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GELFAND CETLIN SYSTEM
and the
QUANTIZATION of THE SYMPLECTIC GROUPOID

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SUMMARY of THE TALK

- Motivation: the symplectic groupoid approach to the quantization of Poisson manifolds
- Bihamiltonian systems in the quantization of symplectic gpd
- A bihamiltonian system defined on compact hermitian symmetric spaces
- Equivalence with Gelfand-Cetlin integrable model

SYMPLECTIC GROUPOIDS

(M, π) $\quad \pi \in \Gamma(\Lambda^2 TM)$ Poisson mfld

$[\pi, \pi] = 0$ Jacobi identity

$$(M,\pi) \text{ integrable} \iff \Big(\underset{\substack{\downarrow\downarrow \\ M}}{\overset{\text{Lie groupoid}}{\mathcal{G}(M,\pi)}},\ \overset{\text{symplectic struct.}}{\Omega}\Big) \text{ symplectic groupoid}$$

Definition: A Lie gpd $\mathcal{G}$ is a symplectic gpd if

graph of multiplication $\subset \bar{\mathcal{G}} \times \bar{\mathcal{G}} \times \mathcal{G}$ is lagrangian

Remark:

$$\begin{array}{c} \mathcal{G}(M,\pi) \\ \downarrow\downarrow \\ M \end{array} \xrightarrow[\text{quantization}]{\text{compatible}} \begin{array}{l} \mathcal{H} \equiv \text{space of states} \\ m_{\mathcal{G}} \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H} \end{array} \equiv \text{associative algebra}$$

$(\mathcal{H}, m_{\mathcal{G}})$ should be thought as a quantization of (M, π)

MULTIPLICATIVE INTEGRABILITY

$F = \{ F_i \in C^\infty(\mathcal{G}),\ \{F_i, F_j\} = 0 \}$ integrable model

"polarization" $\mathcal{G} \longrightarrow \mathcal{G}_F \equiv$ contour level set of F

F is multiplicative if $\mathcal{G}_F$ is a topological groupoid and $\mathcal{G} \to \mathcal{G}_F$ is a gpd map

$\mathcal{G}_F^{bs} \subset \mathcal{G}_F$ subgroupoid of Lagrangian leaves (contour sets) satisfying Bohr-Sommerfeld leaves

if $\mathcal{G}_F^{bs}$ admits Haar system

we can consider

$C(\mathcal{G}_F^{bs})$ convolution algebra

as a "quantization" of the underlying Poisson mfld

How to construct such F ?

Poisson Bruhat structure on $\mathbb{C}P_n$

$(\mathbb{C}P_n, \pi)$ $\qquad \mathbb{C}P_n = su(n+1)/u(n)$

standard Poisson structure

- $T^n \subset su(n+1)$ acts on $\mathbb{C}P_n$ by Poisson diffeo

$\forall\, X \in \mathrm{Lie}\, T^n$ let $\sigma_X \in \mathrm{Vect}(\mathbb{C}P_n)$ be the fundamental vector field

- $[\pi, \sigma_X] = 0 \Rightarrow h_X \in Z^1(g(\mathbb{C}P_n, \pi))$

- $\sigma_X = \Omega_{FS}^{-1}\, dc_X \qquad c_X \in C^\infty(\mathbb{C}P_n)$

Fubini - Study

$F = \{h, c\} \equiv$ multiplicative inseparable model

why ?

$$\left[\Omega_{FS}^{-1}, \pi\right] = 0$$

BIHAMILTONIAN STRUCTURE (Magri, Morosi)

definition: A Poisson Nijenhuis structure is a triple (M, π_0, N) where

- (M, π_0) is a Poisson mfld
- $N : TM \to TM$ Nijenhuis tensor

$$0 = [NV, NW] - N\left([NV, W] + [V, NW] - N[V, W]\right)$$

$V, W \in \Gamma(TM)$

- π_0 and N are compatible $(\pi_1 = N \circ \pi_0)$
  - $\pi_0 \circ N^t = N \circ \pi_0$
  - $[\alpha, \beta]_{\pi_1} = [N^t\alpha, \beta]_{\pi_0} + [\alpha, N^t\beta]_{\pi_0} - N^t[\alpha, \beta]_{\pi_0}$

consequences

1) $\pi_j \equiv N^j \pi_0$ are Poisson

2) $[\pi_j, \pi_k] = 0$

For $\mathbb{C}P_n$ $\quad \pi_0 = \Omega_{FS}^{-1}$ Fubini-Study

$\pi_1 = \pi$ Poisson-Bruhat

$N = \pi \cdot \Omega_{FS}$

INTEGRABLE MODEL FROM (M, π_0, N)

$$F_k \equiv \frac{1}{k!} \operatorname{Tr} N^k$$

fundamental
Lenart
hierarchy

- $\{F_k, F_l\}_{\pi_0} = \{F_k, F_l\}_{\pi_1} = 0$
- $N^t dF_k = dF_{k+1}$

EIGENVALUES of N (action variables)

Let us consider $\pi_0 = \Omega^{-1}$ non degenerate

$\lambda_m \in \mathbb{R}$ $\quad \det(N_m - c_m) = 0$

$\forall m \in M$ $\quad \#$ distinct eigenvalues $\leq \frac{1}{2} \dim M$

Let $c \in C^\infty(U)$ $\quad U \subset M$

$\det(N_m - c(m)) = 0 \quad m \in U$

$$\boxed{N^t dc = c\, dc}$$

MODULAR CLASS (Damianou - Fernandes)

Let (M, Ω^{-1}, N) be a Poisson-Nijenhuis structure

Fix the symplectic volume form Ω^n

$\chi = \Omega^{-1}(d\,\mathrm{Tr} N)$ is the modular vector field of $\pi_1 = N \cdot \Omega^{-1}$ (with respect to Ω^n)

χ is the first vector field of the fundamental Lenart hierarchy of (M, Ω^{-1}, N)

BACK TO QUANTIZATION of (M, π_1)

M, Ω, π_1 $\qquad [\pi_1, \Omega^{-1}] = 0$

symplectic $\quad$ Poisson

$\Rightarrow \quad (M, \Omega^{-1}, N = \pi_1 \circ \Omega)$ is Poisson Nijenhuis

Let $N^t dc = c\, dc$

$$\Updownarrow$$

$$\pi dc = \frac{1}{2} \Omega^{-1} dc^2$$

$$\Updownarrow$$

$$[\pi, c] = [\Omega^{-1}, dc^2/2]$$

$$\sigma_c \equiv \Omega^{-1} dc = [\Omega^{-1}, c]$$

$$[\pi, \sigma_c] = [\pi, [\Omega^{-1}, c]] = [\Omega^{-1}, [\pi, c]]$$

$$= [\Omega^{-1}, [\Omega^{-1}, dc^2/2]] = 0$$

σ_c is Poisson vector field

$\downarrow$ lift

$h_c \in Z^1(\mathcal{G}(M, \pi_1))$ groupoid 1-cocycle

MULTIPLICATIVE INTEGRABLE MODEL

$$F = \left\{ c,\ h_c \in C^\infty(\mathcal{G}(M,\pi)),\ c \equiv \text{eigenvalues di } N \right\}$$
$$\left(c \equiv \ell^*(c) \right)$$

proposition: F is a multiplicative integrable model

$\left(\text{proof of multiplicativity}\right)$: $\gamma \in \mathcal{G}$ $\quad c(\ell(\gamma)) = c$ let us compute $c(r(\gamma))$

when $c > 0$ $\quad \pi_1 d \log c = \Omega^{-1} dc \equiv \sigma_c$

$\log c$ is a local hamiltonian

$$h_c = \ell^*(\log c) - r^*(\log c)$$

$$h = h_c(\gamma) = \log c - \log c(r(\gamma))$$

$$c(r(\gamma)) = c\, e^{-h} \text{ depends only on } (c,h)$$

$L_{c,h} \in \mathcal{S}_F$ contour level set

$$\ell(L_{ch}) = L_c = \{ \ell(\gamma),\ \gamma \in L_{ch} \}$$
$$r(L_{ch}) = L_{ce^{-h}}$$

$$L_{ch} \cdot L_{c'h'} = L_{c,h+h'} \qquad \blacksquare$$

COMPACT HERMITIAN SYMMETRIC SPACES

$\mathfrak{g}$ compact Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ Cartan

$\Sigma(\mathfrak{g}) = \{\varphi, \alpha_1, \dots, \alpha_{r-1}\}$ simple roots

$\Pi(\mathfrak{g}) \equiv$ roots

$\varphi \in \Sigma(\mathfrak{g})$ is of non compact type

if $\forall \lambda \in \Pi(\mathfrak{g})$ $\quad \mathfrak{g}_\lambda \equiv$ root space

$$\lambda = \begin{cases} \pm\varphi + \sum_i \lambda^i \alpha_i & \varphi\text{-non compact} \\ \sum_i \lambda^i \alpha_i & \varphi\text{- compact} \end{cases}$$

$$\mathfrak{h}_\varphi \equiv \mathfrak{t} \bigoplus_{\lambda\ \varphi\text{-compact}} (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{g}$$

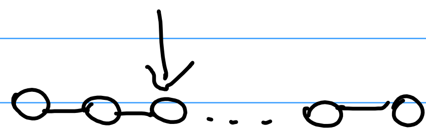
$$\mathfrak{h}_\varphi^\perp = \bigoplus_{\lambda\ \varphi\text{-non compact}} (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{g}$$

$$\mathfrak{g} = \mathfrak{h}_\varphi + \mathfrak{h}_\varphi^\perp$$

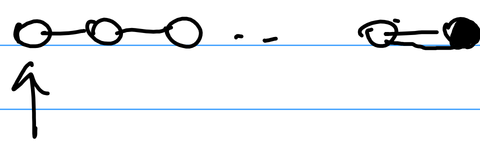
$$[\mathfrak{h}_\varphi, \mathfrak{h}_\varphi] \subset \mathfrak{h}_\varphi \qquad [\mathfrak{h}_\varphi, \mathfrak{h}_\varphi^\perp] \subset \mathfrak{h}_\varphi^\perp \qquad [\mathfrak{h}_\varphi^\perp, \mathfrak{h}_\varphi^\perp] \subset \mathfrak{h}_\varphi$$

$H_\varphi \subset G$ (simply connected) Lie $H_\varphi = \mathfrak{h}_\varphi$

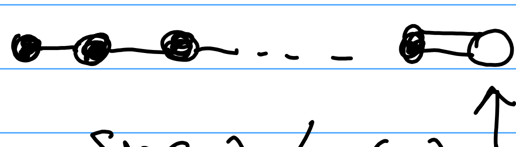
FACT: $M_\varphi = G/H_\varphi$ is a compact hermitian symmetric space

A_n 

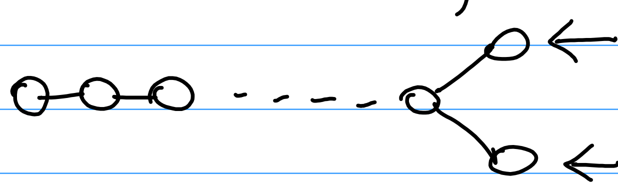
$$M = SU(n+1)/S(U(n+1-k)\times U(k)) = Gr(k,n)$$

B_n 

$$M = SO(n+2)/SO(n)\times SO(2)$$

C_n 

$$M = Sp(n)/U(n)$$

D_n 

$$M = SO(2n)/U(n)$$

BRUHAT POISSON STRUCTURE

$$r = i \sum_{\lambda \in \Pi_+(\mathfrak{g})} E_\lambda \wedge E_{-\lambda} \qquad \begin{cases} E_\lambda \in \mathfrak{g}_\lambda \\ \mathrm{Tr}(E_\lambda E_{-\lambda}) = 1 \end{cases}$$

$$\pi_G = r^L - r^R \in \Gamma(\Lambda^2 TG) \qquad \left(X^L \text{ left inv vect field of } X \in \mathfrak{g} \right)$$

$$[\pi_G, \pi_G] = 0$$

$(G, \pi_G) \equiv$ standard Poisson-Lie group

FACT: $H_\varphi \subset G$ is a Poisson subgroup

- π_G projects to $\pi_\varphi \in \Gamma(\Lambda^2 T M_\varphi)$ $\quad M_\varphi = G/H_\varphi$

We call

(M_φ, π_φ) . Bruhat - Poisson structure

- the homogeneous action $G \times M_\varphi \longrightarrow M_\varphi$ is Poisson, i.e. $\forall\, x \in \mathfrak{g}$

$\sigma \equiv$ infinitesimal action

$$\mathcal{L}_{\sigma_x}(\pi_\varphi) = \sigma \wedge \sigma(\delta(x))$$

bialgebra $\rightarrow$ $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$

A POISSON PENCIL ON $G/H_\varphi = M_\varphi$

G/H_φ as a (co)adjoint orbit

Let $H_\varphi \in \mathfrak{h}$ s.t. $\begin{cases} \varphi(H_\varphi) = i \\ \alpha_i(H_\varphi) = 0 \end{cases}$

$$M_\varphi = G/H_\varphi \simeq Ad_G(H_\varphi) \subset \mathfrak{g} \underset{\text{Killing Form}}{\simeq} \mathfrak{g}^*$$

$\Omega_\varphi \equiv$ KKS symplectic form

THEOREM (Koroshkin, Radul, Rubtsov '93)

$$[\pi_\varphi, \Omega_\varphi^{-1}] = 0$$

proof. $X \in \mathfrak{g}$

$$\mathcal{L}_{\sigma_X}[\pi_\varphi, \Omega_\varphi^{-1}] = [\mathcal{L}_{\sigma_X}\pi_\varphi, \Omega_\varphi^{-1}] + [\pi_\varphi, \overset{0}{\overset{\|}{\mathcal{L}_{\sigma_X}\Omega_\varphi^{-1}}}]$$

$$= [\sigma\wedge\sigma(\delta(X)), \Omega_\varphi^{-1}] = 0$$

$$[\pi_\varphi, \Omega_\varphi^{-1}] \in \Gamma(\Lambda^3 TM_\varphi)^G = (\Lambda^3 \mathfrak{h}_\varphi^\perp)^{H_\varphi} = 0. \blacksquare$$

$\pi_t = \pi_\varphi + t\,\Omega_\varphi^{-1}$ Poisson pencil

BIHAMILTONIAN STRUCTURE

$$N_\varphi \equiv \Pi_\varphi \circ \Omega_\varphi : TM_\varphi \longrightarrow TM_\varphi$$

Bruhat-Poisson

KKS symplectic structure

$(M_\varphi, \Omega_\varphi^{-1}, N_\varphi)$ Poisson-Nijenhuis

Solve the eigenvalue problem for N_φ

find $c \in C^\infty(\mathcal{U})$ $\quad \mathcal{U} \subset M_\varphi$

s.t. $N_\varphi^* dc = c\, dc$

1) Are the eigenvalues (generically) independent?
2) Do they exist as smooth global functions?

GELFAND - CETLIN INTEGRABLE MODEL

(Guillemin, Sternberg '83)

- Lie algebra representation theory: GC basis

$$G_n \subset G_{n-1} \subset \cdots G_1 \subset G \quad \text{nested subgroups}$$

idea: fix a basis of a representation of G by reducing the rep of the subgroups

- Let f be a Casimir of $\mathfrak{g}_k^*$

  g be a Casimir of $\mathfrak{g}_e^*$

$$p_k : \mathfrak{g}^* \to \mathfrak{g}_k^* \qquad p_e : \mathfrak{g}^* \to \mathfrak{g}_e^*$$

$$\{ p_k^* f, p_e^* g \}_{\mathfrak{g}^*} = 0$$

Fix $\lambda \in \mathfrak{t}^*$ coadjoint orbit $\mathcal{O}_\lambda \subseteq \mathfrak{gl}$

Casimirs of $\mathfrak{g}_k^*$, $k=1, \dots$ form a set of commuting hamiltonians for Ω_{KKS}^{-1}

$$G = U(n+1)$$

$$G_k = U(k) \times \mathbb{T}^{n+1-k} \qquad k=1,\dots,n$$

$$= \left\{ \begin{pmatrix} A & \\ & e^{i\varphi_{k+1}} & & \\ & & \ddots & \\ & & & e^{i\varphi_{n+1}} \end{pmatrix} \in U(n+1) \right\}$$

$$\mathbb{T} = G_1 \dots G \subset G_n \subset U(n+1)$$

Cartan subalgebra

$$\mu_G : \mathcal{O}_\lambda \to \mathfrak{g}^* \simeq \mathrm{Herm}_{n+1}$$

$$\mu_{G_k} : \mathcal{O}_\lambda \to \mathfrak{g}_k^*$$

$$\mu_{G_k} = \begin{pmatrix} (\mu_G)_{ij} & 0 & & \\ 0 & * & 0 & 0 \\ & 0 & * & \\ & & 0 & \ddots \\ & 0 & & * \end{pmatrix} \quad i,j=1\dots k$$

The Gelfand - Cetlin variables are the eigenvalues $\mu_i^{(k)}$ of μ_{G_k} $\forall k$

Gelfand - Cetlin cone $\quad C_{GC} \subset \mathbb{R}^{\frac{\dim \mathcal{O}}{2}}$

$$\lambda_1 \geq \mu_1^{(n)} \geq \lambda_2 \geq \mu_2^{(n)} \geq \cdots \geq \lambda_n$$

$$\mu_1^{(n)} \geq \mu_1^{(n-1)} \geq \mu_2^{(n)} \geq \cdots \geq \mu_n^{(n)}$$

$$\cdots \quad - - - - \quad \cdots$$

$$\mu_1^{(k)} \geq \mu_1^{(k-1)} \geq \cdots \geq \mu_k^{(k)}$$

If $\lambda_k = (\overbrace{1, \ldots, 1}^{k}, 0 \ldots 0) = -iH_k$

$$\mathcal{O}_{\lambda_k} = \mathrm{Ad}_{U(n)} \lambda_k \simeq Gr(k,n)$$

We are going to show that

$$\{ \mu_i^{(k)}, \mu_j^{(k')} \}_{\text{Poisson Bruhat}} = 0$$

THEOREM The Gelfand Cetlin Variables for $Gr(k,n)$ are the eigenvalues of the Nijenhuis tensor $N_\varphi = \pi_\varphi \circ \Omega_{KKS}$

SKETCH of THE CONSTRUCTION

$$\lambda = \left(\begin{array}{c|c} 1_k & 0 \\ \hline 0 & 0 \end{array}\right) \in Herm_n \qquad \lambda^2 = \lambda$$

$$\Rightarrow \quad \mu_G(g) = g\lambda g^{-1} \quad \Rightarrow \quad \mu_G^2 = \mu_G$$

$$\begin{array}{c} \mathcal{E}_{\mu_G} = \mathrm{Im}\,\mu_G \subset \mathbb{C}^n \\ \downarrow{\scriptstyle \mu_G} \\ Gr(k,n) \end{array}$$

hermitian vector bundle

$\mathcal{E}_{\mu_G}$ is the associated vector bundle of

$$\begin{array}{c} S^{k,n} = U(n+1)/U(n-k) \longleftarrow U(k) \\ \downarrow \\ Gr(k,n) \end{array}$$

Stiefel bundle

Consider the standard Poisson structure $\pi_{U(n)}$

All subgroups are Poisson subgroups

$$(S^{k,n}, \pi_{S^{k,n}}) \longrightarrow (Gr(k,n), \pi_G) \text{ is Poisson}$$

$\mathcal{E}_k = S^{k,n} \times_f \mathbb{C}^k$ is a Poisson vector bundle

meaning

$$\{\ ,\ ]: C^\infty(Gr(k,n)) \otimes \Gamma(\mathcal{E}_k) \longrightarrow \Gamma(\mathcal{E}_k)$$

satisfying ----

$$\Updownarrow$$

$\nabla_{df}(-) = \{f, -]$ is a flat contravariant connection

$f \in C^\infty(Gr(k,n))$

$$\sigma_i : S^{k,n} \longrightarrow \mathbb{C}^k \qquad \sigma_i = (g_{i,n+1-k}, \dots, g_{i,n+1})$$

Proposition (A formula for ∇)

$$\nabla_N \sigma_i = \Omega \cdot \nabla \sigma_i = \sum_r d(C_-(\mu_G)_{ir}) \otimes \sigma_r$$

where $C_-(\mu_G) = \begin{pmatrix} 2(\mu_G)_{11} & 0 & \cdots & 0 \\ (\mu_G)_{12} & 2(\mu_G)_{22} & & 0 \\ & & \ddots & \end{pmatrix}$

Since
$$(\mu_G)_{ij} = \langle \sigma_i , \sigma_j \rangle$$
we compute

- $$N_\varphi^* \, d\mu_G = C_-(d\mu_G)\, \mu_G' + \mu_G\, C_+(d\mu_G)$$
- the same formula is valid for upper left minors μ_{G_r}

$$\mu_{G_r} v_i^{(r)} = \mu_i^{(r)} v_i^{(r)} \qquad v_i^{(r)} \in \mathbb{C}^r$$
$$(v_i^{(r)}\, v_i^{(r)}) = 1$$

we finally get

$$N_\varphi^t \, d\mu_i^{(r)} = \mu_i^{(r)} \, d\mu_i^{(r)}$$

i.e. Gelfand-Cetlin variables diagonalize N_φ

CONSEQUENCES

- Gelfand-Cetlin system on flag manifolds is a completely integrable model: GC variables ($\equiv$ eigenvalues of N_φ) are independent

- For $k=1$, $Gr(1,n)=\mathbb{CP}_n$ the GC variables are the components of the moment map for the torus action. They are global smooth functions

- For $k>1$, $Gr(k,n)$ is not a toric mfld. GC variables are global continuous functions that are not smooth when they coincide.

- Their hamiltonian vector fields are defined only on the interior of the GC cone and so their lifted groupoid cocycles.

next step:

find a way to embody continuous but not smooth eigenvalues in the groupoid picture