# Groups Actions on Deformation Quantization

Niek de Kleijn

August 24, 2014

## 1 Introduction

In these notes we will take a look at the extension of symplectic group action to deformation quantizations. This is mainly motivated for two reasons. The first is that classification of such extensions of group action, particularly in cohomological form might lead to explicit formulas for algebraic index theorems for deformation quantization in the presence of group actions. The second is related to our idea of the extension of a group action and the construction of deformation quantization applied in these notes. Fedosov notes in the paper in which he introduce his simple geometric construction for formal deformation quantization of symplectic manifolds[1] that one can in this construction always find a way to relate symplectic automorphisms of the manifold to automorphisms of a deformation quantization. We do this by simply composing the push-forward with a certain inner automorphism (which need not be unique). We see this as the extension of the symplectic automorphism. In the case of a group action we consider the case when all the autmorphisms defined by the elements of the group are extended while preserving the group structure an extension of the group action. The motivation is that although Fedosov noted this idea he also notes that there is no guarantee such extensions exist. We will use mainly the conventions and some results as laid out in [2].

## 2 The Fedosov Construction

### 2.1 Weyl Bundle

The fedosov construction is a geometric construction for deformation quantizations of symplectic manifolds. It identifies these deformations as subalgebras of the sections of a bundle of algebras called the Weyl bundle.

**Definition 2.1** (Weyl Functor).
*Denote by* $\mathbb{W} : SymVec_{\mathbb{C}} \to GrAss_{\mathbb{C}}$ *the Weyl functor from the category of complex symplectic vector spaces to the category of complex graded associa-*

*tive algebras given by*

$$\mathbb{W}(V,\omega) = \widehat{(TV)[\![\hbar]\!]}/I,$$

*where the ideal $I \subset TV[\![\hbar]\!]$ is generated by the elements*

$$v \otimes w - w \otimes v - i\hbar\omega(v,w)$$

*for $v, w \in V$ and the hat signifies completion in the $\langle V \rangle$-adic topology. The grading is given by the assertion that $|\hbar| = 2$ and $|v| = 1$ for all $v \in V$.*

Since taking the tensor algebra, formal power series etc. are all covariant functors so is $\mathbb{W}$. Note especially that the ideals corresponding to different symplectic spaces map into each other, since symplectic maps preserve the symplectic form. We will call the algebra $\mathbb{W}(V,\omega)$ the Weyl algebra associated to $(V,\omega)$. In fact a more general definition is possible by replacing $\mathbb{C}$ with a field containing the square root of $-1$. In the complex case the existence of symplectic bases shows that the isomorphism classes of (finite dimensional) symplectic vector spaces are classified by $2\mathbb{N}$ in terms of the dimension. Consider then the symplectic vector space $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_{st} = \sum_{i=1}^{n} \xi^i \wedge x^i$ corresponding to the symplectic basis $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ (where the subscripts imply duality). By complexification we obtain the symplectic vector space $(\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}, \omega := \omega_{st} \otimes_{\mathbb{R}} 1)$. In the following we will simply denote $\mathbb{W}_{\hbar,n} = \mathbb{W}(\mathbb{C}^{2n},\omega)$ and even omit the $n$ if it is implied (for instance by dimension of the manifold).

Now let us apply this construction in the case of a symplectic manifold $(M,\omega)$ of dimension $2n$. Recall that $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and $\omega_x : T_xM \otimes T_xM \to \mathbb{R}$ is a symplectic form. This means in particular that $\omega_x$ is non-degenerate for all $x \in M$ and therefore defines an isomorphism $I_{\omega_x} : T^*_xM \to T_xM$ for every $x \in M$, which we can group into the isomorphism $I_\omega : T^*M \to TM$. Thus we obtain the corresponding symplectic vector spaces $(T^*_xM, \bar{\omega}_x)$ for each $x \in M$, where $\bar{\omega} = (I_\omega \otimes I_\omega)^*\omega$.

**Lemma 2.2.**
*There exists a system of coordinate neighborhoods $\mathcal{U}$ on $M$ such that the corresponding trivializations of $TM$ become symplectomorphisms. Here, for every $U \in \mathcal{U}$, we consider the standard symplectic structure $(U \times \mathbb{R}^{2n}, \omega_{st})$.*

*Proof.*
Recall that for every $x \in M$ there exists a Darboux coordinate neighborhood $(U_x, q^1, \ldots, q^n, p^1, \ldots, p^n)$ such that $\frac{\partial}{\partial q^1}|_y, \ldots, \frac{\partial}{\partial q^n}|_y, \frac{\partial}{\partial p^1}|_y, \ldots, \frac{\partial}{\partial p^n}|_y$ forms a symplectic basis for every $y \in U_x$. Then define

$$\phi_x : (TU_x, \omega|_{U_x}) \to (U_x \times \mathbb{R}^{2n}, \omega_{st})$$

by

$$\phi_x(\sum_{i=1}^{n} a^i \frac{\partial}{\partial q^i}|_y + b^i \frac{\partial}{\partial p^i}|_y) = (y, \sum_{i=1}^{n} a^i x_i + b^i \xi_i).$$

Since this map sends a symplectic basis to a symplectic basis we see that the $\phi_x$ are indeed symplectomorphisms. $\square$

An immediate corollary of the lemma is that the transition functions of $TM$ take values in $Sp(2n, \mathbb{R}) \subset GL(2n, \mathbb{R})$. Therefore the same is true when we consider $T^*M$ instead of $TM$, so by functoriality of complexification and $\mathbb{W}$ we find the bundle of algebras $\mathcal{W}$ with fibers isomorphic to $\mathbb{W}_\hbar$. Where the vector bundle structure is given by the one on $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ (here we use the lemma to assert that all the necessary maps are in the category of symplectic vector spaces and thus have counterparts in the category of graded algebras).

**Definition 2.3.**
*For $(M, \omega)$ a symplectic manifold we call the bundle $\mathcal{W}$, with fiber $\mathbb{W}(T_x^*M \otimes_{\mathbb{R}} \mathbb{C}, \omega_x \otimes 1)$ at $x \in M$, the Weyl bundle on $M$.*

We will now use the Fedosov construction to identify subalgebras of the sections of the Weyl bundle that are isomorphic to formal deformation quantizations. In the Fedosov construction this subalgebra is obtained as the kernel of a particular kind of connection. Then let us introduce some notation for relevant objects and also the only constant "part" of the connection. First of all let us be precise about what we mean by a connection on the Weyl bundle.

**Definition 2.4.**
*A connection* $\nabla$ *on* $\mathcal{W} \to M$ *is a* $\mathbb{C}[\![\hbar]\!]$*-linear map* $\Gamma(\mathcal{W}) \to \Gamma(\mathcal{W}) \otimes \Omega^1(M)$ *that respects the algebra structure of* $\mathcal{W}$*. That is we have*

$$\nabla(\tau\sigma) = \sigma\nabla\tau + (\nabla\sigma)\tau \quad \forall \sigma, \tau \in \Gamma(\mathcal{W})$$

*and*

$$\nabla f = df \quad \forall f \in C^\infty(M) \subset \Gamma(\mathcal{W}).$$

Thus we see that a connection is indeed a map from vector fields into derivations of the algebra $\Gamma(\mathcal{W})$. That is to say for each point $x \in M$ and vector $X \in T_xM$ $\nabla$ assign a derivation of the Weyl-algebra $\mathbb{W}_\hbar$ with smoothness and linearity conditions. Thus let us denote this last Lie algebra of derivations by $\mathfrak{g}$. Now we can use the following result about the standard notion of Weyl-algebra.

**Lemma 2.5.**
*Denote by* $\mathbb{A}_n$ *the algebra over* $\mathbb{C}$ *given by the following construction (the resulting algebra is usually called the nth Weyl algebra over* $\mathbb{C}$*). First let* $V = \langle x^1, \ldots, x^n, y^1, \ldots, y^n \rangle$ *and let* $TV$ *be the tensor algebra generated by* $V$*. Then impose on* $TV$ *the relations* $[x^i, x^j] = [y^i, y^j] = 0$ *and* $[y^i, x^j] = \delta_{ij}$ *for* $i, j = 1, \ldots, n$*, where the brackets denote the commutator and* $\delta_{ij}$ *denotes the Kronecker delta. Then all derivations of* $\mathbb{A}_n$ *are inner for all* $n \in \mathbb{N}$*.*

We will omit the proof because it is quite long, but it comes down to observing a sort of formal integration of polynomials in the sense that we may identify $[y^i, -]$ as $\partial_{x^i}$ Now it is easy to see that this lemma carries over in slightly altered form for $\mathbb{W}_\hbar$ (because of the $\hbar$ and grading9). Denoting by $\tilde{\mathfrak{g}}$ the Lie algebra given by $\frac{1}{\hbar}\mathbb{W}_\hbar$ under commutation (the $\hbar$ will cancel against the $\hbar$ in the commutator) we can summarize the result in the statement that the sequence

$$\begin{array}{ccccccccc} 0 & \to & \frac{1}{\hbar}\mathcal{Z} & \hookrightarrow & \tilde{\mathfrak{g}} & \to & \mathfrak{g} & \to & 0 \\ & & & & \frac{1}{\hbar}\Phi & \mapsto & \frac{1}{\hbar}[\Phi, -] & & \end{array}$$

is short exact, where $\mathcal{Z} = \mathbb{C}[\![\hbar]\!]$ denotes the center of $\mathbb{W}_\hbar$. Note that we retain still the grading on $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ (where the lowest order in the first is $-2$ and in the second $-1$ since the scalars commute). We will denote the homogeneous elements of degree $i$ in $\mathfrak{g}$ or $\tilde{\mathfrak{g}}$ by $\mathfrak{g}_i$ and $\tilde{\mathfrak{g}}_i$ respectively.

There is one last standard element of the symplectic geometry. Namely the canonical form $A_{-1} : \Gamma(TM) \to \Gamma(T^*M) \subset \Gamma(\mathcal{W})$, given as

$$A_{-1}(v)|_x = \omega_x(v, -)$$

for all $v \in TM$ and $x \in M$.

## 2.2 Fedosov connection

The main object of the Fedosov construction is the notion of Fedosov connection which determines the deformation quantization in terms of flat sections in the Weyl-bundle.

**Definition 2.6.**
*We call a flat connection $\nabla : \Gamma(\mathcal{W}) \to \Gamma(\mathcal{W}) \otimes \Omega^1(M)$ a Fedosov connection if it is of the form*

$$\nabla = \frac{1}{\hbar}[A_{-1}, -] + \nabla_0 + \frac{1}{\hbar}[A_1, -] + \frac{1}{\hbar}[A_2, -] + \ldots,$$

*where $\nabla_0$ denotes the lift of a symplectic connection and $A_i$ is a one-form with values in the degree $i+2$ of section of the Weyl bundle for each $i$.*

Note that we can view a symplectic connection as taking values in derivations of $\Gamma(T^*M) \subset \Gamma(\mathcal{W})$, we mean the lift in this sense, and so $\nabla_0$ takes values in $\mathfrak{g}_0$. Thus we see that every summand in the Fedosov connection takes values in the degree of it's subscript that is $\frac{1}{\hbar}[A_i, -]$ takes values in $\mathfrak{g}_i$ for each $i \in \mathbb{N} \cup \{-1\}$ and $\nabla_0$ takes values in $\mathfrak{g}_0$.

As mentioned a Fedosov connection allows us to construct and classify the deformation quantizations associated to $(M, \omega)$ as follows.

**Theorem 2.7.** *(R. Nest, B. Tsygan; 1999)*
*Suppose $\theta \in (i\hbar)^{-1}\omega + \Omega^2(M, \mathbb{C}[\![\hbar]\!])$ with $d\theta = 0$ and $\nabla_0$ any lift of a symplectic connection as above, then there is a $\tilde{\mathfrak{g}}$ valued one form $\frac{1}{\hbar}A_\theta$ such that $\nabla_\theta = \nabla_0 + \frac{1}{\hbar}[A_\theta, -]$ is a Fedosov connection on $M$ and $\tilde{\nabla}_\theta^2 = \theta$, where $\tilde{\nabla}_\theta$ is any lift to $\tilde{\mathfrak{g}}$. Furthermore* $\operatorname{Ker} \nabla_\theta \simeq C^\infty(M)[\![\hbar]\!]$ *as vector spaces and promoting this isomorphism to an algebra isomorphism defines a star-product. Overmore if $[\theta] = [\theta'] \in (i\hbar)^{-1}\omega + H^2(M, \mathbb{C}[\![\hbar]\!])$ we have* $\operatorname{Ker} \nabla_\theta \simeq \operatorname{Ker} \nabla_{\theta'}$.

We will omit the proof here, but the idea is to construct $A_\theta$ starting with $A_{-1}$ and continuing by induction on the degree. Note also that we could write $\frac{1}{2}[\tilde{\nabla}_\theta, \tilde{\nabla}_\theta] = \tilde{\nabla}_\theta^2$ and that the assertion is that this is a *scalar* valued one form. So we see that the deformation quantizations of $(M, \omega)$ correspond exactly to the Fedosov connections and that they are classified by classes in $(i\hbar)^{-1}\omega + H^2(M, \mathbb{C}[\![\hbar]\!])$. Here the $(i\hbar)^{-1}\omega$ term shows up since $\frac{1}{\hbar^2}[A_{-1}, A_{-1}] = (i\hbar)^{-1}\omega$.

Now let us determine how Fedosov connections corresponding to the same deformation quantization are related. So let $\nabla$ and $\nabla'$ be Fedosov connections such that $\operatorname{Ker} \nabla \simeq \operatorname{Ker} \nabla'$ then we have corresponding lifts and forms $\theta = \tilde{\nabla}^2$ and $\theta' = \tilde{\nabla}'^2$, with $[\theta] = [\theta']$. Then suppose $\theta' = \theta + d\alpha$ for some (scalar valued) one-form $\alpha$. Then we might consider the Fedosov connection $\nabla + [\alpha, -]$ (flatness is immediate since $\alpha$ is scalar valued). Then we see that $\tilde{\nabla} + \alpha$ lifts this connection and $\mathrm{Ker}(\nabla + [\alpha, -]) = \mathrm{Ker}\,\nabla$, but we also have $(\tilde{\nabla} + \alpha)^2 = \theta + d\alpha = \theta'$. Thus we might as well assume that $\tilde{\nabla}^2 = \tilde{\nabla}'^2$ as forms. Then we find by induction on the degree and the Baker-Campbell-Hausdorff formula and acyclicity of a certain complex[2] that there exists an element $\phi \in \Gamma(\mathcal{W})$ such that $\nabla' = \mathrm{Ad}\, e^{\phi} \circ \nabla \circ \mathrm{Ad}\, e^{-\phi}$. So we may conclude that for two Fedosov connections as above we have in general that $\nabla' = \mathrm{Ad}\, e^{\phi} \circ \nabla \circ \mathrm{Ad}\, e^{-\phi} + [\alpha, -]$ for some scalar valued one-form $\alpha$ and a section $\phi$ of the Weyl bundle. Note also that this induces the automorphism $\mathrm{Ad}\, e^{\phi} \in \mathrm{Aut}(\Gamma(\mathcal{W}))$ which is an isomorphism of the kernels, sending the $\nabla$ flat section $\sigma$ to $e^{\phi}\sigma e^{-\phi}$ which are $\nabla'$ flat.

# 3 Extending automorphisms of $(M, \omega)$ to algebra automorphisms

Now let us turn to the matter of symplectic group actions on the manifold. We would like to know whether and in what way such actions persist in the deformation quantization. Let us then fix a deformation quantization, i.e. a Fedosov connection and denote the kernel $\mathbb{A}_\hbar$ when this is convenient. Obviously the first step is to find out how a general symplectomorphism (by symplectomorphism we will always mean a symplectic automorphism of the manifold) can induce an automorphism of the deformation quantization. In the language above this means that given a deformation quantization, i.e. a Fedosov connection $\nabla$, and a symplectomorphism $\gamma : (M, \omega) \to (M, \omega)$ we would like to find an induced map $A_\gamma$ on $\Gamma(\mathcal{W})$ which gives an automorphism of $\mathrm{Ker}\,\nabla$ in a sensible way.

Note first of all that by functoriality of the Weyl-bundle construction the pull-back $\gamma^*$ and push-forward $\gamma_* = (\gamma^*)^{-1}$ are automorphisms of $\Gamma(\mathcal{W})$.

**Lemma 3.1.**
*Identifying for the moment* $\mathrm{Ker}\,\nabla = (C^\infty(M)[\![\hbar]\!], \star)$, *the symplectomorphism induces another equivalent star-product* $\bullet_\gamma$ *by the rule*

$$f \bullet_\gamma g = (\gamma^{-1})^*(\gamma^*(f) \star \gamma^*(g)).$$

*Proof.*
The lemma follows immediately from the fact that $\gamma^*\omega = \omega$ and that pullbacks under an automorphism of differential operators are still differential operators. Note that the isomorphism in the lemma is given simply by $\gamma_*$. $\square$

Note also that on the level of the sections of the Weyl bundle the isomorphism in the lemma is deduced from the connection $\gamma^*\nabla$ given by $\gamma^*\nabla\sigma = \gamma^*(\nabla\gamma_*\sigma)$ for all $\sigma \in \Gamma(\mathcal{W})$, where we follow the convention that $\gamma^*\alpha(X) = \alpha(\gamma_* X)$ for all one-forms $\alpha$ and vector fields $X$. So we have then the deformation quantizations $\operatorname{Ker}\nabla$ and $\operatorname{Ker}\gamma^*\nabla$ and the isomorphism $\gamma^*$ between them.

**Lemma 3.2.**
*$\gamma^*\nabla$ is a Fedosov connection*

*Proof.*
Explicitely we have $\nabla = \frac{1}{\hbar}[A_{-1}, -] + \nabla_0 + \frac{1}{\hbar}[A, -]$ as above. But then

$$\gamma^*\nabla\sigma = \frac{1}{\hbar}\gamma^*([A_{-1}, \gamma_*\sigma] + \gamma^*\nabla_0\sigma + \frac{1}{\hbar}\gamma^*[A, \gamma_*\sigma] =$$

$$= \frac{1}{\hbar}[A_{-1}, \sigma] + \gamma^*\nabla_0\sigma + \frac{1}{\hbar}[\gamma^*(A), \sigma].$$

Note that

$$\gamma^*\nabla_0\omega = \gamma^*(\nabla_0\gamma_*\omega) = \gamma^*(\nabla_0\omega) = 0,$$

so $\gamma^*\nabla_0$ is again a symplectic connection. Lastly we have

$$\gamma^*\nabla\sigma\tau = \gamma^*(\nabla\gamma_*(\sigma\tau)) =$$

$$= \gamma^*(\gamma_*\sigma\nabla\gamma_*\tau) + \gamma^*(\nabla(\gamma_*\sigma)\gamma_*\tau) =$$
$$= \sigma\gamma^*\nabla\tau + (\gamma^*\nabla\sigma)\tau$$

and

$$(\gamma^*\nabla)^2\sigma = \gamma^*\nabla\gamma^*(\nabla\gamma_*\sigma) = \gamma^*(\nabla^2\gamma_*\sigma) = 0,$$

so, by noting that the push-forward and pull-back of symplectomorphisms preserve the degree in the sections of the Weyl bundle, the lemma is proved. □

The isomorphism above then implies that $[\tilde{\gamma^*\nabla}^2] = [\tilde{\nabla}^2]$ for any lifts to $\tilde{\mathfrak{g}}$. So by the last section we have $\gamma^*\nabla = \operatorname{Ad} e^{\phi_\gamma}\nabla\operatorname{Ad} e^{-\phi_\gamma} + [\alpha_\gamma, -]$ for some section $\phi_\gamma$ and some scalar valued one form $\alpha_\gamma$. Also we have immediately the induced automorphism of the sections of the Weyl bundle $\operatorname{Ad} e^{\phi_\gamma}$ which maps $\operatorname{Ker}\nabla$ isomorphically onto $\operatorname{Ker}\nabla$. So we have now

$$\operatorname{Ker}\nabla \overset{\operatorname{Ad} e^{\phi_\gamma}}{\underset{\gamma^*}{\rightrightarrows}} \operatorname{Ker}\gamma^*\nabla,$$

where both arrows denote isomorphisms. Thus we can compose them in order to obtain the automorphism $A_\gamma := \gamma_* \circ \operatorname{Ad} e^{\phi_\gamma} \in \operatorname{Aut}(\Gamma(\mathcal{W})|\operatorname{Ker}\nabla)$. Note also that in the case that $\gamma^*$ preserves the connection (up to a scalar valued one-form) we obtain again the isomorphism from the lemma.

# 4 Group actions

Now let us move on to group actions. Suppose we have a symplectic group action $\Gamma \curvearrowright (M, \omega)$ and a deformation quantization $\nabla$. Then by the preceding section this means that we have for each $\gamma \in \Gamma$ an automorphism $A_\gamma$, which corresponds to a map (of sets) $\phi : \Gamma \to \Gamma(\mathcal{W})$ such that for all $\gamma \in \Gamma$ we have

$$\mathrm{Ad}\, e^{\phi_\gamma} \circ \nabla \circ \mathrm{Ad}\, e^{-\phi_\gamma} = \nabla.$$

Since we would like to have another group action $\Gamma \curvearrowright \mathrm{Ker}\, \nabla$ this means we should have $A_\gamma \circ A_\mu = A_{\gamma\mu}$ for each pair $\gamma, \mu \in \Gamma$ (note that indeed the unit $e \in \Gamma$ will preserve $\nabla$ and therefore $A_e = \mathrm{Id}$).

Note that

$$\begin{aligned} A_\gamma \circ A_\mu &= \gamma_* \circ \mathrm{Ad}\, e^{\phi_\gamma} \circ \mu_* \circ \mathrm{Ad}\, e^{\phi_\mu} = \\ &= \gamma_* \circ \mu_* \circ \mathrm{Ad}\, \mu^*(e^{\phi_\gamma}) \circ \mathrm{Ad}\, e^{\phi_\mu} = \\ &= (\gamma\mu)_* \circ \mathrm{Ad}\, \mu^*(e^{\phi_\gamma}) e^{\phi_\mu}, \end{aligned}$$

since for each $\sigma, \phi \in \Gamma(\mathcal{W})$ we have

$$\gamma^* \circ \mathrm{Ad}\, e^{\phi}(\sigma) = \gamma^*(e^{\phi}\sigma e^{-\phi}) = e^{\gamma^*(\phi)}\gamma^*(\sigma)e^{-\gamma^*(\phi)} = \mathrm{Ad}\, e^{\gamma^*(\phi)} \circ \gamma^*(\sigma).$$

On the other hand $A_{\gamma\mu} = (\gamma\mu)_* \circ \mathrm{Ad}\, e^{\phi_{\gamma\mu}}$. Thus we see that the $A_\gamma$ define a group action if and only if

$$\mathrm{Ad}\, e^{-\phi_{\gamma\mu}} \mu^*(e^{\phi_\gamma}) e^{\phi_\mu} = \mathrm{Id},$$

so if and only if

$$A_{\gamma,\mu} := e^{-\phi_{\gamma\mu}} e^{\mu^*(\phi_\gamma)} e^{\phi_\mu} \in \Gamma(\mathcal{Z}) \subset \Gamma(\mathcal{W}),$$

for all pairs $\gamma, \mu \in \Gamma$. We have slightly abused notation here by writing $\mathcal{Z}$ both for the bundle and the fiber of the bundle, to be completely clear by $\Gamma(\mathcal{Z})$ we mean the center of $\Gamma(\mathcal{W})$.

This expression might seem familiar to one who is used to dealing with group cohomology. At least if one would assume commutation of the exponents the logarithm of $A_{\gamma,\mu}$ would read exactly as a 2-coboundary in group cohomology. However, we cannot assume them to commute. So we will proceed by using the non-abelian group cohomology.

# 5 Cohomological Obstruction and Classification

## 5.1 Cohomological Approach

Let us fix the function $\phi$ mentioned above. Note that this function is not unique. Then we can identify a cochain corresponding to $\phi$, which determines whether the corresponding $A_\gamma$ define a group action or not. Namely let the function $\tilde{\Phi} : \Gamma \to \Gamma(\mathcal{W})^\times$ be given by $\tilde{\Phi}(\gamma) = \gamma_*(e^{\phi_\gamma})$. Denoting by $\pi$ the quotient map to $\Gamma(\mathcal{W})^\times/\Gamma(\mathcal{Z})^\times$ we find a 1-cochain

$$\Phi := \pi \circ \tilde{\Phi} \in C^1(\Gamma, \Gamma(\mathcal{W})^\times/\Gamma(\mathcal{Z})^\times),$$

where the action is given by the push-forward of elements of $\Gamma$. Then $\Phi$ is a cocycle if

$$\gamma_*(e^{\phi_\gamma})(\gamma\mu)_*(e^{\phi_\mu}) = (\gamma\mu)_*(e^{\phi_{\gamma\mu}}) \mod \Gamma(\mathcal{Z})^\times$$

by definition. But then we have also that

$$e^{-\phi_{\gamma\mu}}\mu^*(e^{\phi_\gamma})e^{\phi_\mu} \in \Gamma(\mathcal{Z})^\times,$$

so the $A_\gamma$ define an action. Thus finding an extension of the symplectic action is equivalent to finding a 1-cocycle as above. A better characterization of such 1-cocycles is needed and future research is aimed in that direction.

This is not the end of the story however. Let us turn from existence to classification of extensions of symplectic group actions. In the definition of the $A_\gamma$ we have the invertible elements $e^{\phi_\gamma}$. We have not shown however that these choices were unique. In fact it is easy to see that we can move from one choice to another in the following way. Suppose that $\phi, \psi \in \Gamma(\mathcal{W})$ such that

$$\mathrm{Ad}\, e^{\phi} \circ \nabla \circ \mathrm{Ad}\, e^{-\phi} = \mathrm{Ad}\, e^{\psi} \circ \nabla \circ \mathrm{Ad}\, e^{-\psi},$$

then we see that

$$\mathrm{Ad}\, U \circ \nabla \circ \mathrm{Ad}\, U^{-1} = \nabla,$$

where $U = e^{-\psi}e^{\phi}$. So suppose $\sigma \in \Gamma(\mathcal{W})$ then

$$U(\nabla(U^{-1}\sigma U))U^{-1} = \nabla(\sigma) + [U\nabla U^{-1}, \sigma],$$

where we used that $\nabla(U)U^{-1} = -U\nabla U^{-1}$. So we see that the above identity holds iff $U \in \mathcal{G}_\nabla := \{U \in \Gamma(\mathcal{W})^\times | U^{-1}\nabla U \in \Gamma(\mathcal{Z}) \otimes \Omega^1(M)\}$.

Note first of all that $\mathcal{G}$ is a group with the multiplication of $\Gamma(\mathcal{W})$. Using the fact that the centralizer of $\mathbb{A}_\hbar$ is the entire center of $\Gamma(\mathcal{W})$ (this follows from the fact that for each element in the a fiber of $\mathcal{W}$) one can find a flat section passing through it) it is easy to see that both $\Gamma(\mathcal{Z})^\times$ and $\mathbb{A}_\hbar^\times$ are normal subgroups of $\mathcal{G}$.

Now we see that the group $\mathcal{G}$ encodes exactly our freedom of choice for the $e^{\phi_\gamma}$. Let us fix a choice of $e^{\phi_\gamma}$. Then we see that the group action persists if there exists a map $U : \Gamma \to \mathcal{G}$ such that $A_\gamma \circ \mathrm{Ad}\, U_\gamma \circ A_\mu \circ \mathrm{Ad}\, U_\mu = A_{\gamma\mu} \circ \mathrm{Ad}\, U_{\gamma\mu}$ or in other terms such that $\exists C_{\gamma,\mu} \in \Gamma(\mathcal{Z})^\times$ with

$$C_{\gamma,\mu}e^{\phi_{\gamma\mu}}U_{\gamma\mu} = \mu^*(e^{\phi_\gamma}U_\gamma)e^{\phi_\mu}U_\mu,$$

for all $\gamma, \mu \in \Gamma$. This last expressions seems to imply again some kind of cohomological statement.

From now on we will write simply $\mathcal{Z}$ instead of $\Gamma(\mathcal{Z})$ for notational simplicity.

**Lemma 5.1.**
*We have $\mathcal{G}/\mathcal{Z}^\times \simeq Inn(\Gamma(\mathcal{W})| \operatorname{Ker} \nabla)$ and $A_\gamma(\mathcal{G}) \subset \mathcal{G}$ for all $\gamma \in \Gamma$.*

*Proof.*
Denote by Ad the map that sends an element of $\mathcal{G}$ to the inner automorphism of the sections of the weyl bundle given by conjugation. Obviously the map is a group homomorphism and $\operatorname{Ker} \operatorname{Ad} = \mathcal{Z}^\times$. Suppose now that $\operatorname{Ad} U \in \operatorname{Inn}(\Gamma(\mathcal{W})| \ker \nabla)$ then for all flat sections $\sigma$ we have

$$0 = \operatorname{Ad} U \circ \nabla \circ \operatorname{Ad} U^{-1}(\sigma) = [U \nabla U^{-1}, \sigma].$$

So by the comment about the centralizer of $\operatorname{Ker} \nabla$ above we see that $U \in \mathcal{G}$. Thus the first claim is shown. Suppose now that $U \in \mathcal{G}$ then using the first claim it is sufficient to show that $\operatorname{Ad} A_\gamma(U)$ preserves $\operatorname{Ker} \nabla$ for all $\gamma \in \Gamma$. Suppose then that $\sigma \in \operatorname{Ker} \nabla$, we have

$$\nabla \circ \operatorname{Ad} A_\gamma(U)(\sigma) = \gamma_*(\gamma^* \nabla(\operatorname{Ad} e^{\phi_\gamma} U \sigma U^{-1})) = \gamma_* \circ \operatorname{Ad} e^{\phi_\gamma} \circ \nabla \circ \operatorname{Ad} U(\sigma).$$

Thus the second claim is also shown. □

**Proposition 5.2.**

$$\mathcal{Z}^1_\Phi(\Gamma, \mathcal{G}/\mathcal{Z}^\times) := \{U : \Gamma \to \mathcal{G}/\mathcal{Z}^\times | U_\gamma A_\gamma(U_\mu)\Phi(\gamma)\gamma_*(\Phi(\mu)) = U_{\gamma\mu}\Phi(\gamma\mu)\}$$

*classifies the extensions of symplectic groups actions on $\mathbb{A}_\hbar$.*

*Proof.*

Suppose $\psi : \Gamma \to \Gamma(\mathcal{W})$ defines an extension of the symplectic group action $\Gamma \curvearrowright (M, \omega)$. Then let us denote by $U$ the map given by
$U_\gamma = \gamma_*(e^{\psi_\gamma} e^{-\phi_\gamma})$. We have

$$\operatorname{Ad} U_\gamma A_\gamma(U_\mu)\Phi(\gamma)\gamma_*(\Phi(\mu))\Phi(\gamma\mu)^{-1} U^{-1}_{\gamma\mu} =$$

$$= \gamma_* \circ \operatorname{Ad} e^{\psi_\gamma} \circ \mu_* \circ \operatorname{Ad} e^{\psi_\mu} \circ \operatorname{Ad} e^{-\psi_{\gamma\mu}} \circ (\gamma\mu)^* = \operatorname{Id},$$

since $\psi$ defines a group action. But the above equation shows that $\pi(U)$ is in $\mathcal{Z}^1_\Phi(\Gamma, \mathcal{G}/\mathcal{Z}^\times)$. Conversely suppose $\tilde{U} \in \mathcal{Z}^1_\Phi(\Gamma, \mathcal{G}/\mathcal{Z}^\times)$ then denote by $U$ an arbitrary lift to $\mathcal{G}$. Then by direct computation we see that $B_\gamma := \operatorname{Ad} U_\gamma \circ \gamma_* \circ \operatorname{Ad} e^{\phi_\gamma}$ defines an action. □

Then we see that if we assume the existence of an extension of an action (i.e. a $\phi$ that satisfies the cocycle condition) then in fact $\mathcal{Z}^1_\Phi(\Gamma, \mathcal{G}/\mathcal{Z}^\times) = \mathcal{Z}^1(\Gamma, \mathcal{G}/\mathcal{Z}^\times)$ with the action given by the $A_\gamma$. In this case we can pass to cohomology in order to be able to use the homological algebra[3]. By homological algebra we mean here that we can obtain from $\Gamma$-equivariant short exact sequences of coefficient groups a truncated long exact sequence of cohomology pointed sets, as examplified below. The corresponding $H^1(\Gamma, \mathcal{G}/\mathcal{Z}^\times)$ then classifies extensions of actions up to conjugation by inner automorphism.

### 5.2 Computing Cohomology

Now we would like to show that computation of this non-abelian first group cohomology pointed set is possible. The idea is to bring it down to the problem of computing cohomology with values in $\mathbb{A}_\hbar^\times/\mathbb{C}[\![\hbar]\!]^\times$ and use a certain sequence of subgroups to approximate the cohomology pointed set here. The best way to show this approach is through example. So let us present the very simplest of examples here.

Our example will be of the manifold $\mathbb{R}^2$ with the standard symplectic structure and action of $\mathbb{Z}/2\mathbb{Z}$ given by rotation. In this case the action preserves the Fedosov connection since the pushforward and pullback are given by constant matrices. So we indeed have an action on $\mathbb{A}_\hbar$ and $\mathcal{G}$ given by the pushforward of the action.

Note that the sequence (we will always mean $\Gamma$-equivariant sequence)

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}[\![\hbar]\!]^\times & \hookrightarrow & \mathcal{Z}^\times & \to & \mathcal{Z}^1_{dR}(\mathbb{R}^2)[\![\hbar]\!] & \to & 0, \\ & & & & f & \mapsto & \frac{\mathrm{d}f}{f} & & \end{array}$$

is exact, where we used the identification of $\mathcal{Z}$ with power series in complex valued smooth functions on the manfiold. Surjectivity of the last map can be shown by induction on the degree and the fact that $e^{-g}\mathrm{d}e^g = \mathrm{d}g$. Since the group in this case is finite we find already that $H^1(\Gamma, \mathcal{Z}^\times/\mathbb{C}[\![\hbar]\!]^\times) = 0$ (note that this is even a statement in Abelian group cohomology). Here we use the fact that first de Rham cohomology vanishes to show surjectivity. Now let us promote the above sequence to the sequence

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{A}_\hbar^\times & \hookrightarrow & \mathcal{G} & \to & \mathcal{Z}^1_{dR}(\mathbb{R}^2)[\![\hbar]\!] & \to & 0, \\ & & & & U & \mapsto & U^{-1}\nabla U & & \end{array}$$

which is of course still exact. It remains exact after modding out $\mathbb{C}[\![\hbar]\!]^\times$ on the right. After modding out we even find the section $\mathrm{d}f \mapsto e^f$ from $\mathcal{Z}^1_{dR}(\mathbb{R}^2)[\![\hbar]\!]$ to $\mathcal{G}$. So we find that $H^1(\Gamma, \mathbb{A}_\hbar^\times/\mathbb{C}[\![\hbar]\!]^\times)$ surjects on $H^1(\Gamma, \mathcal{G}/\mathbb{C}[\![\hbar]\!]^\times)$ which surjects on $H^1(\Gamma, \mathcal{G}/\mathcal{Z}^\times)$ since we have the exact sequence

$$1 \to \mathcal{Z}^\times/\mathbb{C}[\![\hbar]\!]^\times \to \mathcal{G}/\mathbb{C}[\![\hbar]\!]^\times \to \mathcal{G}/\mathcal{Z}^\times \to 1.$$

Lastly we consider the descending chain of normal subgroups of $\mathbb{A}_\hbar^\times$ given by $G_n := 1 + \hbar\mathbb{A}_\hbar$. Then we have first of all the exact sequence

$$1 \to G_1 \to G_0 \to C^\infty(\mathbb{R}^2)^\times \to 1,$$

which shows that cohomology with values in $G_1$ surjects on cohomology with values in $G_1$, for the higher orders in $n$ we have

$$1 \to G_{n+1} \to G_n \to C^\infty(\mathbb{R}^2) \to 0,$$

note that these are exact sequences of groups. So again since $\mathbb{Z}/2\mathbb{Z}$ is finite we see that we have a chain of surjections which in the limit form a surjection from cohomology with values in the trivial group to the cohomology we want to compute. Therefore we see that in this simplest of examples the only extension of the group action is the natural one.

The answer to this example may not be very exciting, however we do see from the computation that the only things that are essentially used are surjectivity of the last map in the first sequence, existence of a section after modding out scalars and finiteness of the group. So we do have some control over the computability of our cohomology pointed sets.

## References


[1] Boris V. Fedosov, *A simple geometric construction of deformation quantization*. J. Differential Geom. Volume 40, Number 2 (1994), 213-238.

[2] R. Nest and B. Tsygan, *Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems*. Preprint math/9906020, 1999.

[3] P. Gille and T. Szamuely, *Central simple algebras and Galois Cohomology*. Cambridge Studies in advanced mathematics, 2006.