The Bohr-Sommerfeld groupoid of quantum $\mathbb{CP}^n$

joint with F. Bonechi, J. Qiu, M. Tarlini

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N. Ciccoli

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Let (M, π) be an *integrable* Poisson manifold with symplectic groupoid

$$\mathcal{G}(M) \underset{l}{\overset{r}{\rightrightarrows}} M \; : \qquad m : \mathcal{G}_2(M) \to \mathcal{G}(M)$$

Karasev-Weinstein-Zakrzewski

Apply geometric quantization to $\mathcal{G}(M)$ and compare the outcome with deformation quantization of (M, π).

Symplectic integration

For a Poisson manifold (M, π) the cotangent bundle T^*M has a natural structure of Lie algebroid (i.e. Lie bracket between 1–forms + Lie map between 1-forms and vector fields).

A symplectic groupoid is a Lie groupoid integrating this Lie algebroid (much as Lie groups integrate Lie algebras - but... possible obstructions).

If the obstruction is not present (meaning of the word *integrable*) then the groupoid has also a symplectic manifold *compatible* with the Lie groupoid structure.

KWZ program

1. Prequantum line bundle (L, ∇) + σ covariantly constant normalized 2–cocycle in L;
2. *Multiplicative* polarization $\mathcal{F}$: set of leaves $\mathcal{G}(M)/\mathcal{F}$ is a groupoid inheriting (reduced) 2–cocycle σ_0;
3. Bohr-Sommerfeld condition identifying a subgroupoid $(\mathcal{G}(M)/\mathcal{F})^{bs}$;
4. (Twisted) convolution C^*–algebra $C^*((\mathcal{G}(M)/\mathcal{F})^{bs}; \sigma_0)$.

Motivating example

Let $M = \mathbb{T}^2$ with constant symplectic structure

$$\pi = \theta\partial_1 \wedge \partial_2$$

$\mathcal{G}(\mathbb{T}^2) = T^*\mathbb{T}^2$ (change in grpd + sympl.)
Prequantum bundle= trivial line bundle + 2–cocycle;

- Horizontal polarization $\Rightarrow C^*(\mathbb{Z}^2; \sigma_0)$ with $\sigma_0 = e^{\pi}$ (Weyl);
- Cylindrical polarization $\Rightarrow C^*(\mathbb{Z} \rtimes \mathbb{S}^1)$ action groupoid with trivial cocycle (irrational rotation algebra).

Outcome

Quantum torus $p \star q = e^{\hbar}\, q \star p$.

Multiplicative polarization

A groupoid polarization $\mathcal{F} \subseteq T^{\mathbb{C}}\mathcal{G}$ is *multiplicative* (Hawkins JSG 2008) if, letting

$$\mathcal{F}_2 = (\mathcal{F} \times \mathcal{F}) \cap T^{\mathbb{C}}\mathcal{G}_2$$

then

$$m_*(\mathcal{F}_2(\gamma, \eta)) = \mathcal{F}(m(\gamma, \eta))$$

for any composable pair $(\gamma, \eta) \in \mathcal{G}_2$.

Problem: there are topological obstructions to the existence of real multiplicative polarizations

$\mathbb{CP}^1$-obstruction

Let π be **any** integrable Poisson structure on $\mathbb{CP}^1$, then there are no real multiplicative polarizations on its symplectic groupoid (linked to non existence of rank 1 foliations on $\mathbb{CP}^1$).

Bruhat-Poisson structure on $\mathbb{CP}^1$:

$$\pi_B = \begin{cases} -\imath(1+|z|^2)\partial_z \wedge \partial_{\overline{z}} & \text{on } \mathbb{CP}^1 \setminus [1,0] \\ -\imath|w|^2(1+|w|^2)\partial_w \wedge \partial_{\overline{w}} & \text{on } \mathbb{CP}^1 \setminus [0,1] \end{cases}$$

Still possibile to perform KWZ procedure with a **singular** multiplicative polarization (Bonechi, C., Staffolani, Tarlini JGP 2012).

Loosening requirements

What do we really need for a C^*–groupoid convolution algebra?

- $\mathcal{G} \to \mathcal{G}_{\mathcal{F}}$ **Lagrangian fibration of topological groupoids;**
- $\mathcal{G}_{\mathcal{F}}^{bs}$ **Bohr–Sommerfeld subgroupoid carrying a left Haar measure;**
- **the prequantization cocycle descending to** $\mathcal{G}_{\mathcal{F}}^{bs}$**;**
- **the *modular* 1–cocycle descending to** $\mathcal{G}_{\mathcal{F}}^{bs}$**;**

(M, π) Poisson, V volume form on $M \Rightarrow \chi_V$ *modular* vector field (divergence of π w.r.t. to V) defines a class in $H^1_\pi(M)$. $\chi_V \Rightarrow f_V$

(van Est map) 1–cocycle on $\mathcal{G}$; f_V should be quantizable,

coincide with the modular function of the quasi invariant measure on the base space, implement **KMS condition.**

Multiplicative integrable system

integrable

A family $F = \{f_1, \dots f_N\}$ of functions, $N = \frac{1}{2}\dim \mathcal{G}$, is an integrable system if are in involution $\{f_i, f_j\} = 0$ and $df_1 \wedge \dots \wedge df_N \neq 0$ on a dense open subset of M.

multiplicative

The integrable system is called *multiplicative* if the distribution $\mathcal{F} = \langle X_{f_1}, \dots X_{f_N} \rangle$ is multiplicative, or, more generically, if the topological space of level sets of $f_1, \dots f_N$ inherits a topological groupoid structure from $\mathcal{G}$.

modular

The integrable system is called *modular* if the modular function f_V is in involution with all f_i's.

Multiplicative integrable system

Consider the level sets of a multiplicative integrable system

$$\mathcal{G}_F(M) = \mathcal{G}(M)/\mathcal{F}$$

It is **well behaved** if:

1. $\mathcal{G}_F(M)$ is a topological groupoid and $\mathcal{G}(M) \to \mathcal{G}_F(M)$ a topological groupoid epimorphism;
2. For each pair l_1, l_2 of composable leaves $m : l_1 \times l_2 \to l_1 l_2$ induces a surjective map in homology ($\Rightarrow$ subgroupoid $\mathcal{G}^{bs}_{\mathcal{F}}(M)$).
3. $\mathcal{G}^{bs}_F(M)$ admits a left Haar system (guaranteed if it is étale).

Let $SU(n+1)$ be given the *standard* Poisson–Lie structure π_{std}.

There is a one–parameter family of *covariant* $(\mathbb{CP}^n, \pi_t)$, non symplectic when $t \in [0,1]$.

Non symplectic are all quotient by coisotropic subgroups:

$$U_t(n) = \sigma_t S(U(1) \times U(n))\sigma_t^{-1} \subseteq SU(n+1)$$

where

$$\sigma_t = \begin{pmatrix} \sqrt{1-t} & 0 & \sqrt{t} \\ 0 & \mathrm{id}_{n-1} & 0 \\ -\sqrt{t} & 0 & \sqrt{1-t} \end{pmatrix}$$

Some equivalences. In fact:

$$\psi : \mathbb{CP}^n \to \mathbb{CP}^n; \qquad \psi(\pi_t) = -\pi_{1-t}$$

- π_0, π_1, *standard* or *Bruhat–Poisson*
- π_t, $t \in]0, 1[$, *non standard*.

Poisson pencil

Let π_λ be the Fubini-Study bivector. Then $[\pi_\lambda, \pi_0] = 0$ (Koroshkin-Radul-Rubtsov CMP '93) and $\pi_t = \pi_0 + t\pi_\lambda$.

Standard $\mathbb{CP}^n$: symplectic foliation

Projecting the chain of Poisson subgroups

$$SU(1) \subseteq SU(2) \subseteq \ldots \subseteq SU(n)$$

one gets the chain of Poisson submanifolds

$$\{*\} \subseteq \mathbb{CP}^1 \subseteq \ldots \subseteq \mathbb{CP}^{n-1}$$

In homogeneous coordinates

$$P_k = \{[X_1, \ldots, X_k, 0, \ldots, 0]\}$$

is a Poisson submanifold. All symplectic leaves are contractible and symplectomorphic to standard $\mathbb{C}^k$.

Non standard $\mathbb{CP}^n$: symplectic foliation

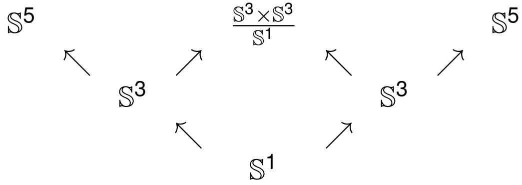
singular locus

Let

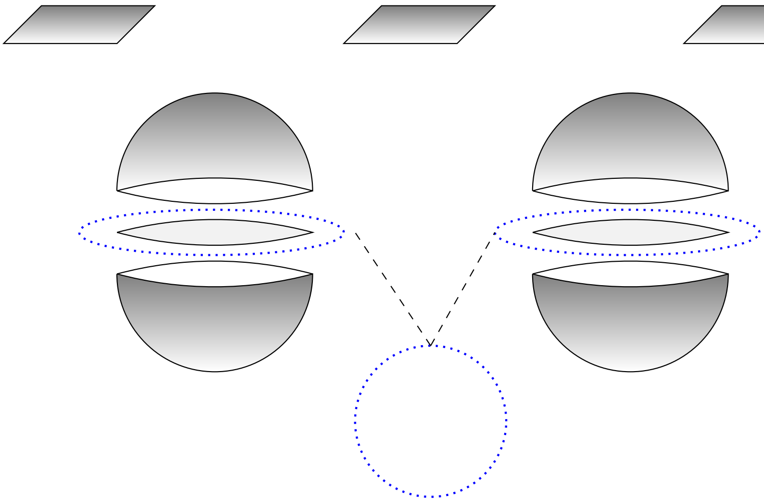
$$P_k(t) = \left\{ F_{k,t} = t \sum_{i=1}^{k} |X_i|^2 - (1-t) \sum_{i=k+1}^{n} |X_i|^2 = 0 \right\}$$

Then $\bigcup_{i=1}^{n} P_i(t)$ is the singular part; complement has $n+1$ connected contractible leaves $\simeq \mathbb{C}^n$.

Scheme of the singular part for $\mathbb{CP}^3$:



symplectic foliation of $\mathbb{CP}^2_t$



The symplectic groupoid

$$\mathcal{G}(\mathbb{CP}^n, \pi_t) = \{[g\gamma] \,:\, g \in SU(n+1), \gamma \in SB(n+1, \mathbb{C}), {}^{g}\gamma \in U_t(n)^{\perp}\}$$

is a fibre bundle over $\mathbb{CP}^n$ with contractible fibre $U_t(n)^{\perp}$.

It is an exact symplectic manifold.

It carries a hamiltonian $\mathbb{T}^n$–action with momentum map

$$h([g\gamma]) = \log p_{A_{n+1}}(\gamma)$$

torus action

The Cartan $\mathbb{T}^n \subseteq SU(n+1)$ acts on $(\mathbb{CP}^n, \pi_\lambda)$ with momentum map

$$c : \mathbb{CP}^n \to \mathfrak{t}_n^*; \qquad \operatorname{Im} c = \Delta_n$$

The action is Poisson w. r. to π_t.

Suitable basis H_k of $\mathfrak{t}_n$ such that

1. infintesimal vector fields σ_{H_k} are eigenvalues of the Nijenhuis operator with eigenvector $(c_k - 1)$;
2. $\sigma_{H_k} = \{b_k, --\}$, with $b_k = \log|c_k - 1 + t|$.

Summarizing actions

- Hamiltonian $\mathbb{T}^n$–action on $\mathbb{CP}^n$ with momentum map $c : \mathbb{CP}^n \to \mathbb{R}^n$;
- Hamiltonian $\mathbb{T}^n$–action on $\mathcal{G}(\mathbb{CP}^n, \pi_t)$ with momentum map $h : \mathcal{G}(\mathbb{CP}^n) \to \mathbb{R}^n$ by groupoid 1–cocycles;

Let us consider

$$\mathcal{F} = \{I^* c_i, h_i \ \ldots i = 1, \ldots, n\}$$

Theorem

$\mathcal{F}$ is a multiplicative modular integrable system on $\mathcal{G}(\mathbb{CP}^n, \pi_t)$ with:

$$f_{FS} = \sum_{i=1}^{n} h_i$$

Aim: prove this integrable system is well behaved.

The topological groupoid of level sets

Let $\mathbb{R}^n$ act on $\mathbb{R}^n$ via

$$c \cdot h = (1 - t + e^{-h}(c + t - 1))$$

and let $\mathbb{R}^n \rtimes \mathbb{R}^n\big|_{\Delta_n}$ be the action groupoid restricted to the standard simplex. Then:

$$\mathcal{G}_{\mathcal{F}}(t) = \{(c, h) \in \mathbb{R}^n \rtimes \mathbb{R}^n\big|_{\Delta_n} \,:\, c_i = c_{i+1} = 1 - t \Rightarrow h_i = h_{i+1}\}$$

is the topological groupoid of level sets.

Level sets L_{ch} are connected with: $H_1(L_{ch};\mathbb{Z})$ generated by hamiltonian flows of $h_j, l^* c_j$;

Theorem

BS conditions select a discret subset of lagrangians

$$\mathcal{G}^{bs}_{\mathcal{F}}(t) = \{(c,h) \in \mathcal{G}_{\mathcal{F}}(t)\ : h_k \in \hbar\mathbb{Z}, \log|c_k - 1 + t| \in \hbar\mathbb{Z}\}$$

This is an étale subgroupoid with a unique left Haar system.

The modular function f_{FS} is quantized to

$$f_{FS}(c,h) = \sum_{i=1}^{n} h_i$$

The space of units is

$$\Delta_n^{\mathbb{Z}}(t) = \{c \in \Delta_n \,:\, c_k = 1 - t + e^{-\hbar n_k}\}$$

The quasi invariant measure associated to f_{FS} is:

$$\mu_{fs}(c) = exp(-\hbar \sum_{k=1}^{n} n_k)$$

Groupoid orbits are labelled by $(r, s) : r + s \leq n$. Each is a transitive subgroupoid over

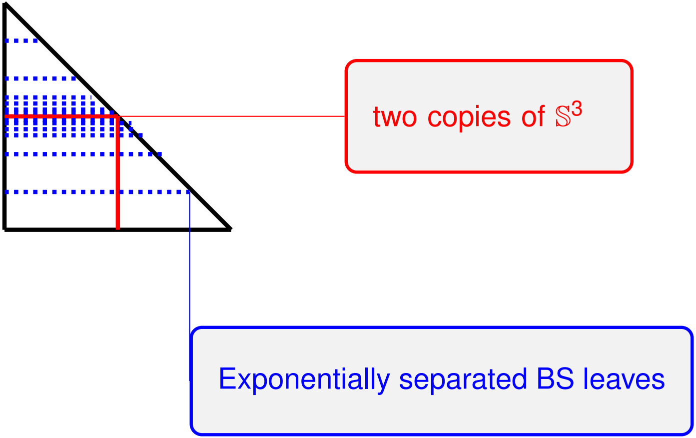
$$\Delta_{r,s}^{\mathbb{Z}}(t) = \left\{ (m, \infty, n) \in \overline{\mathbb{Z}}^r \times \infty \times \overline{\mathbb{Z}}^s : \begin{array}{ccc} -\frac{\log(1-t)}{\hbar} & \leq m_i & \leq m_{i+1} \\ n_i & \geq n_{i+1} & \geq -\frac{\log(t)}{\hbar} \end{array} \right\}$$

1. The Poisson antiautomorphism ψ lifts to a groupoid isomorphism;
2. Poisson submanifolds are quantized by topological subgroupoids

$$P_k(t) = \{(c, h) \,|\, c_k = 1 - t\}$$

3. Groupoids thus obtained concide with:
   - Sheu for $(\mathbb{CP}^n, \pi_0)$;
   - Sheu for $\mathbb{S}^{2n-1}$ as Poisson sbmfld of $\mathbb{CP}^n, \pi_t$, $t \neq 0, 1$;
   - Sheu for $(\mathbb{CP}^1, \pi_t)$.

Example: $\mathbb{C}P^2$



1–cocycle $c = \log D \in Z^1(\mathcal{G};\mathbb{R})$

$A_c(t) = e^{\imath tc}$ map in $Aut(C^*\mathcal{G})$

$\phi_\mu(f \star A_c(\imath\beta)g) = \phi_\mu(g \star f)$

D modular function w.r.t. to μ

μ quasi-invariant measure on $\mathcal{G}_0$

$\phi_\mu : C^*(\mathcal{G}) \to \mathbb{R}$

Modular class in $H^1_\pi(M)$

Van den Bergh bimodule

Functoriality: $(\mathbb{CP}^1, \pi_t)$ are all Poisson–Morita equivalent for $0 < t < 1$ and the nonstandard groupoid does not depend on t[1].

- Are groupoids for $n > 1$ independent of t?
- Is $(\mathbb{CP}^n, \pi_t)$ Poisson-Morita to $(\mathbb{CP}^n, \pi_s)$ for $t, s \in]0, 1[$?
- Can we characterize Poisson submanifolds which functorially quantize?

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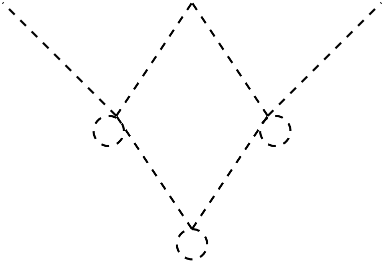
[1] Bursztyn-Radko, Ann. Inst. Fourier 2003

Nonstandard quantum $\mathbb{CP}^1$ is not only a groupoid but also a graph C^*–algebra [2].

Is nonstandard quantum $\mathbb{CP}^n$ the graph C^*–algebra of the following graph?



[2] Hong–Szymański, CMP 2002