\colon\ C(D_q) \to C_a$ and $\pi_b\colon\ C(D_p) \to C_b$. On the other hand, $\rho_{2\theta} \circ \pi_a$ and $\rho_{1\theta} \circ \pi_b$ coincide with the faithful representation π^I [KL93, p. 14], so that $\rho_{2\theta}|_{C_a}$ and $\rho_{1\theta}|_{C_b}$ are injective. On the other hand,

$$\begin{aligned}\rho_{2\theta}\big((\beta \circ \alpha)(a)\big) &= \rho_{2\theta}\big(\beta(\pi(\rho_{2\theta}(a) \otimes 1))\big)\\ &= \sum_{n=0}^{\infty}(\sqrt{1-q^{n+1}} - \sqrt{1-q^n})\rho_{2\theta}\big(\tilde{\beta}(s^{n+1}s^{*n} \otimes 1)\big)\end{aligned}$$

$$= \sum_{n=0}^{\infty} (\sqrt{1-q^{n+1}} - \sqrt{1-q^n}) s^{n+1} s^{*n}$$
$$= \rho_{2\theta}(a). \tag{3.6}$$

Similarly, $\rho_{1\theta}((\beta \circ \alpha)(b) - b) = 0$. Consequently, $\beta \circ \alpha = \mathrm{id}$. □

As shown in the foregoing lemma, there exists the following short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{K} \xrightarrow{j} \mathcal{T} \otimes \mathcal{T} \xrightarrow{\pi} C(S^3_{pq}) \longrightarrow 0, \tag{3.7}$$

where j is the inclusion map. Applying the Künneth formula and remembering $K_0(\mathcal{K}) \cong K_0(\mathcal{T}) \cong \mathbb{Z}$, $K_1(\mathcal{K}) \cong K_1(\mathcal{T}) \cong 0$, reduces the six-term exact sequence corresponding to (3.7) to

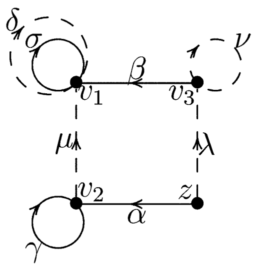
$$0 \longrightarrow K_1(C(S^3_{pq})) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{j_*} \mathbb{Z} \xrightarrow{\pi_*} K_0(C(S^3_{pq})) \longrightarrow 0. \tag{3.8}$$

Due to the exactness of this sequence, to end the proof it suffices to show that $j_* = 0$. Note that j_* goes from $K_0(\mathcal{K} \otimes \mathcal{K})$ to $K_0(\mathcal{T} \otimes \mathcal{T})$. Put $p = 1 - ss^*$. Then $p \otimes p \in \mathcal{K} \otimes \mathcal{K}$ is a minimal projection generating $K_0(\mathcal{K} \otimes \mathcal{K})$, and we have $j_*([p \otimes p]) = [p \otimes p] \in K_0(\mathcal{T} \otimes \mathcal{T})$. In $K_0(\mathcal{T} \otimes \mathcal{T})$ we obviously have $[p \otimes p] + [(1-p) \otimes p] = [1 \otimes p]$. On the other hand, $(s \otimes p)(s \otimes p)^* = (1-p) \otimes p$ and $(s \otimes p)^*(s \otimes p) = 1 \otimes p$, so that $(1-p) \otimes p$ and $1 \otimes p$ are Murray–von Neumann equivalent. Hence $[1 \otimes p] = [(1-p) \otimes p]$, which implies that $j_* = 0$. □

It was observed in [HS02, Proposition 2.1] that the C^*-algebra $C(\mathrm{SU}_q(2))$ of the quantum 3-sphere of Woronowicz can be realized as the Cuntz–Krieger algebra corresponding to a certain finite directed graph. Similarly, $C(S^3_{pq})$ can be identified with the algebra of a higher rank graph. We sketch this identification below, referring the interested reader to [KP00] and [RSY03] for details on higher rank graphs.

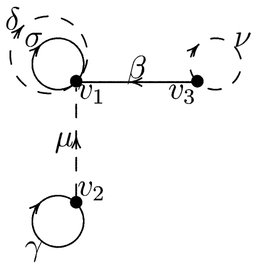
Remark 3.3.★ It is well known that the Toeplitz algebra $\mathcal{T}$ is isomorphic to the C^*-algebra of the 1-graph Λ with vertices $\Lambda^0 = \{v_1, v_2\}$ and oriented edges $\Lambda^1 = \{e_1, e_2\}$ such that $r(e_1) = s(e_1) = r(e_2) = v_1$ and $s(e_2) = v_2$. (We use the edge-orientation convention of [KP00].) The product $\Lambda \times \Lambda$ is a graph of rank 2 as in [RSY03, Definitions 2.1 and Definition 3.9], and the argument of [KP00, Corollary 3.5(iv)] combined with [RSY03, Theorem 4.1] shows that $C^*(\Lambda \times \Lambda) \cong C^*(\Lambda) \otimes C^*(\Lambda) \cong \mathcal{T} \otimes \mathcal{T}$. The Cartesian product 2-graph $\Lambda \times \Lambda$ has the 1-skeleton (see [RSY03, §2])

★ The following argument is due to Aidan Sims.



(3.9)

Here solid edges have degree (1, 0) and dashed edges have degree (0, 1). Let $\{s_\tau : \tau \in \Lambda \times \Lambda\}$ denote the universal Cuntz–Krieger $(\Lambda \times \Lambda)$-family. Now $\{z\}$ is a saturated hereditary subset of $(\Lambda \times \Lambda)^0$ in the sense of [RSY03, §5]. If I is the closed ideal in $C^*(\Lambda \times \Lambda)$ generated by S_z, then [RSY03, Theorem 5.2] implies that $C^*(\Lambda \times \Lambda)/I$ is isomorphic to the C^*-algebra of the quotient 2-graph, whose 1-skeleton is



(3.10)

It is not difficult to verify that $I \cong \mathcal{K} \otimes \mathcal{K}$. Hence $(\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K})$ is the C^*-algebra of the above 2-graph. The former coincides with $C(S^3_{pq})$ by Lemma 3.2.

To end with, let us compare the geometry behind our computation and the corresponding calculations in [MNW90] and [M-K91a]. As can be expected, in all three cases the K-groups are obtained from the 6-term exact sequence of K-theory. Also in all three cases, they coincide with their classical counterparts. However, the source of the 6-term exact sequence is each time different: [MNW90, (0.2)], [M-K91a, p. 355], (3.7). Considering the geometric meaning of the employed C^*-algebras (e.g., $\mathcal{K}$ corresponds to $C_0(\mathbb{R}^2)$, $\mathcal{T}$ to $C(D)$, where D is the unit disc in $\mathbb{R}^2$; see the introduction in [HMS03a]), we obtain the corresponding classical constructions:

$$0 \longrightarrow C_0(\mathbb{R}^2 \times S^1) \longrightarrow C(S^3) \longrightarrow C(S^1) \longrightarrow 0 \quad \text{(exact sequence)}, \tag{3.11}$$

$$\begin{array}{ccc} C(S^3) & \longrightarrow & C(S^1 \times D) \\ \downarrow & & \downarrow \\ C(D \times S^1) & \longrightarrow & C(T^2) \end{array} \quad \text{(pullback diagram)}, \tag{3.12}$$

$$0 \longrightarrow C_0(\mathbb{R}^4) \longrightarrow C(D \times D) \longrightarrow C(S^3) \longrightarrow 0 \quad \text{(exact sequence)}. \tag{3.13}$$

The first sequence means that removing S^1 from S^3 leaves a boundary-less solid torus. Think of $S^3 = \{(c_1, c_2) \in D \times D \mid |c_1|^2 + |c_2|^2 = 1\}$ as a field of 2-tori over the internal points of [0, 1] bounded by circles at the endpoints (e.g., take $|c_1|^2 \in [0, 1]$ as the interval parameter). Remove S^1 given by $c_1 = 0$ ($|c_2| = 1$). What remains is S^1 times a field of circles over the internal points of (0, 1] shrinking to a point at 1. The latter is an open disc (homeomorphic with $\mathbb{R}^2$). The second sequence depicts gluing of two solid tori along their boundaries, which is known as a Heegaard splitting of S^3. The corresponding six-term exact sequence is the Mayer-Vietoris sequence of K-theory. Finally, to visualize the last sequence, recall that we can think of S^3 as the set $\{(z_1, z_2) \in D \times D \mid (1 - |z_1|)(1 - |z_2|) = 0\}$ (see Proposition 1.7 and divide by $(1 + |z_1|)(1 + |z_2|)$). Removing S^3 from $D \times D$ leaves all points $(z_1, z_2) \in D \times D$ such that $(1 - |z_1|)(1 - |z_2|) \neq 0$, which is precisely the Cartesian product of two open discs (homeomorphic with $\mathbb{R}^4$).

4. Hopf–Galois Aspects of $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$

Our goal now is to prove that the extension $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ is Hopf–Galois, equivariantly projective, and non-cleft. We begin with the following:

LEMMA 4.1. *The locally trivial* $\mathcal{O}(\mathrm{U}(1))$*-extension* $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ *is Hopf–Galois.*

Proof. Note first that, since $\mathcal{O}(\mathrm{U}(1))$ is cosemisimple, the injectivity of the canonical map (see Subsection 1.3)

$$\mathrm{can}\colon\ \mathcal{O}(S^3_{pq}) \otimes_B \mathcal{O}(S^3_{pq}) \longrightarrow \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(\mathrm{U}(1)) \tag{4.1}$$

follows from its surjectivity. Indeed, since there is a Haar functional f_H on $\mathcal{O}(\mathrm{U}(1))$, we get the total integral of Doi by composing it with the unit map: $j := \eta \circ f_H\colon \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(S^3_{pq})$. Therefore, we can apply Remark 3.3 and Theorem I of [S-HJ90]. On the other hand, as explained in Subsection 1.3, to prove the surjectivity of can, it suffices to show that both $1 \otimes u$ and $1 \otimes u^*$ are in its image. (Here u and u^* are the generators of $\mathcal{O}(\mathrm{U}(1))$.) Taking advantage of (1.27), the relations (1.23), (1.25), and the key relation (1.26), we obtain

$$\begin{aligned}
&\mathrm{can}(a^* \otimes_B a + qb \otimes_B b^*(1 - aa^*))\\
&\quad = (a^*a + qbb^* - qaa^*bb^*) \otimes u\\
&\quad = (qaa^* + 1 - q + qbb^* - qaa^*bb^*) \otimes u\\
&\quad = 1 \otimes u.
\end{aligned} \tag{4.2}$$

Analogously, using (1.24) instead of (1.23), we obtain

$$\begin{aligned}
&\mathrm{can}(b^* \otimes_B b + pa \otimes_B a^*(1 - bb^*))\\
&\quad = (b^*b + paa^* - paa^*bb^*) \otimes u^*
\end{aligned}$$

$$= (pbb^* + 1 - p + paa^* - paa^*bb^*) \otimes u^*$$
$$= 1 \otimes u^*. \tag{4.3}$$

Thus we have shown that can is bijective, as needed. □

Our next step is to construct a strong connection ℓ (see Subsection 1.3).

LEMMA 4.2. *The Hopf–Galois $\mathcal{O}(\mathrm{U}(1))$-extension $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ is equivariantly projective.*

Proof. Since, due to [BH04, Lemma 2.2], the Hopf–Galois extension is equivariantly projective if and only if there exists a strong connection, we prove the assertion of the lemma by constructing a strong connection ℓ. We define a linear map $\ell\colon \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(S^3_{pq})$, $\ell(h) = h^{\langle 1\rangle} \otimes h^{\langle 2\rangle}$ (summation understood), by giving its values on the basis elements u^μ, $\mu \in \mathbb{Z}$:

$$\ell(1) = 1 \otimes 1, \tag{4.4}$$

$$\begin{aligned} \ell(u) &= a^* \otimes a + qb(1 - aa^*) \otimes b^*,\\ \ell(u^*) &= b^* \otimes b + pa(1 - bb^*) \otimes a^*, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \ell(u^\mu) &= u^{\langle 1\rangle} \ell(u^{\mu-1}) u^{\langle 2\rangle}, \quad \mu > 0,\\ \ell(u^\mu) &= u^{*\langle 1\rangle} \ell(u^{*(|\mu|-1)}) u^{*\langle 2\rangle}, \quad \mu < 0. \end{aligned} \tag{4.6}$$

Denote by $\widetilde{\mathrm{can}}$ the lifting $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R)$ of the canonical map can. (Here m stands for the multiplication map.) To show that $\widetilde{\mathrm{can}} \circ \ell = 1 \otimes \mathrm{id}$, we first note that (4.2) and (4.3) entail that $(\widetilde{\mathrm{can}} \circ \ell)(u) = 1 \otimes u$ and $(\widetilde{\mathrm{can}} \circ \ell)(u^*) = 1 \otimes u^*$. Assume now that $(\widetilde{\mathrm{can}} \circ \ell)(u^k) = 1 \otimes u^k$. Then

$$\begin{aligned} (\widetilde{\mathrm{can}} \circ \ell)(u^{k+1}) &= \widetilde{\mathrm{can}}(u^{\langle 1\rangle} \ell(u^k) u^{\langle 2\rangle})\\ &= u^{\langle 1\rangle}((\widetilde{\mathrm{can}} \circ \ell)(u^k)) \Delta_R(u^{\langle 2\rangle})\\ &= u^{\langle 1\rangle}(1 \otimes u^k) \Delta_R(u^{\langle 2\rangle})\\ &= 1 \otimes u^{k+1}. \end{aligned} \tag{4.7}$$

The case $(\widetilde{\mathrm{can}} \circ \ell)(u^{*k}) = 1 \otimes u^{*k}$ can be handled in the same way. Therefore, it follows by induction that $\widetilde{\mathrm{can}} \circ \ell = 1 \otimes \mathrm{id}$. Now, let us consider the following diagram (cf. [DGH01, (1.25)]:

$$\begin{array}{ccccc} & & \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(S^3_{pq}) & & \\ & \nearrow^{\ell} & \downarrow{\scriptstyle \pi_B} & \searrow^{\widetilde{\mathrm{can}}} & \\ \mathcal{O}(\mathrm{U}(1)) & \xrightarrow{T} & \mathcal{O}(S^3_{pq}) \otimes_{\mathcal{O}(S^2_{pq})} \mathcal{O}(S^3_{pq}) & \xrightarrow{\mathrm{can}} & \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(\mathrm{U}(1)). \end{array}$$

The right triangle part of the diagram commutes by construction, and we have already shown that the big triangle commutes. Thus the commutativity of the left triangle follows from the injectivity of can. This means that ℓ is a lifting of the translation map. It is by construction unital, so that it remains to show its bicolinearity, which is again done inductively.

First, it is immediate to see the equality $(\ell \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta_R) \circ \ell$ on the generator u. We can write this as

$$u^{\langle 1\rangle} \otimes u^{\langle 2\rangle} \otimes u = u^{\langle 1\rangle} \otimes \Delta_R(u^{\langle 2\rangle}). \tag{4.8}$$

Next, assume that $((\ell \otimes \mathrm{id}) \circ \Delta)(u^k) = ((\mathrm{id} \otimes \Delta_R) \circ \ell)(u^k)$ for some $k > 0$. Then, taking advantage of (4.8), we obtain

$$\begin{aligned}((\mathrm{id} \otimes \Delta_R) \circ \ell)(u^{k+1}) &= (\mathrm{id} \otimes \Delta_R)(u^{\langle 1\rangle}\ell(u^k)u^{\langle 2\rangle}) \\ &= u^{\langle 1\rangle}((\mathrm{id} \otimes \Delta_R)(\ell(u^k)))\Delta_R(u^{\langle 2\rangle}) \\ &= u^{\langle 1\rangle}(\ell \otimes \mathrm{id})(\Delta(u^k))\Delta_R(u^{\langle 2\rangle}) \\ &= u^{\langle 1\rangle}(\ell(u^k) \otimes u^k)\Delta_R(u^{\langle 2\rangle}) \\ &= u^{\langle 1\rangle}\ell(u^k)u^{\langle 2\rangle} \otimes u^{k+1} \\ &= \ell(u^{k+1}) \otimes u^{k+1} \\ &= ((\ell \otimes \mathrm{id}) \circ \Delta)(u^{k+1}). \end{aligned} \tag{4.9}$$

A similar argument can be made with u^* in place of u. This proves the right colinearity of ℓ due to the fact that $\{u^\mu\}_{\mu\in\mathbb{Z}}$ is a basis of $\mathcal{O}(\mathrm{U}(1))$. The proof of the left colinearity (see(1.38)–(1.39)) is fully analogous, now using

$$\begin{aligned} &u \otimes u^{\langle 1\rangle} \otimes u^{\langle 2\rangle} = \Delta_L(u^{\langle 1\rangle}) \otimes u^{\langle 2\rangle}, \\ &\Delta_L := (S^{-1} \otimes \mathrm{id}) \circ (\mathrm{flip}) \circ \Delta_R, \end{aligned} \tag{4.10}$$

and its $*$-version. Thus we have shown that ℓ is a strong connection. □

Remark 4.3. Galois coactions are algebraic incarnations of principal actions in classical geometry. There is a proposal for the corresponding principality or Galois condition for C^*-algebras [E-DA00]. In our situation, Definition 2.4 of [E-DA00] defining principal coactions on C^*-algebras reduces to requiring that $C(S^3_{pq})\Delta_R(C(S^3_{pq}))$ be norm dense in $C(S^3_{pq}) \otimes C(\mathrm{U}(1))$. Since the powers of u span a dense subspace of $C(\mathrm{U}(1))$, it suffices to note that $1 \otimes u^\mu \in C(S^3_{pq})\Delta_R(C(S^3_{pq}))$. This follows from the proof of the foregoing theorem, so that the coaction $\Delta_R\colon C(S^3_{pq}) \to C(S^3_{pq}) \otimes C(\mathrm{U}(1))$ is principal in the sense of [E-DA00].

We now provide an explicit formula for ℓ. Put $x = a^* \otimes a$ and $y = qb(1 - aa^*) \otimes b^*$, and consider them as elements of $\mathcal{O}(S^3_{pq})^{op} \otimes \mathcal{O}(S^3_{pq})$, where the superscript *op* indicates the opposite algebra. Then the formula for $\ell(u^k)$ reads $\ell(u^k) = (x + y)^k$. Due to (1.23), $yx = qxy$, so that we can use the formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \qquad (4.11)$$
$$\binom{n}{k}_q = \frac{(q-1)\cdots(q^n-1)}{(q-1)\cdots(q^k-1)(q-1)\cdots(q^{n-k}-1)}.$$

Consequently, we obtain

$$\ell(u^n) = \sum_{k=0}^{n} \binom{n}{k}_q q^{n-k}(1-aa^*)^{n-k} a^{*^k} b^{n-k} \otimes a^k b^{*^{n-k}}. \qquad (4.12)$$

The formula for $\ell(u^{*^n})$ can be derived exchanging the roles of a and b, and q and p. Notice also that, since $\widetilde{\mathrm{can}} \circ \ell = 1 \otimes \mathrm{id}$, we have

$$m \circ \ell = (\mathrm{id} \otimes \varepsilon) \circ \widetilde{\mathrm{can}} \circ \ell = \varepsilon. \qquad (4.13)$$

Thus (4.12) entails the following identity in $\mathcal{O}(S^3_{pq})$:

$$\sum_{k=0}^{n} \binom{n}{k}_q q^{n-k}(1-aa^*)^{n-k} a^{*^k} b^{n-k} a^k b^{*^{n-k}} = 1. \qquad (4.14)$$

A similar identity follows from an explicit formula for $\ell(u^{*k})$.

Next, consider the 1-dimensional corepresentations of $\mathcal{O}(\mathrm{U}(1))$, $\rho_\mu(1) = 1 \otimes u^{-\mu}$, $\mu \in \mathbb{Z}$. We can identify $\mathrm{Hom}_{\rho_\mu}(\mathbb{C}, \mathcal{O}(S^3_{pq}))$ with $\mathcal{O}(S^3_{pq})_\mu := \{p \in \mathcal{O}(S^3_{pq}) \mid \Delta_R(p) = p \otimes u^{-\mu}\}$. Since the powers of u form a basis of $\mathcal{O}(\mathrm{U}(1))$, we have the direct sum decomposition $\mathcal{O}(S^3_{pq}) = \bigoplus_{\mu\in\mathbb{Z}} \mathcal{O}(S^3_{pq})_\mu$ as $\mathcal{O}(S^2_{pq})$-bimodules. According to the general result referred to in Subsection 1.3, strong connections determine idempotent matrices E_μ of the associated modules ($\mathcal{O}(S^3_{pq})_\mu \cong \mathcal{O}(S^2_{pq})^{\mathrm{size} E_\mu} E_\mu$ as $\mathcal{O}(S^2_{pq})$-modules). Due to Theorem 2.3, one can combine the afore-mentioned idempotent formula of Subsection 1.3 and formula (4.12), to find explicitly the idempotents $E_{-n} = R^{\mathrm{T}}_{-n} L_{-n}$, $n \in \mathbb{N}$, where

$$R_{-n} = (b^{*n}, ab^{*n-1}, \ldots, a^n), \qquad (4.15)$$

$$L_{-n} = \left(q^n(1-aa^*)^n b^n\,,\ \binom{n}{1}_q q^{n-1}(1-aa^*)^{n-1} a^* b^{n-1}\,,\ \ldots\,,\ a^{*n}\right). \qquad (4.16)$$

The idempotents E_n are given by a similar formula, with a and b exchanged and q replaced by p. For $\mu = -1$, we have

$$E_{-1} := \begin{pmatrix} b^* \\ a \end{pmatrix} \begin{pmatrix} qb(1-aa^*) & a^* \end{pmatrix} = \begin{pmatrix} q(1-aa^*)b^*b & a^*b^* \\ qa(1-aa^*)b & aa^* \end{pmatrix}. \qquad (4.17)$$

Now we want to prove that the Hopf–Galois extension $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ is not cleft. We do this by showing that the K_0-class of the idempotent E_{-1} is not trivial.

This, in turn, we prove by computing an appropriate invariant of K-theory. The invariant has a very simple form, namely it is the Chern–Connes pairing of a trace (0-cyclic cocycle) with E_{-1}. It is known [HMS03b, Lemma 3.2] that there exists a trace on $\mathcal{O}(S^2_{pq})$ given by

$$\mathrm{tr}(f) := \mathrm{Tr}(\rho_2(f) - \rho_1(f)). \tag{4.18}$$

Here ρ_1, ρ_2: $\mathcal{O}(S^2_{pq}) \to \mathcal{B}(\mathcal{H})$ are the two infinite-dimensional representations given by (1.21)–(1.22), and Tr is the operator trace. The pairing of tr and E_μ has been computed in [HMS03b, Theorem 3.3] (see [BHMS] for details) for any $\mu \in \mathbb{Z}$. Since the special case $\mu = -1$ computation is straightforward, we enclose it here for the convenience of the reader.

LEMMA 4.4. *Let* $\langle\ ,\ \rangle$ *denote the pairing between the cyclic cohomology* $HC^{even}(\mathcal{O}(S^2_{pq}))$ *and* $K_0(\mathcal{O}(S^2_{pq}))$. *Then* $\langle \mathrm{tr}, [E_{-1}]\rangle = -1$.

Proof. Using (4.17), (1.24) and (1.26), we have

$$\begin{aligned}\langle \mathrm{tr}, [E_{-1}]\rangle &= \mathrm{tr}(\mathrm{Tr}_{M_2}(E_{-1}))\\ &= \mathrm{tr}(aa^* + q(1 - aa^*)b^*b)\\ &= \mathrm{tr}(aa^* + q(1 - aa^*)(p(bb^* - 1) + 1))\\ &= \mathrm{tr}(q + (1 - q)aa^*).\end{aligned} \tag{4.19}$$

Taking advantage of (1.29), this can be expressed in terms of f_0 and f_1. Indeed, using again the commutation relations, we get

$$aa^* = 1 - bb^* + ab(ab)^* = 1 - f_0 + f_1 f_1^* \quad \text{(injection } \iota \text{ suppressed).} \tag{4.20}$$

It is immediate from (4.18) that $\mathrm{tr}(1) = 0$, and it follows from (1.21)–(1.22) that $\mathrm{tr}(f_0 - f_1 f_1^*) = \frac{1}{1-q}$. This yields

$$\mathrm{tr}(q + (1 - q)aa^*) = (1 - q)\mathrm{tr}(aa^*) = (q - 1)\mathrm{tr}(f_0 - f_1 f_1^*) = -1, \tag{4.21}$$

as claimed. □

Since every free module can be represented in K_0 by the identity matrix, the pairing between tr and the K_0-class of any free module always yields zero. Thus the left module $\mathcal{O}(S^3_{pq})_{-1} \cong \mathcal{O}(S^2_{pq})^2 E_{-1}$ is not (stably) free. Now, reasoning as in [HM99, Section 4], we can conclude that the $\mathcal{O}(\mathrm{U}(1))$-extension $\mathcal{O}(S^2_{pq})_{-1} \subseteq \mathcal{O}(S^3_{pq})$ is non-cleft. Combining this with Lemma 4.1 and Lemma 4.2, we obtain:

THEOREM 4.5. *The locally trivial* $\mathcal{O}(\mathrm{U}(1))$*-extension* $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ *is an equivariantly projective non-cleft Hopf–Galois extension.*

Remark 4.6. Note that in [CM02, Proposition 20] it was only shown that $\mathcal{O}(S^3_{pq})$ is not isomorphic to the tensor product $\mathcal{O}(S^2_{pq}) \otimes \mathcal{O}(\mathrm{U}(1))$ (non-triviality). Here we prove that $\mathcal{O}(S^3_{pq})$ is not a crossed product $\mathcal{O}(S^2_{pq}) \rtimes \mathcal{O}(\mathrm{U}(1))$. (For a discussion concerning "non-trivial" versus "non-cleft", see the end of Section 4 in [DHS99].)

Acknowledgements

The authors are indebted to A. Sims and A. Sitarz for their help with Remark 3.3 and Proposition 1.7, respectively, and to T. Brzezinski and P. Schauenburg for their help with the proofreading. P.M.H. has been supported by a Marie Curie Fellowship of the European Community under the contract number HPMF-CT-2000-00523. P.M.H. is also grateful to the Graduiertenkolleg "Quantenfeldtheorie" and the Naturwissenschaftlich-Theoretisches Zentrum der Universität Leipzig, and to the University of Newcastle, Australia, for their hospitality and financing his visit to Leipzig and Newcastle, respectively. R.M. has been supported by the Deutsche Forschungsgemeinschaft, and he likes to thank H.-D. Doebner for providing excellent working conditions at the TU Clausthal. R.M. is also grateful for hospitality and financial support to the Universität München. R.M. and W.S. thank the European Commission for covering the expenses of their stay in Warsaw during the Banach Centre school/conference "Noncommutative Geometry and Quantum Groups." W.S. also would like to thank the Research Management Committee of the University of Newcastle and the Max-Planck-Institut für Mathematik Bonn for their support. All three authors are grateful to the Mathematisches Forschungsinstitut Oberwolfach for support via its Research in Pairs programme.

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