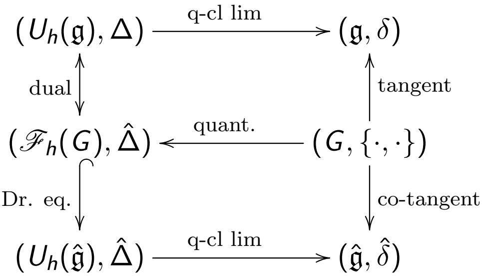
# Poisson-Lie groupoids and the contraction procedure

Kenny De Commer

Free University Brussels

20 August 2014

# Lie bialgebras and quantizations



# Example: Lie algebra level

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$

- Generators
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
- Relations
$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = H.$$
- Cobracket
$$\delta(H) = 0, \qquad \delta(X_\pm) = iX_\pm \wedge H.$$
- QUEA $U_h(\mathfrak{g})$:
$$[H, X_\pm] = \pm 2X_\pm, \qquad [X_+, X_-] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}},$$
Coproduct
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(X_\pm) = X_\pm \otimes e^{\frac{1}{2}hH} + e^{-\frac{1}{2}hH} \otimes X_\pm.$$
- Compact real structure $\mathfrak{su}(2)$: $H^* = H, \quad X_+^* = X_-$.

# Example: Lie group level

- $G = SL(2,\mathbb{C})$,

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\}.$$

- Poisson structure:

$$\{a,b\} = iab, \qquad \{a,d\} = 2ibc, \qquad \{b,c\} = 0, \qquad \ldots$$

- QFSHA $\mathscr{F}_h(G)$:

$$ab = e^h ba, \qquad ad - e^h bc = da - e^{-h} cb = 1, \qquad bc = cb, \qquad \ldots$$

- Compact real form: $G_c = SU(2)$:

$$a^* = d, \quad b^* = -c.$$

# Example: Dual Lie bialgebra

$\hat{\mathfrak{g}} = \mathbb{C} \ltimes (\mathbb{C} \oplus \mathbb{C})$

- Generators

$$\hat{H} = \left(\begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2}\end{pmatrix}, \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2}\end{pmatrix}\right), \quad \hat{X}_+ = \left(\begin{pmatrix} 0 & 0 \\ i & 0\end{pmatrix}, 0\right), \quad \hat{X}_- = \left(0, \begin{pmatrix} 0 & 0 \\ i & 0\end{pmatrix}\right)$$

- Relations

$$[\hat{H}, \hat{X}_\pm] = -i\hat{X}_\pm, \qquad [\hat{X}_+, \hat{X}_-] = 0$$

- Cobracket

$$\hat{\delta}(\hat{H}) = \hat{X}_+ \wedge \hat{X}_-, \quad \hat{\delta}(\hat{X}_\pm) = \pm 2\hat{H} \wedge \hat{X}_\pm.$$

- QUE algebra $U(\hat{\mathfrak{g}})[[h]] = U_h(\hat{\mathfrak{g}}) \supseteq \mathscr{F}_h(G)$

$$\begin{aligned} a &= e^{ih\hat{H}}(1 - h^2 e^{-h}\hat{X}_+\hat{X}_-)^{1/2}, & b &= ihe^{-h/2}\hat{X}_+, \\ d &= (1 - h^2 e^{-h}\hat{X}_+\hat{X}_-)^{1/2} e^{-ih\hat{H}}, & c &= ihe^{h/2}\hat{X}_-. \end{aligned}$$

- Dual real form: $\hat{H}^* = \hat{H}, \quad \hat{X}_+^* = \hat{X}_-.$

# Example: Dual Lie group

- $\hat{G} = \mathbb{C}^\times \ltimes (\mathbb{C} \oplus \mathbb{C})$,

$$\left\{\left(\begin{pmatrix} A & 0 \\ B & A^{-1}\end{pmatrix}, \begin{pmatrix} A & 0 \\ C & A^{-1}\end{pmatrix}\right) \mid A \in \mathbb{C}^\times, B, C \in \mathbb{C}\right\}.$$

- Poisson structure

$$\{A, B\} = iAB, \quad \{A, C\} = -iAC, \quad \{B, C\} = \frac{i}{2}(A^2 - A^{-2}).$$

- QFSHA $\mathscr{F}_h(\hat{G}) \subseteq U_h(\mathfrak{g})$:

$$A = e^{\frac{1}{2}hH}, \quad B = h\left(\frac{e^h - e^{-h}}{2h}\right)^{1/2} X_+, \quad C = h\left(\frac{e^h - e^{-h}}{2h}\right)^{1/2} X_-.$$

- Dual real form: $\hat{G}_c = \mathbb{R}_+^\times \ltimes \mathbb{C}$,

$$A^* = A, \quad B^* = C.$$

# From Lie groups to Lie groupoids

**Definition**

*Lie groupoid*: smooth manifolds $\mathscr{O}, \mathscr{G}$ with groupoid structure $\mathscr{G} \underset{t}{\overset{s}{\rightrightarrows}} \mathscr{O}$ s.t. $s, t$ submersions and structure maps smooth.

**Definition**

*Lie algebroid*: vector bundle $A$ over $\mathscr{O}$ with *anchor map* $\rho : A \to T\mathscr{O}$ and bracket on $\Gamma(A)$ s.t. $[\alpha, \cdot]_A$ satisfy Leibniz rule w.r.t. $\rho(\alpha)$.

$$(\mathscr{O}, \mathscr{G}, s, t) \qquad \Rightarrow \qquad A = \ker(\mathrm{d}s)_{|\mathscr{O}}, \quad \rho = \mathrm{d}t_{|A}.$$

# Lie bialgebroids

## Definition

$\mathscr{G}$ is *Poisson groupoid* if $\mathscr{G}$ is Poisson s.t. $\mathrm{Graph}(\mathrm{mult})$ co-isotropic in $\Gamma \times \Gamma \times \bar{\Gamma}$.

E.g. $G$ Poisson-Lie group: $\mathrm{Graph}(\mathrm{mult}) = \{(g, h, gh) \mid g, h \in G\}$,

$$\begin{aligned}\{(\Delta(f) \otimes 1 - 1 \otimes 1 \otimes f), (\Delta(g) \otimes 1 - 1 \otimes 1 \otimes g)\} \\ = \{\Delta(f), \Delta(g)\} \otimes 1 - 1 \otimes 1 \otimes \{f, g\} \\ = \Delta(\{f, g\}) \otimes 1 - 1 \otimes 1 \otimes \{f, g\}.\end{aligned}$$

## Definition

$(A, \mathscr{O})$ is a Lie *bialgebroid* if also $(A^*, \mathscr{O})$ is Lie algebroid s.t.

$$\delta : \Gamma(A) \to \Gamma(\Lambda^2(A)), \quad \delta(X)(\xi, \eta) = X([\xi, \eta]) - \rho_*(\xi)(X(\eta)) + \rho_*(\eta)(X(\xi))$$

satisfies cocycle condition.

# Example: dynamical $r$-matrix

Data:

- $\mathfrak{g}$ complex Lie algebra
- $\mathfrak{h} \subseteq \mathfrak{g}$ abelian
- Basis $\{h_i\} \subseteq \mathfrak{h}$, dual basis $\{\lambda_i\} \subseteq \mathfrak{h}^*$
- $r : U \subseteq \mathfrak{h}^* \to \Lambda^2(\mathfrak{g})$ s.t. $\sum_i h_i \wedge \frac{\mathrm{d}r}{\mathrm{d}\lambda_i} + \frac{1}{2}[r, r]$ constant in $(\Lambda^3(\mathfrak{g}))^{\mathfrak{g}}$.

Then Lie bialgebroid

$$\mathscr{O} = \mathfrak{h}^*, \qquad A = \mathfrak{g} \oplus T\mathfrak{h}^* \underset{\rho}{\to} T\mathfrak{h}^*, \qquad \delta(\xi) = [r, \xi].$$

# Example: dynamical $SL(2, \mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

- $\mathfrak{h} = \mathbb{C}$ Cartan and

$$r(y) = -\frac{1}{2}\coth(y/2) X_+ \wedge X_-.$$

- $\mathscr{G} = \mathbb{C} \times SL(2, \mathbb{C}) \times \mathbb{C}$.
- Poisson structure

$$\{a, b\} = i \tanh(y) ab, \quad \{a, d\} = i(\tanh(x) + \tanh(y))(ad - 1)$$

$$\{f, a\} = i\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) a, \quad \{f, b\} = i\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) b.$$

# Quantizations

- Lie bialgebra/Poisson-Lie group $\Rightarrow$ Hopf algebra
- Lie bialgebroid/Poisson groupoid $\Rightarrow$ Hopf algebroid
- ↳ with zero bracket on object space $\Rightarrow$ Hopf face algebra

# Hopf face algebras

Data:

- Index set $J$,
- Vector spaces ${}^{x}_{w}A^{y}_{z}$, indices in $J$,
- Partial multiplication and units,

$$M : {}^{x}_{w}A^{y}_{z} \otimes {}^{y}_{z}A^{u}_{v} \to {}^{x}_{w}A^{u}_{v}, \qquad \mathbf{1}\binom{x}{y} \in {}^{x}_{y}A^{x}_{y}.$$

- Partial comultiplication and counits,

$$\Delta_{uv} : {}^{x}_{w}A^{y}_{z} \to {}^{x}_{u}A^{y}_{v} \otimes {}^{u}_{w}A^{v}_{z}, \qquad \epsilon \in \left({}^{x}_{x}A^{y}_{y}\right)^{*}$$

- Antipode

$$S : {}^{x}_{w}A^{y}_{z} \to {}^{z}_{y}A^{w}_{x}$$

s.t. e.g.

$$\Delta_{uv}(xy) = \int_t \Delta_{ut}(x)\Delta_{tv}(y)\mathrm{d}t.$$

# Example: dynamical quantum SU(2)

Dynamical quantum $SL(2,\mathbb{C})$

- Index set $J$: real numbers '+ infinitesimals'.
- Generators:

$$\alpha_{xy} \in {}^{x}_{y}A^{x-h}_{y-h}, \quad \beta_{xy} \in {}^{x}_{y}A^{x-h}_{y+h}, \quad \ldots$$

- Relations:

$$\alpha_{xy}\beta_{x-h,y-h} = \left(\frac{\cosh(y+h)}{\cosh(y-h)}\right)^{1/2} \beta_{xy}\alpha_{x-h,y+h}, \quad \ldots$$

- Comultiplication:

$$\Delta_{z,z-h}(\alpha_{xy}) = \alpha_{xz} \otimes \alpha_{zy}, \quad \Delta_{z,z+h}(\alpha_{xy}) = \beta_{xz} \otimes \gamma_{zy}.$$

# Example: Contraction

Contraction quantum $\widehat{G} = \widehat{SL(2, C)}$.

- Index set $J$: complex numbers.
- $$ {}_y^x A_z^w = \delta_{x,w}\delta_{y,z}\mathscr{F}_h(\widehat{SL(2,\mathbb{C})}; x, y). $$
- Relations:
$$ [B_{xy}, C_{xy}] = h(xA_{xy}^2 - yA_{xy}^{-2}) $$
- Coproduct:
$$ \Delta_{zz}(A_{xy}) = A_{xz} \otimes A_{zy}, \quad \Delta_{zz}(B_{xy}) = B_{xz} \otimes A_{zy} + A_{xz}^{-1} \otimes B_{zy}, \quad \ldots $$
- Real form: index set $J = \mathbb{R}$,
$$ A_{xy}^* = A_{xy}, \quad B_{xy}^* = C_{xy}. $$

# Associated contraction Poisson groupoid

Contraction $\widehat{G} = \widehat{SL(2,\mathbb{C})}$ groupoid

- $\widehat{\mathscr{G}} = \mathbb{C} \times \widehat{G} \times \mathbb{C}$.
- Poisson structure: $\{f,g\} = \{f,A\} = \{f,B\} = \{f,C\} = 0$ and

$$\{A,B\} = iAB, \quad \{A,C\} = -iAC, \quad \{B,C\} = \frac{i}{2}(xA^2 - yA^{-2}).$$

- $\Rightarrow$ Poisson manifolds $(\widehat{G}, \{\cdot,\cdot\}_{x,y})$ with Poisson multiplication

$$(\widehat{G}, \{\cdot,\cdot\}_{x,y}) \times (\widehat{G}, \{\cdot,\cdot\}_{y,z}) \to (\widehat{G}, \{\cdot,\cdot\}_{x,z}).$$

- Real structure:

$$x^* = x, \quad y^* = y, \quad A^* = A, \quad B^* = C.$$

# Isotropy groups and contraction

- Each $(\widehat{SL(2,\mathbb{C})}, \{\cdot,\cdot\}_{y,y})$ is Poisson group.
- Dual Poisson groups?
  - If $y \neq 0$: $SL(2,\mathbb{C})$.
  - If $y = 0$: $E_{\mathbb{C}}(2) = \{\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\right) \mid a \in \mathbb{C}^{\times}, b, c \in \mathbb{C}\}$.
- W.r.t. real form:
  - If $y > 0$: $SU(2)$.
  - If $y = 0$: $E(2) = U(1) \ltimes \mathbb{C} = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in U(1), b \in \mathbb{C}\}$.
  - If $y < 0$: $SU(1,1)$.

# Contraction Lie bialgebroid

Lie bialgebroid associated to $\widehat{\mathscr{G}} = \mathbb{C} \times \widehat{SL(2,\mathbb{C})} \times \mathbb{C}$.

- $\hat{\mathscr{A}} = \widehat{\mathfrak{sl}(2,\mathbb{C})} \oplus T\mathbb{C} \underset{\hat{\rho}}{\to} T\mathbb{C}$ over $\mathscr{O} = \mathbb{C}$.
- Direct sum Lie algebroid structure.
- Cobracket: with $\hat{E} = 2i\frac{\partial}{\partial y}$ we have $\hat{\delta}(y) = 0$ and

$$\hat{\delta}(\hat{X}_\pm) = \pm 2\hat{H} \wedge \hat{X}_\pm, \quad \hat{\delta}(\hat{H}) = y\hat{X}_+ \wedge \hat{X}_-, \quad \hat{\delta}(\hat{E}) = \hat{X}_+ \wedge \hat{X}_-.$$

- Real form: take object set $\mathbb{R}$ and

$$\hat{E}^* = \hat{E}, \quad \hat{H}^* = \hat{H}, \quad \hat{X}_\pm^* = \hat{X}_\mp$$

# Dual contraction Lie bialgebroid

$V$ = vector space generated by $H, X_{\pm}, E$.

- $\mathscr{A} = V \times \mathbb{C} \to \mathbb{C}$ trivial vector bundle.
- Anchor map $\rho = 0$.
- Relations

$$[H, X_{\pm}] = \pm 2X_{\pm}, \quad [X_+, X_-] = yH + E, \quad E \text{ central.}$$

$\Rightarrow$ Bundle of Lie algebras.

- Cobracket:

$$\delta(H) = 0, \quad \delta(X_{\pm}) = -iH \wedge X_{\pm}, \quad \delta(E) = 0, \quad \delta(y) = 2iE.$$

# Dual contraction Poisson groupoid

- $\tilde{G}_y = \{\left(\begin{pmatrix} a & b \\ yc & d \end{pmatrix}, \begin{pmatrix} a & yb \\ c & d \end{pmatrix}\right)\} \subseteq SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$.
- $G = \{(y,\theta,g) \mid y \in \mathbb{C}, \theta \in \mathbb{C}, g \in \tilde{G}_y\}$ with

$$(y,\theta,g)\cdot(y,\theta',g') = (y,\theta+\theta'+\Omega_y(g,g'),gg')$$

w.r.t. cocycle

$$\Omega_y(g,g') = \frac{1}{y}\ln\left(\frac{a(gg')}{a(g)a(g')}\right) = \frac{1}{y}\ln\left(1+y\frac{bc'}{aa'}\right), \quad y \neq 0,$$

$$\Omega_0(g,g') = \frac{bc'}{aa'}.$$

- Poisson structure:

$$\{a,b\} = iab, \quad \{a,d\} = 2iybc, \quad \{b,c\} = 0, \ldots$$

$$\{y,-\} = 2i\frac{\partial}{\partial\theta}, \quad \{\theta,-\} = -2i\frac{\partial}{\partial y}.$$

# Contraction of representations

$\mathrm{Hol}_0(\mathbb{C})$ = space of germs of holomorphic functions around zero.

- Infinitesimal representation on $\mathrm{Hol}_0(\mathbb{C})$:

$$\begin{aligned} (X_+ f)(z) &= zf(z) + yz^2 f'(z), \\ (X_- f)(z) &= -f'(z), \\ (Hf)(z) &= 2zf'(z), \\ (Ef)(z) &= f(z). \end{aligned}$$

- Locally integrated representation:

$$(\pi_{(y,\theta,g)} f)(z) = e^{-\theta}(1 + yb(az - c))^{-1/y} f\left(-\frac{az - c}{ybz - d}\right).$$