Quantization of Poisson-Lie Hamiltonian systems

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Outline

Symmetries and Conserved quantities

How to obtain conserved quantities for systems with symmetries?

- system?
- symmetries?
- conserved quantity?

Semi-classical Step

Let's put a Poisson structure on our Lie group!

New structures:

- Poisson Lie groups
- Lie bialgebras

What is a Hamiltonian action in this context?

Poisson action

Definition

The action of (G, π_G) on (M, π) is called Poisson action if the map $\Phi : G \times M \to M$ is Poisson, where $G \times M$ is a Poisson manifold with structure $\pi_G \oplus \pi$.

Generalization of canonical action! If $\pi_G = 0$, the action is Poisson if and only if it preserves π.

Momentum map

Definition (Lu)

A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\widehat{X} = \pi^\sharp(\boldsymbol{\mu}^*(\theta_X))$$

where θ_X is the left invariant 1-form on G^* defined by the element $X \in \mathfrak{g} = (T_e G^*)^*$ and $\boldsymbol{\mu}^*$ is the cotangent lift $T^*G^* \to T^*M$.

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A Hamiltonian action is a Poisson action induced by an equivariant momentum map.

Infinitesimal momentum map

The forms $\alpha_X = \boldsymbol{\mu}^*(\theta_X)$ satisfy

$$\alpha_{[X,Y]} = [\alpha_X, \alpha_Y]_\pi \quad \text{and} \quad d\alpha_X + \alpha \wedge \alpha \circ \delta(X) = 0$$

Definition

Let M be a Poisson manifold and G a Poisson Lie group. An infinitesimal momentum map is a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\ ,\ ]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [\ ,\ ]_\pi).$$

Steps in formal approach

Goal: quantize Hamiltonian actions

1. Quantize structures
2. Quantize Poisson action
3. Quantize Momentum map

Quantum action

How can we define a quantum action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{A}_\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action

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Definition

The quantum action is a linear map

$$\Phi_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to End\ \mathcal{A}_\hbar : X \mapsto \Phi_\hbar(X)(f)$$

such that

1. Hopf algebra action
2. Algebra homomorphism

Quantum Hamiltonian action

1. Quantum momentum map which, as in the classical case, generates the quantum action
2. $\hbar \to 0$ classical momentum map

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2. $\hbar \to 0$ classical momentum map

Definition

A quantum momentum map is defined to be a linear map

$$\boldsymbol{\mu}_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to \Omega^1(\mathcal{A}_\hbar) : X \mapsto a_X db_X.$$

General idea

joint with R. Nest and P. Bieliavsky

- Formal Drinfeld twist
- Non-formal Drinfeld twists (Bieliavsky, Gayral)

Triangular Lie biagebras

Consider a particular class of Lie bialgebras $(\mathfrak{g}, \delta)$ with

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Theorem (Drinfeld)

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, with r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$. There exists a deformation $\mathcal{U}_\hbar(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ whose classical limit is $\mathfrak{g}$ with Lie bialgebra structure defined by r. Furthermore, $\mathcal{U}_\hbar(\mathfrak{g})$ is a triangular Hopf algebra and isomorphic to $\mathcal{U}(\mathfrak{g})[[\hbar]]$

Drinfeld Twist

- giving a twist on $\mathcal{U}_\hbar(\mathfrak{g})$ is equivalent to give an associative star product on $C^\infty(G)$

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Question: does twist produce quantum Hamiltonian action?

Bieliavsky-Gayral construction

Triangular structures associated to Kähler Lie groups: non formal approach!

Explicit construction of families of kernels

$$\{\kappa_t \in C^\infty(G \times G)\}_t$$

such that for "any" action of G on a $C^\star$-algebra $\mathcal{A}$ by $C^\star$-algebra automorphisms, κ_t defines an star product on $\mathcal{A}$

Non formal Twist?

If $\mathcal{A}$ is the algebra of (complex valued continuous) functions on G, which G acts on via the right-regular representation, then asymptotic expansion automatically yields a left-invariant formal $\star$-product on (G, ω^G):

$$f_1 \star_t f_2 \;:=\; f_1 f_2 + \sum_{k\geq 1} \left(\frac{t}{2i}\right)^k \tilde{F}_k^{(\kappa)}(f_1, f_2) \quad (f_1, f_2 \in C_0^\infty(G))$$

F defines formal twist quantization of our triangular Lie bialgebra!