EXTREMA OF THE EINSTEIN-HILBERT ACTION FOR NONCOMMUTATIVE 4-TORI

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joint with
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The Heat Kernel of a Riemannian Manifold (M, g)

$$\triangle_g : C^\infty(M) \to C^\infty(M),$$

$$K : \mathbb{R}_{>0} \times M \times M \to \mathbb{C},$$

$$\big(e^{-t\triangle_g} f\big)(x) = \int_M K(t,x,y)\, f(y)\, dvol(y).$$

$$K(t,x,y) \sim \frac{e^{-\mathrm{dist}(x,y)^2/4t}}{(4\pi t)^{n/2}} \Big(\sum_{i=0}^{\infty} U_i(x,y)\, t^i \Big) \qquad (t \to 0),$$

$U_i : N(Diag(M \times M)) \to \mathbb{C}$ (geometric information),

$U_0(x,x) = 1$ ($\Rightarrow$ Weyl's law), $U_1(x,x) =$ scalar curvature.

Spectral Triples

$$(\mathcal{A}, \mathcal{H}, D),$$

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \qquad (*\text{-representation}),$$

$$D = D^* : Dom(D) \subset \mathcal{H} \to \mathcal{H},$$

$$D\,\pi(a) - \pi(a)\,D \in \mathcal{L}(\mathcal{H}).$$

Examples.

$$\big(C^\infty(M), L^2(M,S), D = \text{Dirac operator}\big).$$

$$\Big(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), \frac{1}{i}\,\frac{\partial}{\partial x}\Big).$$

Local Geometric Invariants of $(\mathcal{A}, \mathcal{H}, D)$

These invariants such as scalar curvature can be computed by considering small time heat kernel expansions of the form

$$\mathsf{Trace}\big(\pi(a)\, e^{-tD^2}\big) \sim_{t \to 0^+} \sum_{n=0}^{\infty} a_n(a, D)\, t^{(n-d)/2},$$

where d is the spectral dimension.

Noncommutative 4-Torus $\mathbb{T}^4_\theta$

$C(\mathbb{T}^4_\theta)$ is the universal C^*-algebra generated by 4 unitaries

$$U_1, U_2, U_3, U_4,$$

satisfying

$$U_k U_\ell = e^{2\pi i \theta_{k\ell}} U_\ell U_k,$$

for a skew symmetric matrix

$$\theta = (\theta_{k\ell}) \in M_4(\mathbb{R}).$$

Action of $\mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ on $C(\mathbb{T}^4_\theta)$

$$\mathbb{R}^4 \ni s \mapsto \alpha_s \in \mathsf{Aut}\Big(C(\mathbb{T}^4_\theta)\Big),$$

$$\alpha_s(U^m) := e^{is\cdot m}\, U^m, \quad U^m := U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}, \quad m_j \in \mathbb{Z}.$$

$$\delta_j = \frac{\partial}{\partial s_j}\Big|_{s=0}\, \alpha_s \; : C^\infty(\mathbb{T}^4_\theta) \to C^\infty(\mathbb{T}^4_\theta),$$

$$\begin{aligned} \delta_j(U_k) &:= U_k \qquad \text{if} \qquad k = j, \\ &:= 0 \qquad \text{if} \qquad k \neq j. \end{aligned}$$

Complex Structure on $\mathbb{T}^4_\theta$

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar\partial = \bar\partial_1 \oplus \bar\partial_2,$$

$$\partial_1 = \frac{1}{2}\,(\delta_1 - i\delta_3), \qquad \partial_2 = \frac{1}{2}\,(\delta_2 - i\delta_4),$$

$$\bar\partial_1 = \frac{1}{2}\,(\delta_1 + i\delta_3), \qquad \bar\partial_2 = \frac{1}{2}\,(\delta_2 + i\delta_4).$$

Volume Form on $\mathbb{T}^4_\theta$

$$\varphi_0 : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi_0(1) := 1,$$

$$\varphi_0(U_1^{m_1}\, U_2^{m_2}\, U_3^{m_3}\, U_4^{m_4}) := 0, \qquad (m_1, m_2, m_3, m_4) \neq (0,0,0,0).$$

$$\varphi_0(a\, b) = \varphi_0(b\, a), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

$$\varphi_0(a^*\, a) > 0, \qquad a \neq 0.$$

Conformal Perturbation (Connes-Tretkoff)

Let $h = h^* \in C^\infty(\mathbb{T}^4_\theta)$ and replace the trace φ_0 by

$$\varphi : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-2h}), \qquad a \in C(\mathbb{T}^4_\theta).$$

φ is a KMS state with the modular group

$$\sigma_t(a) = e^{2ith}\, a\, e^{-2ith}, \qquad a \in C(\mathbb{T}^4_\theta),$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-2h}\, a\, e^{2h}, \qquad a \in C(\mathbb{T}^4_\theta).$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

Perturbed Laplacian on $\mathbb{T}^4_\theta$

$$d = \partial \oplus \bar{\partial} : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

Remark. If $h = 0$ then $\varphi = \varphi_0$ and

$$\triangle_{\varphi_0} = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = \partial^* \partial$$

(the underlying manifold is Kähler).

Explicit Formula for $\triangle_\varphi$

Lemma. Up to an anti-unitary equivalence $\triangle_\varphi$ is given by

$$e^h\bar{\partial}_1 e^{-h}\partial_1 e^h + e^h\,\partial_1 e^{-h}\bar{\partial}_1\, e^h + e^h\bar{\partial}_2 e^{-h}\,\partial_2 e^h + e^h\partial_2 e^{-h}\bar{\partial}_2 e^h,$$

where ∂_1, ∂_2 are analogues of the Dolbeault operators.

Connes' Pseudodifferential Calculus (1980)

A smooth map $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ is a symbol of order $m \in \mathbb{Z}$, if for any $i, j \in \mathbb{Z}^4_{\geq 0}$, there exists a constant c such that

$$||\partial^j \delta^i \big(\rho(\xi)\big)|| \leq c(1 + |\xi|)^{m - |j|},$$

and if there exists a smooth map $k : \mathbb{R}^4 \setminus \{0\} \to C^\infty(\mathbb{T}^4_\theta)$ such that

$$\lim_{\lambda \to \infty} \lambda^{-m} \rho(\lambda \xi) = k(\xi), \qquad \xi \in \mathbb{R}^4 \setminus \{0\}.$$

- Given a symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$, the corresponding ψDO is:

$$P_\rho(a) = (2\pi)^{-4} \int \int e^{-is.\xi} \, \rho(\xi) \, \alpha_s(a) \, ds \, d\xi, \qquad a \in C^\infty(\mathbb{T}^4_\theta).$$

- Differential operators:

$$\rho(\xi) = \sum a_\ell \, \xi^\ell, \quad a_\ell \in C^\infty(\mathbb{T}^4_\theta) \quad \Rightarrow \quad P_\rho = \sum a_\ell \, \delta^\ell.$$

- ΨDO's on $\mathbb{T}^4_\theta$ form an algebra:

$$\sigma(P\,Q) \sim \sum_{\ell \in \mathbb{Z}^4_{\geq 0}} \frac{1}{\ell!} \, \partial^\ell_\xi \rho(\xi) \, \delta^\ell(\rho'(\xi)).$$

• A symbol $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$ of order m is elliptic if $\rho(\xi)$ is invertible for any $\xi \neq 0$, and if there exists a constant c such that

$$||\rho(\xi)^{-1}|| \leq c(1+|\xi|)^{-m},$$

when $|\xi|$ is sufficiently large.

• Example of an elliptic operator:

$$\triangle_\varphi = e^h \bar{\partial}_1 e^{-h} \partial_1 e^h + e^h \partial_1 e^{-h} \bar{\partial}_1 e^h + e^h \bar{\partial}_2 e^{-h} \partial_2 e^h + e^h \partial_2 e^{-h} \bar{\partial}_2 e^h.$$

Symbol of $\triangle_\varphi$

Lemma. The symbol of $\triangle_\varphi$ is equal to

$$a_2(\xi) + a_1(\xi) + a_0(\xi),$$

where

$$a_2(\xi) = e^h \sum_{i=1}^{4} \xi_i^2, \qquad a_1(\xi) = \sum_{i=1}^{4} \delta_i(e^h)\, \xi_i,$$

$$a_0(\xi) = \sum_{i=1}^{4} \big(\delta_i^2(e^h) - \delta_i(e^h)\, e^{-h}\, \delta_i(e^h)\big).$$

Mellin Transform and Asymptotic Expansions

$$\triangle_\varphi^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\triangle_\varphi}\, t^s \,\frac{dt}{t},$$

$$\mathsf{Trace}(a\, e^{-t\triangle_\varphi}) \sim_{t\to 0^+} \; t^{-2} \sum_{n=0}^\infty B_n(a, \triangle_\varphi)\, t^{n/2}.$$

Approximate $e^{-t\triangle_\varphi^2}$ by pseudodifferential operators:

$$e^{-t\triangle_\varphi} = \frac{1}{2\pi i} \int_C e^{-t\lambda}\, (\triangle_\varphi - \lambda)^{-1}\, d\lambda,$$

$$B_\lambda\, (\triangle_\varphi - \lambda) \approx 1,$$

$$\sigma(B_\lambda) = b_0 + b_1 + b_2 + \cdots .$$

Analogue of Weyl's Law for $\mathbb{T}^4_\theta$

Theorem. For the eigenvalue counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}$$

of the Laplacian $\triangle_\varphi$ on $\mathbb{T}^4_\theta$, we have

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2} \lambda^2 \qquad (\lambda \to \infty).$$

Corollary.

$$\lambda_j \sim \frac{\sqrt{2}}{\pi \varphi_0(e^{-2h})^{1/2}}\, j^{1/2} \qquad (j \to \infty),$$

$$\mathrm{Tr}_\omega\Big((1 + \triangle_\varphi)^{-2}\Big) = \frac{\pi^2}{2} \varphi_0(e^{-2h}).$$

Dixmier Trace $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \to \mathbb{C}$

For any $T \in \mathcal{K}(\mathcal{H})$, let

$$\mu_1(T) \geq \mu_2(T) \geq \cdots \geq 0$$

be the sequence of eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$.

- $$\mathcal{L}^{1,\infty}(\mathcal{H}) := \Big\{T \in \mathcal{K}(\mathcal{H}); \qquad \sum_{n=1}^{N} \mu_n(T) = O\left(\log N\right)\Big\}.$$

- $$\mathrm{Tr}_\omega(T) := \lim_\omega \Big(\frac{1}{\log N} \sum_{n=1}^{N} \mu_n(T)\Big), \qquad 0 \leq T \in \mathcal{L}^{1,\infty}(\mathcal{H}).$$

Noncommutative Residue (Wodzicki)

Let P be a classical ψDO acting on smooth sections of a vector bundle E over a closed smooth manifold M of dimension n.

- **Definition:**

$$\mathsf{Res}(P) = (2\pi)^{-n} \int_{S^*M} \mathsf{tr}(\rho_{-n}(x,\xi))\, dx\, d\xi,$$

where $S^*M \subset T^*M$ is the unit cosphere bundle on M and ρ_{-n} is the component of order $-n$ of the complete symbol of P.

- **Theorem:** Res is the unique trace on $\Psi(M,E)$.

A Noncommutative Residue for $\mathbb{T}^4_\theta$

Classical symbols: $\rho : \mathbb{R}^4 \to C^\infty(\mathbb{T}^4_\theta)$

$$\rho(\xi) \sim \sum_{i=0}^{\infty} \rho_{m-i}(\xi) \qquad (\xi \to \infty),$$

$$\rho_{m-i}(t\,\xi) = t^{m-i}\,\rho_{m-i}(\xi), \qquad t > 0, \quad \xi \in \mathbb{R}^4.$$

Theorem. The linear functional

$$\mathsf{Res}(P_\rho) := \int_{\mathbb{S}^3} \varphi_0\big(\rho_{-4}(\xi)\big)\, d\xi$$

is the unique trace on classical pseudodifferential operators on $\mathbb{T}^4_\theta$.

Analogue of Connes' Trace Theorem for $\mathbb{T}^4_\theta$

Theorem. For any classical symbol ρ of order -4 on $\mathbb{T}^4_\theta$, we have

$$P_\rho \in \mathcal{L}^{1,\infty}(\mathcal{H}_0),$$

and

$$\mathrm{Tr}_\omega(P_\rho) = \frac{1}{4}\mathsf{Res}(P_\rho).$$

Remark. Weyl's law is a special case of this theorem: let

$$\rho(\xi) = \frac{1}{(1+|\xi|^2)^2}.$$

Scalar Curvature for $\mathbb{T}^4_\theta$

It is the unique element $R \in C^\infty(\mathbb{T}^4_\theta)$ such that

$$\mathrm{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\, R), \qquad a \in C^\infty(\mathbb{T}^4_\theta),$$

$$\zeta_a(s) := \mathrm{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

Connes' Rearrangement Lemma

For any $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$, $\rho_1, \ldots, \rho_\ell \in C^\infty(\mathbb{T}_\theta^4)$:

$$\int_0^\infty \frac{u^{|m|-2}}{(e^h u + 1)^{m_0}} \prod_1^\ell \rho_j \, (e^h u + 1)^{-m_j} \, du$$

$$= e^{-(|m|-1)h} \, F_m(\Delta, \ldots, \Delta)\Big(\prod_1^\ell \rho_j \Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \Big(x \prod_1^j u_k + 1 \Big)^{-m_j} \, dx.$$

Examples of F_m

$$F_{(3,4)}(u) = \frac{60u^3 \log(u) + (u-1)(u(u(3(u-9)u-47)+13)-2)}{6(u-1)^6 u^3}$$

$$F_{(2,2,1)}(u,v) =$$

$$\frac{(v-1)\Big((u-1)(uv-1)(u(u(v-1)+v)-1)-u^2(v-1)(2uv+u-3)\log(uv)\Big)+(u(2v-3)+1)(uv}{(u-1)^3u^2(v-1)^2(uv-1)^2}$$

Identities Relating $\delta_i(e^h)$ and $\delta_i(h)$

$$e^{-h}\,\delta_i(e^h) = g_1(\Delta)\big(\delta_i(h)\big),$$

$$e^{-h}\,\delta_i^2(e^h) = g_1(\Delta)\big(\delta_i^2(h)\big) + 2\,g_2(\Delta_{(1)},\Delta_{(2)})\big(\delta_i(h)\,\delta_i(h)\big),$$

where

$$g_1(u) = \frac{u-1}{\log u},$$

$$g_2(u,v) = \frac{u(v-1)\log(u) - (u-1)\log(v)}{\log(u)\log(v)(\log(u)+\log(v))}.$$

Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^4$

Theorem.

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

where

$$\nabla(a) := \frac{1}{2}\log\Delta(a) = [-h,a], \qquad a \in C(\mathbb{T}_\theta^4),$$

$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\left((-e^s-3)\,s\,(e^t-1) + (e^s-1)\,(3e^t+1)\,t\right)}{4\,s\,t\,(s+t)}.$$

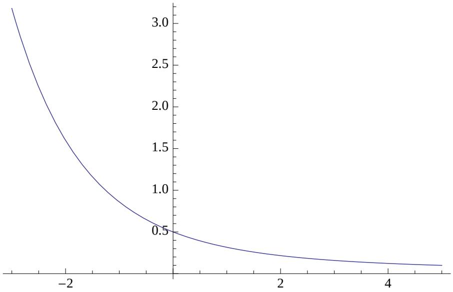
Recalling the Scalar Curvature of $\mathbb{T}_\theta^2$

Theorem. (Connes-Moscovici; Khalkhali-F.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, the scalar curvature of $\mathbb{T}_\theta^2$ is equal to

$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$
$$+R_2(\nabla,\nabla)\Big(\delta_1(\frac{h}{2})^2 + |\tau|^2\,\delta_2(\frac{h}{2})^2 + \Re(\tau)\,\big\{\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})\big\}\Big)$$
$$+i\,W(\nabla,\nabla)\Big(\Im(\tau)\,[\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})]\Big).$$

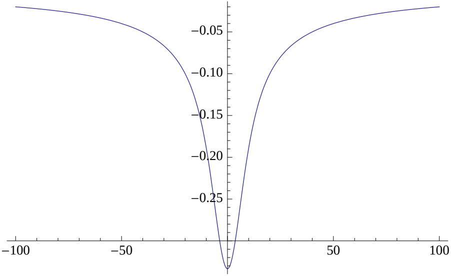
The One Variable Function for $\mathbb{T}^4_\theta$

$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



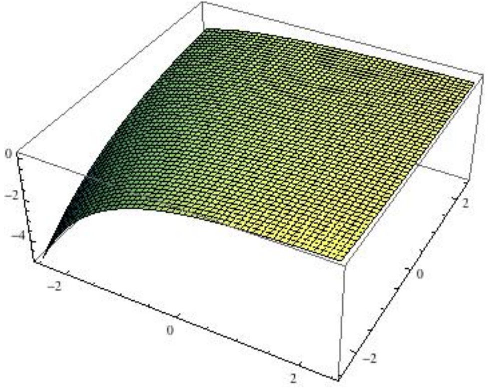
The One Variable Function for $\mathbb{T}_\theta^2$

$$R_1(x) = \frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}.$$

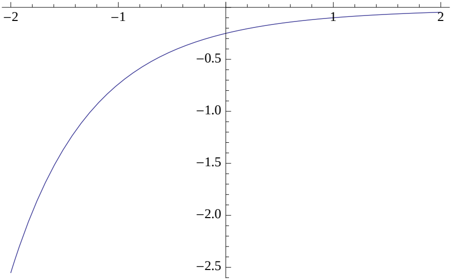


The Two Variable Function for $\mathbb{T}^4_\theta$

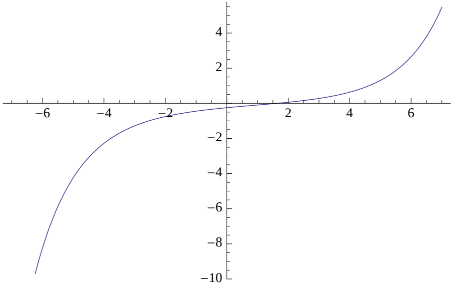
$$H(s,t) = \left(-\frac{1}{4} + \frac{t}{24} + O\left(t^3\right)\right) + s\left(\frac{5}{24} - \frac{t}{16} + \frac{t^2}{80} + O\left(t^3\right)\right) + s^2\left(-\frac{1}{12} + \frac{7t}{240} - \frac{t^2}{144} + O\left(t^3\right)\right) + O\left(s^3\right).$$



$$
\begin{aligned}
H(s,s) &= -\frac{e^{-2s}\left(e^{s}-1\right)^{2}}{4s^{2}} \\
&= -\frac{1}{4}+\frac{s}{4}-\frac{7s^{2}}{48}+\frac{s^{3}}{16}-\frac{31s^{4}}{1440}+\frac{s^{5}}{160}+O\left(s^{6}\right).
\end{aligned}
$$



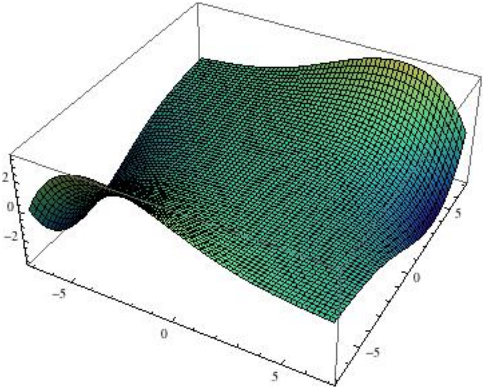
$$
\begin{aligned}
G(s) \quad := \quad H(s,-s) \quad &= \quad \frac{-4s - 3e^{-s} + e^{s} + 2}{4s^2} \\
&= \quad -\frac{1}{4} + \frac{s}{6} - \frac{s^2}{48} + \frac{s^3}{120} - \frac{s^4}{1440} + \frac{s^5}{5040} + O\left(s^6\right).
\end{aligned}
$$



The First Two Variable Function for $\mathbb{T}_\theta^2$

$R_2(s,t) =$

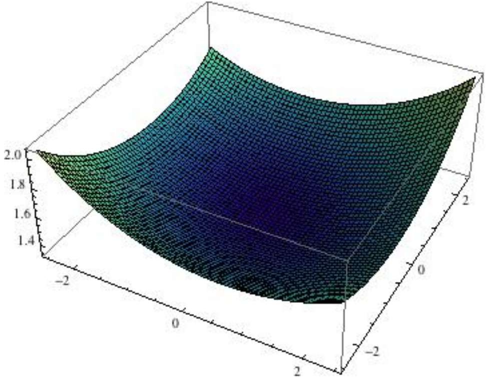
$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



The Second Two Variable Function for $\mathbb{T}^2_\theta$

$W(s,t) =$

$$\frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



Commutative Case $\theta = 0 \in M_4(\mathbb{R})$

We have

$$k(0) = 1/2, \qquad H(0,0) = -1/4.$$

Therefore, in the commutative case $\theta = 0$, since $\nabla = 0$, the formula for the scalar curvature of $\mathbb{T}^4_\theta$ reduces to

$$R = \frac{\pi^2}{2} \sum_{i=1}^{4} \big(\delta_i^2(h) - \frac{1}{2} \delta_i(h)^2 \big).$$

This, up to a normalization factor, is the scalar curvature of the ordinary 4-torus equipped with the metric

$$ds^2 = e^{-h} \, (dx_1^2 + dx_2^2 + dx_3^3 + dx_4^2),$$

where $h \in C^\infty(\mathbb{T}^4, \mathbb{R})$.

Motivation for the Computations

In the 2-dimensional case:

P. B. Cohen, A. Connes, *Conformal geometry of the irrational rotation algebra*, MPI preprint 1992-93

$$\zeta_h(0) + 1 = \varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big).$$

Two theories were developed: the spectral action principle (Chamseddine-Connes) and twisted spectral triples (Connes-Moscovici);

A. Connes, P. Tretkoff, *The Gauss-Bonnet Theorem for the Non- commutative Two Torus*, 2009

$$\zeta_h(0) + 1 = 0 \qquad\qquad (\tau = i).$$

This created the need to investigate the Gauss-Bonnet for general conformal structures (Khalkhali-F) and stimulated the computation of scalar curvature for $\mathbb{T}^2_\theta$ (Connes-Moscovici; Khalkhali-F).

Einstein-Hilbert Action for $\mathbb{T}_\theta^4$

Theorem. We have the local expression (up to a factor of π^2):

$$\begin{aligned}\varphi_0(R) &= \frac{1}{2}\sum_{i=1}^{4}\varphi_0\Big(e^{-h}\delta_i^2(h)\Big) \\ &\quad + \sum_{i=1}^{4}\varphi_0\Big(G(\nabla)(e^{-h}\delta_i(h))\,\delta_i(h)\Big).\end{aligned}$$

Extrema of the Einstein-Hilbert Action

Theorem. For any Weyl factor $e^{-h} \in C^\infty(\mathbb{T}^4_\theta)$, we have:

$$\varphi_0(R) \leq 0,$$

and the equality happens if and only if h is a constant.

Proof.

$$\varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h} T(\nabla)(\delta_i(h))\,\delta_i(h)\big),$$

where

$$T(s) = \frac{1}{2}\frac{e^{-s}-1}{-s} + G(s) = \frac{-2s + e^{s} - e^{-s}(2s+3) + 2}{4s^2}.$$

$$T(s) = \frac{1}{4} - \frac{s}{12} + \frac{s^2}{16} - \frac{s^3}{80} + \frac{s^4}{288} - \frac{s^5}{2016} + O\left(s^6\right).$$