BRAIDED NONCOMMUTATIVE JOIN ALGEBRA OF GALOIS OBJECTS

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Goal and plan

Motivation: Extend the noncommutative join for compact quantum groups (Hopf algebras) to include Galois objects (quantum torsors). Then to quantum principal bundles.

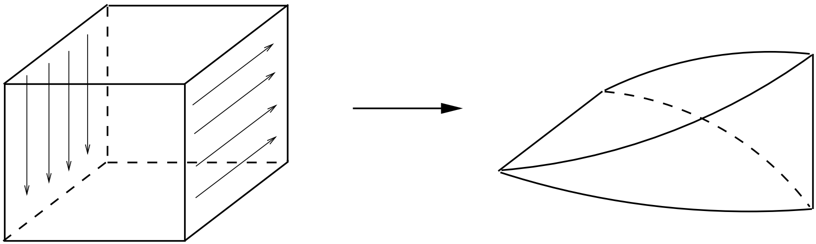
Applications:

1. Quantum coverings from anti-Drinfeld doubles that are used in Hopf-cyclic theory with coefficients.
2. Quantum torus-bundles with potential for constructing new Dirac operators [L.D., A. Sitarz, A. Zucca].

Plan:

1. Recall the basics: classical joins, braidings, Galois objects.
2. Show that the diagonal coaction of noncommutative Hopf algebras on the braided tensor product of Galois objects is a homomorphism of algebras.
3. Construct a braided noncommutative join algebra of Galois objects, and show that it is a principal comodule algebra for the diagonal coaction.
4. Apply to noncommutative tori (& tackle *-structure) and to anti-Drinfeld doubles.

Classical join



The join $X * Y$ of compact Cartan principal G-bundles X and Y (local triviality not assumed) is again a compact Cartan principal G-bundle for the diagonal G-action on $X * Y$:

In particular $G * G$ is a non-trivializable principal G-bundle over ΣG for any compact Hausdorff topological group $G \neq 1$.

For example, we get this way $S^1 \to \mathbb{R}P^1$, $S^3 \to S^2$ and $S^7 \to S^4$, using $G = \mathbb{Z}/2\mathbb{Z}$, $U(1)$ and $SU(2)$, respectively.

Quantum join I

Definition

Let A_1 and A_2 be unital C*-algebras. We call the unital C*-algebra

$$A_1 * A_2 := \left\{ x \in C([0,1]) \underset{\min}{\otimes} A_1 \underset{\min}{\otimes} A_2 \;\middle|\; \begin{array}{l} (\mathrm{ev}_0 \otimes \mathrm{id})(x) \in \mathbb{C} \otimes A_2\,, \\ (\mathrm{ev}_1 \otimes \mathrm{id})(x) \in A_1 \otimes \mathbb{C} \end{array} \right\}$$

join C*-algebra of A_1 and A_2.

Quantum group actions ?
Oops... no diagonal action, i.e. (dually) the diagonal coaction

$$\Delta(a \otimes a') = a_{(0)} \otimes a'_{(0)} \otimes a_{(1)}\, a'_{(1)}$$

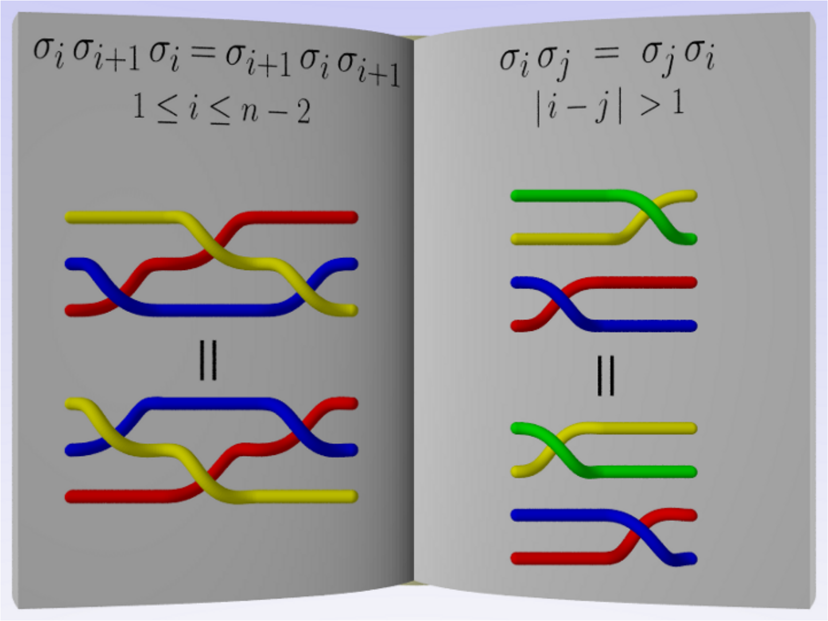
is not a homomorphism of algebras.

Two possible ways out (classically insignificant):

1. gauge coaction
2. braid multiplication

Today about the second option:

Braids


$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
$1 \leq i \leq n-2$
$\sigma_i \sigma_j = \sigma_j \sigma_i$
$|i-j| > 1$

Braidings of algebras

Definition

A factorization of two algebras A and A' is a linear map $\sigma : A' \otimes A \longrightarrow A \otimes A'$ such that

1. $\forall\, a \in A,\ a' \in A'$: $\sigma(1 \otimes a) = a \otimes 1$ and $\sigma(a' \otimes 1) = 1 \otimes a'$,
2. $\sigma \circ (m' \otimes \mathrm{id}) = (\mathrm{id} \otimes m') \circ \sigma_{12} \circ \sigma_{23}$,
   $\sigma \circ (\mathrm{id} \otimes m) = (m \otimes \mathrm{id}) \circ \sigma_{23} \circ \sigma_{12}$.

Here m and m' are multiplications in A and A' respectively.
If in addition $A' = A$ and the braid equation

$$\sigma_{12} \circ \sigma_{23} \circ \sigma_{12} = \sigma_{23} \circ \sigma_{12} \circ \sigma_{23}$$

is satified, we call σ a braiding.

Factorizations classify all associative multiplications on $A \otimes A'$ s.t. A and A' are included in $A \otimes A'$ as unital subalgebras:

$$m_\sigma = (m \otimes m') \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) .$$

Left and right Hopf-Galois extensions

Let H be a Hopf algebra, and P a left (right) H-comodule algebra with coaction

$$ {}_P\Delta(x) = x_{(-1)} \otimes x_{(0)} \quad \text{(left)}, $$
$$ \Delta_P(x) = x_{(0)} \otimes x_{(1)} \quad \text{(right)}. $$

Def. of the left (right) coaction-invariant subalgebra:

$$ B := {}^{co\,H}P := \{x \in P \mid {}_P\Delta(x) = 1 \otimes x\} \quad \text{(left)}, $$
$$ B := P^{co\,H} := \{x \in P \mid \Delta_P(x) = x \otimes 1\} \quad \text{(right)}. $$

Def. of the canonical maps:

$$ can_L : P \underset{B}{\otimes} P \ni x \otimes y \longmapsto x_{(-1)} \otimes x_{(0)}y \in H \otimes P \quad \text{(left)}, $$
$$ can_R : P \underset{B}{\otimes} P \ni x \otimes y \longmapsto xy_{(0)} \otimes y_{(1)} \in P \otimes H \quad \text{(right)}. $$

Definition

P is called left (right) H-Galois extension of B iff can_L (can_R) is a bijection.

Durdevic braiding

Theorem (M. Durdevic)

Let P be a left H-Galois extension of B. Then the linear map

$$\sigma\colon P \underset{B}{\otimes} P \ni x \otimes y \longmapsto {y_{(-1)}}^{[1]} \otimes {y_{(-1)}}^{[2]}\, x\, y_{(0)} \in P \underset{B}{\otimes} P$$

is a braiding. Here $h^{[1]} \otimes h^{[2]} := can_L^{-1}(h \otimes 1)$.

- σ is called Durdevic braiding.
- σ becomes a flip when P is commutative.

Special cases:

1. $B = \mathbb{C}$ (i.e. left Galois object).
   This is the case we are to explore.
2. $P = H$ (a Hopf algebra).
   Then the Durdevic σ coincides with the Yetter-Drinfeld σ:
   $$\sigma(a \otimes b) = b_{(1)} \otimes S(b_{(2)})ab_{(3)}\,,$$
   where S is the antipode of H.

Braiding left Galois objects

Let $\sigma\colon A\otimes A\to A\otimes A$ be a braiding. We call $A\otimes A$ with multiplication m_σ braided tensor product algebra and denote it $A\underline{\otimes}A$.

Lemma (Key lemma)

Let H be a Hopf algebra and A a bicomodule algebra over H (left and right coactions commute). Assume that A is a left Galois object over H, and that $A\underline{\otimes}A$ is the tensor product algebra braided by the Durdevic braiding. Then the right diagonal coaction

$$\Delta_{A\underline{\otimes}A} : A\underline{\otimes}A \ni a\underline{\otimes}a' \longmapsto a_{(0)}\underline{\otimes}a'_{(0)}\otimes a_{(1)}a'_{(1)}\in A\underline{\otimes}A\otimes H$$

is an algebra homomorphism.

Pf. In fact $m_{A\underline{\otimes}A}$ is just the 'pullback' by can_L of the tensor multiplication on $H\otimes A$, and so $can_L : A\underline{\otimes}A\to A\otimes H$ becomes a colinear algebra isomorphism.

Braided noncommutative join construction

Definition

Let H be a Hopf algebra and A a bicomodule algebra over H. Assume that A is a left Galois object over H. We call

$$A\underline{*}A := \left\{ x \in C([0,1]) \otimes A\underline{\otimes}A \;\middle|\; \begin{matrix} (\mathrm{ev}_0 \otimes \mathrm{id})(x) \in \mathbb{C} \otimes A\,, \\ (\mathrm{ev}_1 \otimes \mathrm{id})(x) \in A \otimes \mathbb{C} \end{matrix} \right\}$$

the H-braided noncommutative join algebra of A.

Lemma

The map

$$\begin{aligned} C([0,1]) \otimes A\underline{\otimes}A &\longrightarrow C([0,1]) \otimes A\underline{\otimes}A \otimes H\,, \\ f \otimes a \otimes b &\longmapsto f \otimes a_{(0)} \otimes b_{(0)} \otimes a_{(1)} b_{(1)}\,, \end{aligned}$$

restricts and corestricts to $\delta\colon A\underline{*}A \to (A\underline{*}A) \otimes H$ *making* $A\underline{*}A$ *a right* H*-comodule algebra.*

Main theorem

Theorem

Let $A \underline{*} A$ be the H-braided noncommutative join algebra of A. Assume that the antipode of H is bijective and that A is also a right Galois object. Then the coaction

$$\delta \colon A \underline{*} A \longrightarrow (A \underline{*} A) \otimes H$$

is principal, i.e. the canonical map it induces is bijective and P is H-equivariantly projective as a left B-module. Furthermore, the coaction-invariant subalgebra B is the unreduced suspension ΣH.

Pf. goes by exhibiting $A \underline{*} A$ to be isomorphic to the pullback of two pieces which are shown to be pricipal, and using the fact [HKMZ11] that pullbacks preserve the principality.

Quantum-torus example

Take $A := \mathcal{O}(\mathbb{T}^2_\theta)$, generated by unitaries U and V; and the Hopf algebra $H := \mathcal{O}(\mathbb{T}^2)$ generated by (commuting) unitaries u and v. With the usual coactions, A is an H-bicomodule and a left Galois object. Setting

$$U_L := U \underline{\otimes} 1, \quad V_L := V \underline{\otimes} 1, \quad U_R := 1 \underline{\otimes} U, \quad V_R := 1 \underline{\otimes} V,$$

we can write the linear basis of $A \underline{\otimes} A$ as $\{U_L^k V_L^l U_R^m V_R^n\}_{k,l,m,n \in \mathbb{Z}}$.

The H-braided join comodule algebra of A

$$A \underline{*} A = \left\{ \sum_{\text{finite}} f_{klmn} \otimes U_L^k V_L^l U_R^m V_R^n \in C([0,1]) \otimes A \underline{\otimes} A \;\middle|\; k,l,m,n \in \mathbb{Z}, \; \begin{matrix} f_{klmn}(0)=0 \text{ for } (k,l)\neq(0,0), \\ f_{klmn}(1)=0 \text{ for } (m,n)\neq(0,0) \end{matrix} \right\}$$

is a θ-deformation of a nontrivial $\mathbb{T}^2$-principal bundle $\mathbb{T}^2 * \mathbb{T}^2$ preserving the structure group, the base space, and principality.

The *-structure $U^* = U^{-1}, V^* = V^{-1}$ of $\mathcal{O}(\mathbb{T}^2_\theta)$ matches too:

*-structures

If H is a *-Hopf algebra, we call a *-algebra A right H *-comodule algebra iff

$$(* \otimes *) \circ \Delta_A = \Delta_A \circ * \, .$$

Then on $A \underline{\otimes} A$ we use the pullback by can_L of $* \otimes *$ on $H \otimes A$

$$\begin{aligned}(a \underline{\otimes} b)^* &:= (can_L^{-1} \circ (* \otimes *) \circ can_L)(a \underline{\otimes} b) \\ &= {a^*}_{(-1)}{}^{[1]} \underline{\otimes} {a^*}_{(-1)}{}^{[2]} \, b^* \, {a^*}_{(0)} = (1 \underline{\otimes} b^*) \cdot (a^* \otimes 1).\end{aligned}$$

This, combined with the c.c. on $C([0,1])$, restricts to $A \underline{*} A$. Furthermore, since

$$\Delta_{A \underline{\otimes} A} = {}_A can^{-1} \circ (\mathrm{id} \otimes \Delta_A) \circ {}_A can,$$

is a composition of *-homomorphisms, so is $\Delta_{A \underline{\otimes} A}$, as well as $\Delta_{A \underline{*} A}$ as a restriction of $\mathrm{id} \otimes \Delta_{A \underline{\otimes} A}$. Thus

Theorem

*If A is an H bicomodule and right *-comodule algebra, and a left H-Galois object, then the braided join algebra $A \underline{*} A$ is a right H *-comodule algebra for the diagonal coaction.*

Non-cosemisimple example

Let $q \in \mathbb{C}$ such that $q^3 = 1$, and let H denote the (9-dim) Hopf algebra generated by a and b with relations

$$ab = qba, \qquad a^3 = 1, \qquad b^3 = 0.$$

The comultiplication Δ, counit ε, and antipode S are

$$\begin{aligned} \Delta(a) &= a \otimes a, & \varepsilon(a) &= 1, & S(a) &= a^2, \\ \Delta(b) &= a \otimes b + b \otimes a^2, & \varepsilon(b) &= 0, & S(b) &= -q^2 b. \end{aligned}$$

Set $\alpha_L := a \underline{\otimes} 1,\ \beta_L := b \underline{\otimes} 1,\ \alpha_R := 1 \underline{\otimes} a,\ \beta_R := 1 \underline{\otimes} b$.

The H-braided join of H

$$H \underline{*} H = \left\{ \sum_{k,l,m,n=0}^{2} f_{klmn} \otimes \alpha_L^k \beta_L^l \alpha_R^m \beta_R^n \in C([0,1]) \otimes H \underline{\otimes} H \;\middle|\; \begin{array}{l} f_{klmn}(0)=0 \text{ for } (k,l)\neq(0,0), \\ f_{klmn}(1)=0 \text{ for } (m,n)\neq(0,0) \end{array} \right\}$$

is a finite quantum covering encapsuling the nontrivial $\mathbb{Z}/3\mathbb{Z}$-principal bundle $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ over $\Sigma(\mathbb{Z}/3\mathbb{Z})$.

Anti-Drinfeld doubles

Let H be a finite-dimensional Hopf algebra. The multiplication of the anti-Drinfeld double algebra $AD(H) := H^* \otimes H$ is

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)}))\, \varphi\, \varphi'_{(2)} \otimes h_{(2)}h'.$$

$D(H)$ is a Hopf algebra with $\Delta(\varphi \otimes h) = \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}$.
$AD(H)$-modules $\longleftrightarrow$ *anti-Yetter-Drinfeld modules over H*.

Theorem

Let H be a finite-dimensional Hopf algebra. Then the anti-Drinfeld double $AD(H)$ is a bicomodule algebra and a left and right Galois object over the Drinfeld double $D(H)$ for coactions, respectively,

$$\Delta(\psi \otimes k) = \psi_{(2)} \otimes S^2(k_{(1)}) \otimes \psi_{(1)} \otimes k_{(2)}\,,$$
$$\Delta(\varphi \otimes h) = \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}\,.$$

With H as before, $\dim D(H) = \dim AD(H) = 81$, and we get a neat example of $AD(H) \underline{*} AD(H)$ as a $D(H)$-bundle over $\Sigma D(H)$.

?

Are the semiclassical aspects of the above interesting ?

Thanks !

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JOIN !