A decomposition theorem for quantum groups

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(joint work with Matthew Daws)

From Poisson Brackest to Universal Quantum Symmetry
IMPAN, Warszawa

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G: a locally compact group $\mathcal{C}$: a category

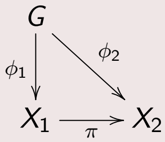
Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact group
2. $\phi : G \longrightarrow X$ is a continuous group homomorphism
3. $\phi(G)$ is dense in X

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Continuous group homomorphism $\pi : X_1 \to X_2$ such that



Initial object: $(\mathrm{Bohr(G)}, \phi_u)$ is Bohr compactification of G

Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact semi-topological semigroup
2. $\phi : G \longrightarrow X$ is a continuous semigroup homomorphism
3. $\phi(G)$ is dense in X

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

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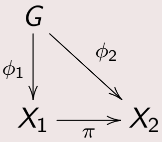


Initial object: (G^ω, ϕ_u) is W.A.P. compactification of G

G: a locally compact group $\mathcal{C}$: a category

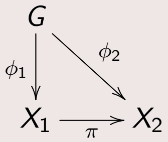
Obj $(\mathcal{C})$:

A pair (X, ϕ) where

1. X is a compact semi-topological (CH)-semigroup
2. $\phi : G \longrightarrow X$ is a continuous semigroup homomorphism
3. $\phi(G)$ is dense in X

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Continuous semigroup homomorphism $\pi : X_1 \to X_2$ such that



Initial object: $(\mathrm{E(G)}, \phi_u)$ is Eberlein compactification of G

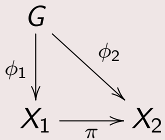
Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact quantum group
2. $\phi : G \longrightarrow X$ a quantum group homomorphism
3. $\phi(G)$ is dense in X (in a suitable sense)

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Quantum group homomorphism $\pi : X_1 \to X_2$ such that



Initial object is Quantum Bohr compactification of G (P. Sołtan)

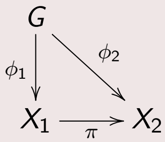
Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact quantum semi-topological (CH)-semigroup
2. $\phi : G \longrightarrow X$ a quantum semigroup homomorphism
3. $\phi(G)$ is dense in X (in a suitable sense)

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Quantum semigroup homomorphism $\pi : X_1 \to X_2$ such that



Do we still have an initial object?

Reduced locally compact quantum group G

Definition (A) (Woronowicz; 1996)

A C* bi-algebra $(C_0(G), \Delta)$ such that

1. There exists a *manageable multiplicative unitary* $W \in B(H \otimes H)$ with $C_0(G) = [(\iota \otimes \omega)(W) : \ \omega \in B(H)_*]$
2. $\Delta(a) = W^*(1 \otimes a)W$ for all $a \in C_0(G)$

Definition (B) (Kustermans & Vaes; 2000)

A C* algebra $C_0(G)$ and a non-degenrate *-homomorphism $\Delta : C_0(G) \longrightarrow M(C_0(G) \otimes C_0(G))$ satisfying:

1. $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$
2. $[\Delta(C_0(G))(1 \otimes C_0(G))] = C_0(G) = [\Delta(C_0(G))(C_0(G) \otimes 1)]$
3. There exists a faithful left-invariant approximate KMS weight ϕ on $(C_0(G), \Delta)$
4. There exists a right-invariant approximate KMS weight ψ on $(C_0(G), \Delta)$

Universal locally compact quantum group

G $\longrightarrow$ reduced locally compact quantum group ; S $\rightarrow$ the antipode

$L^1_*(\widehat{G}) := \{\omega \in L^1(\widehat{G}) \colon \overline{\omega} \circ S \subset f \text{ for some } f \in L^1(\widehat{G})\}$

Definition (Kustermans; 2001)

A C* bialgebra $(C_0^u(G), \Delta_u)$ such that

1. $C_0^u(G)$ is the universal enveloping C* algebra of $L^1_*(\widehat{G})$
2. $\Delta_u : C_0^u(G) \longrightarrow M(C_0^u(G) \otimes C_0^u(G))$ is a non-degenerate *-homomorphism
3. $(\Delta_u \otimes \iota) \circ \Delta_u = (\iota \otimes \Delta_u) \circ \Delta_u$
4. $(\Lambda_G \otimes \Lambda_G) \circ \Delta_u = \Delta \circ \Lambda_G$

where $\Lambda_G : C_0^u(G) \longrightarrow C_0(G)$ is the reducing morphism and Δ is the coproduct of $C_0(G)$

Compact quantum semi-topological (CH)-semigroup:

Definition:

(A, Φ, V, H) is called a *compact quantum (CH)-semigroup* where:

1. A: a unital C* algebra
   $\Phi : A \to A^{**}\overline{\otimes}A^{**}$: a unital *-homomorphism
   $V \in A^{**}\overline{\otimes}B(H)$: a contraction
   H: a Hilbert space
2. $A_V := \{(\iota \otimes \omega)(V) : \omega \in B(H)_*\} \subset A$ and is *norm-dense* in A
3. $(\Phi \otimes \iota) \circ \Phi = (\iota \otimes \Phi) \circ \Phi$ where Φ is “lifted"
4. $(\Phi \otimes \iota)(V) = V_{13}V_{23}$ where Φ is “lifted"

Compact quantum semi-topological (CH)-semigroup

Theorem:

1. Let (A, Φ, V, H) be a compact *quantum* semi-topological (CH)-semigroup with A abelian.

   Then $A = C(S)$ for a compact semi-topological (CH)-semigroup $S \subset B(H)_{\|\cdot\| \leq 1}$.

2. Let S be a compact semi-topological (CH)-semigroup acting on a Hilbert space H. For $s, t \in S$ let

   $$\Phi : C(S) \longrightarrow C(S)^{**}\overline{\otimes}C(S)^{**} : f \mapsto \Phi(f)(s,t) := f(s \cdot t).$$

   Then there exists a $V \in C(S)^{**}\overline{\otimes}B(H)$ with $\|V\| \leq 1$ such that
   $(C(S), \Phi, V, H)$ is a compact quantum semi-topological (CH)-semigroup.

Invariant means on (A, Φ, V, H)

Theorem (Existence of bi-invariant mean):

There exists a state $M : A \longrightarrow \mathbb{C}$

$$(\iota \otimes \widetilde{M})(\widetilde{\Phi}(a)) = (\widetilde{M} \otimes \iota)(\widetilde{\Phi}(a)) = M(a) \quad (a \in A)$$

where $\widetilde{M}$ and $\widetilde{\Phi}$ are the lifts of M and Φ to A^{**}.

G: a locally compact quantum group $\qquad \mathcal{C}$: a category

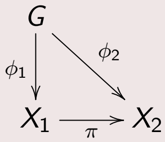
Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact quantum semi-topological (CH)-semigroup
2. $\phi : G \longrightarrow X$ a quantum semigroup homomorphism
3. $\phi(G)$ is dense in X (in a suitable sense)

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Quantum semigroup homomorphism $\pi : X_1 \to X_2$ such that



Morphisms: $S_1 \longrightarrow S_2$

$$S_1 := (A_1, \Phi_1, V_1, H_1) \; ; \; S_2 := (A_2, \Phi_2, V_2, H_2)$$

A morphism $\theta : S_1 \longrightarrow S_2$:

A unital *-homomorphism:

$$\theta : A_2 \longrightarrow A_1$$

satisfying

$$(\widetilde{\theta} \otimes \widetilde{\theta}) \circ \Phi_2 = \Phi_1 \circ \theta$$

where $\widetilde{\theta} : A_2^{**} \longrightarrow A_1^{**}$ is the lift of $\theta : A_2 \longrightarrow A_1$

Morphisms: G $\longrightarrow$ S

G is a locally compact *universal* *quantum* group

S is a compact quantum semi-topological (CH)-semigroup
$S := (A, \Phi, V, H)$

Morphism $\Theta : G \longrightarrow S$

A non-degenerate *-monomorphism

$$\Theta : A \longrightarrow M(C_0^u(G))$$

satisfying

1. $\Theta^* : C_0^u(G)^* \longrightarrow A^*$ is a homomorphism of Banach algebras
2. $(\widetilde{\Theta} \otimes \iota)(V)$ is a unitary representation of G

   The representation can be degenerate

Eberlein compactification of quantum group G

$\mathcal{C}$ is a category:

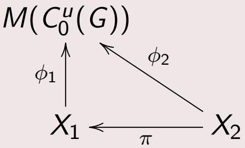
Obj ($\mathcal{C}$):

A pair (X, ϕ) where

1. X is a compact quantum semi-topological (CH)-semigroup
2. A morphism $\phi : G \longrightarrow X$
   ($\phi(G)$ is dense in X $\equiv$ ϕ is a C*-monomorphism)

Morphisms from (X_1, ϕ_1) to (X_2, ϕ_2):

Morphisms $\pi : X_1 \to X_2$ such that



There exists an initial object in this category (BD & Daws)

The initial object: $(E(G), \Phi, V_u, H_u)$

$(E(G), \Phi, V_u, H_u)$ is called the Eberlein compactification of G:

1. $(E(G), \Phi, V_u, H_u)$ is a compact quantum semi-topological (CH)-semigroup
2. In general $E(G) \subsetneq B(G)$; $B(G)$: Fourier-Stieltjes algebra of G
   e.g. this is the case for G $:= \widehat{SU_q(2)}$
3. If G is Kac-type then $E(G) = B(G)$
4. $V_u \in M(C_0^u(G) \otimes B_0(H_u))$ is a unitary representation of G
5. G is classical $\implies$ $E(G)$ is the continuous function algebra over the Eberlein compactification of G

Bohr compactification vs Eberlein compactification

de Leeuw & Glicksberg (1961); Spronk & Stokke (2012)

Let G be a locally compact group.

- Bohr(G) is the Bohr compactification of G
- E(G) is the continuous function algebra over the Eberlein compactification of G

Then

$$\mathbf{E(G)} = \mathbf{C(Bohr(G))} \oplus \mathcal{I}_{\mathbf{0}} \quad \text{(as Banach spaces)}$$

where $\mathcal{I}_0$ is the kernel of the bi-invariant mean of $E(G)$.

Quantum Bohr compactification vs Quantum Eberlein compactification

- G is a locally compact quantum group.
- E(G) is the Eberlein compactification of G
- $C(G^{\mathrm{SAP}})$ is the reduced version of AP(G)

Theorem-(A):

If G is of Kac-type, then

- The bi-invariant mean M on E(G) is tracial
- Let $\mathcal{I}_0$ be the kernel of M. Then $E(G)/\mathcal{I}_0$ can be given a compact quantum group structure
- The compact quantum group $E(G)/\mathcal{I}_0$ is quantum group isomorphic to $C(G^{\mathrm{SAP}})$

Theorem-(A): (contd.)

Quantum Bohr compactification vs Quantum Eberlein compactification

- G is a locally compact quantum group of Kac-type
- E(G) is the Eberlein compactification of G
- $\mathcal{I}_0$ is the kernel of the bi-invariant mean on E(G)
- $C(G^{\mathsf{SAP}})$ is the reduced version of AP(G)

Decomposition Theorem for quantum groups:

If AP(G) is reduced then:

$$\mathbf{E(G)} = \mathbf{AP(G)} \ \oplus \ \mathcal{I}_{\mathbf{0}}$$

e.g.

- If G is a classical locally compact group
  (de Leeuw & Glicksberg decomposition theorem)

- If G is the dual of a classical locally compact group H such that H_d (H with discrete topology) is amenable

THANK YOU FOR YOUR ATTENTION

Dziękuję za uwagę