Introduction

$\mathbb{G}$ - locally compact quantum group

$\mathbb{G} = (A, \Delta)$ — Δ: morphism from A to $A \otimes A$; A: C^*-algebra

$\otimes$ - bifunctor on $\mathcal{C}^*$

$\mathrm{Obj}\,\mathcal{C}^*$ - C^* - algebras

$A, B \in \mathrm{Obj}(\mathcal{C}^*)$ then $\pi \in \mathrm{Mor}(A, B)$

is a map $\pi : A \to M(B)$ s.t. $B = \left\{\pi(a) \cdot b : \begin{matrix} a \in A \\ b \in B \end{matrix}\right\}^{\mathrm{cls}}$

$M(B)$: multipliers of B

$A, B \in \mathrm{Obj}(\mathcal{C}^*) \Rightarrow A \otimes B \in \mathrm{Obj}(\mathcal{C}^*)$

$\otimes$: spatial tensor product

$$\left.\begin{matrix} \pi_1 \in \mathrm{Mor}(A_1, B_1) \\ \pi_2 \in \mathrm{Mor}(A_2, B_2) \end{matrix}\right\} \rightsquigarrow \pi_1 \otimes \pi_2 \in \mathrm{Mor}(A_1 \otimes A_2, B_1 \otimes B_2)$$

$(\mathcal{C}^*, \otimes)$ - monoidal category

Category $\mathcal{C}^*_G$ of G-C^*-algebras

locally compact group

$Obj(\mathcal{C}^*_G)$ - G-C^*-algebras (A, ρ^A)

where A - C^*-algebra

$\rho^A : G \to Aut(A)$

(A, ρ^A), $(B, \rho^B) \in Obj$ then $\pi \in Mor_G(A, B)$

is covariant : $\forall_{g \in G} \quad \rho^B_g \circ \pi = \pi \circ \rho^A_g$

Monoidal structure on $\mathcal{C}^*_G$

$(A, \rho^A) \otimes (B, \rho^B) = (A \otimes B, \rho^{A \otimes B})$

where $\rho^{A \otimes B}_g \equiv \rho^A_g \otimes \rho^B_g \in Aut(A \otimes B)$

$(\mathcal{C}^*_G, \otimes)$ - monoidal category

Crossed product functor

Γ - loc comp ab.

$\hat{\Gamma}$ - Pontryagin dual $\begin{cases} C^*(\Gamma) \simeq C_0(\hat{\Gamma}) \\ \forall_{\gamma\in\Gamma} \ \lambda_\gamma \in M(C^*(\Gamma)) \end{cases}$ unitary generat.

$\mathcal{CP} : \mathcal{C}^*_\Gamma \longrightarrow \mathcal{C}^*_{\hat{\Gamma}}$ - cr. prod. functor

$\mathcal{CP}(A, \rho^A) = (\Gamma \ltimes A, \hat{\rho}^{\Gamma\ltimes A}) \in \mathrm{Obj}\, \mathcal{C}^*_{\hat{\Gamma}}$

crossed product

dual action

$\Gamma \ltimes A$:
ⓐ $C^*(\Gamma)$, $A \in M(\Gamma \ltimes A)$
ⓑ $\{a \cdot x : a \in A, x \in C^*(\Gamma)\}^{\mathrm{cls}} = \Gamma \ltimes A$

$\hat{\rho}^{\Gamma\ltimes A}$:
ⓐ $a \in A \Rightarrow \hat{\rho}_{\hat{\gamma}}(a) = a$
ⓑ $x \in C^*(\Gamma) = C_0(\hat{\Gamma}) \Rightarrow \hat{\rho}_{\hat{\gamma}}(x) = \tau_{\hat{\gamma}}(x)$
where $\tau_{\hat{\gamma}}$ is the right shift action

$\pi \in \mathrm{Mor}_\Gamma(A, B) \rightsquigarrow \mathcal{CP}(\pi) \in \mathrm{Mor}_{\hat{\Gamma}}(\Gamma\ltimes A, \Gamma \ltimes B)$

$\mathcal{CP}(\pi)\big|_A = \pi$

$\mathcal{CP}(\pi)\big|_{C^*(\Gamma)} = \mathrm{id}$

CP and natural transformations of functors

General remarks:

$\overset{F}{\underset{G}{}} : \mathcal{C} \to \mathcal{D}$

functors   categories

a natural transformation φ from F to G

ⓐ $\forall_{c \in \mathrm{Obj}(\mathcal{C})} \quad \varphi^c \in \mathrm{Mor}_{\mathcal{D}}(F(c), G(c))$

ⓑ $\forall_{\alpha \in \mathrm{Mor}_{\mathcal{C}}(c_1, c_2)}$

$$\begin{array}{ccc} F(c_1) & \xrightarrow{F(\alpha)} & F(c_2) \\ \downarrow \varphi^{c_1} & & \downarrow \varphi^{c_2} \\ G(c_1) & \xrightarrow{G(\alpha)} & G(c_2) \end{array}$$

Crossed product examples of nat. trans.

$CP : \mathcal{C}^*_{\Gamma} \longrightarrow \mathcal{C}^*_{\hat{\Gamma}}$

$\mathcal{F} : \mathcal{C}^*_{\Gamma} \longrightarrow \mathcal{C}^*_{\hat{\Gamma}}$

$(A, \rho^A) \mapsto (A, \mathrm{triv})$

$\mathcal{G} : \mathcal{C}^*_{\Gamma} \longrightarrow \mathcal{C}^*_{\hat{\Gamma}}$

$(A, \rho^A) \mapsto (C_0(\hat{\Gamma}), \tau)$

$\alpha : \mathcal{F} \to CP$

$\alpha^A \in \mathrm{Mor}_{\hat{\Gamma}}(C_0(\hat{\Gamma}), CP(A, \rho^A))$

embeds $C_0(\hat{\Gamma})$ into $M(\Gamma \ltimes A)$

$\beta : \mathcal{G} \to CP$

$\beta^A \in \mathrm{Mor}(A_{\mathrm{triv}}, CP(A))$

embeds A into $M(\Gamma \ltimes A)$

Landstad theorem

$(A, \rho^A) \in Obj(\mathcal{C}^*_\Gamma)$

$(\Gamma \ltimes A, \hat{\rho}, \lambda)$

$\nearrow \quad \lambda_\gamma a \lambda_\gamma^* = \rho^A_\gamma(a)$

$\rightarrow \quad \hat{\rho}_{\hat{\gamma}}(a) = a$

$\searrow \quad \lambda : \Gamma \to M(\Gamma \ltimes A)$ unit. rep.

extends to $C^*(\Gamma) \hookrightarrow M(\Gamma \ltimes A)$

$\searrow \quad \hat{\rho}_{\hat{\gamma}}(\lambda_\gamma) = \underset{\mathbb{T}^1}{\underset{\cap}{\langle \hat{\gamma}, \gamma \rangle}} \lambda_\gamma$

Thm (Landstad)

$(D, \hat{\rho}^D) \in \mathcal{C}^*_{\hat{\Gamma}}$; $\lambda : \Gamma \to M(D)$ unit rep

$$\hat{\rho}_{\hat{\gamma}}(\lambda_\gamma) = \langle \hat{\gamma}, \gamma \rangle \lambda_\gamma$$

then $\quad D = \Gamma \ltimes A$ where

$$A = \left\{ d \in M(D) \;\middle|\; \begin{array}{l} (a) \ \hat{\rho}_{\hat{\gamma}}(d) = d \\ (b) \ \Gamma \ni \gamma \mapsto \lambda_\gamma d \lambda_\gamma \in M(D) \text{ - } \| \ \| \text{ cont} \\ (c) \ \forall x, y \in C^*(\Gamma) \\ \quad \lambda(x) d, \ d \lambda(y) \in D. \end{array} \right.$$

$(D, \hat{\rho}^D) \simeq \mathcal{CP}(A, \rho^A)$ where

$\rho^A(d) = \lambda_\gamma d \lambda_\gamma^* \quad \forall d \in A$

Rieffel def functor

$\psi : \hat{\Gamma} \to \Gamma$ - homomorphism

$(A, \rho^A) \in \mathrm{Obj}(\mathcal{C}^*_\Gamma) \rightsquigarrow (\Gamma \ltimes A, \hat{\rho}, \lambda)$

$$\hat{\rho}^{\Psi}_{\hat{\gamma}}(d) = \lambda_{\psi(\hat{\gamma})} \hat{\rho}_{\hat{\gamma}}(d) \lambda_{\psi(\hat{\gamma})}^*$$

$(\Gamma \ltimes A, \hat{\rho}^{\psi}, \lambda_\gamma)$ satisf. Landstad theorem.

$\Big\downarrow$

A^{Ψ} - Landstad algebra

$\pi \rightsquigarrow \pi^{\psi} \equiv \mathcal{C}\mathcal{J}(\pi)\Big|_{A^{\psi}} \in \mathrm{Mor}_\Gamma(A^{\psi}, B^{\psi})$

$\mathcal{RD}^{\psi} \quad \mathcal{C}^*_\Gamma \longrightarrow \mathcal{C}^*_\Gamma$ - deformation functor

$\mathcal{RD}^{\psi}$ is invertible $(\mathcal{RD}^{\psi})^{-1} = \mathcal{RD}^{\Phi}$

where $\Phi(\hat{\gamma}) = \psi(\gamma)^{-1}$

$\mathcal{RD}^{\psi}$ is not monoidal

we define $\otimes_\psi : \mathcal{C}^*_\Gamma \times \mathcal{C}^*_\Gamma \to \mathcal{C}^*_\Gamma$ where

$$\otimes_\psi = \mathcal{RD}^{\psi} \circ \otimes \circ \mathcal{RD}^{\Phi} \times \mathcal{RD}^{\Phi}$$

Properties of $\otimes_\Psi$

(a) $A^\Psi \otimes_\Psi B^\Psi = (A \otimes B)^\Psi$

(b) $i^A \in \mathrm{Mor}_\Gamma(A, A \otimes B)$ $\quad i^A(a) = a \otimes 1\!\!1$

$i^B \in \mathrm{Mor}_\Gamma(B, A \otimes B)$ $\quad i^B(b) = 1\!\!1 \otimes b$

$RD^\Psi(i^A) = i^{A^\Psi} \in \mathrm{Mor}_\Gamma(A^\Psi; A^\Psi \otimes_\Psi B^\Psi)$

$RD^\Psi(i^B) = i^{B^\Psi} \in \mathrm{Mor}_\Gamma(B^\Psi; A^\Psi \otimes B^\Psi)$

Proposition

$A^\Psi \underset{\Psi}{\otimes} B^\Psi = \{ i^{A^\Psi}(a)\, i^{B^\Psi}(b) \quad a \in A^\Psi, b \in B^\Psi \}^{\mathrm{cls}}$.

(c) $\pi_1 \in \mathrm{Mor}_\Gamma(A_1, B_1)$ $\quad \rightsquigarrow \pi_1^\Psi \otimes_\Psi \pi_2^\Psi$

$\pi_2 \in \mathrm{Mor}_\Gamma(A_2, B_2)$ $\quad \cap$

$\mathrm{Mor}(A_1^\Psi \otimes_\Psi A_2^\Psi, B_1^\Psi \otimes_\Psi B_2^\Psi)$

(d) we may form

$\pi_1^\Psi \otimes_\Psi id$

$\pi_2^\Psi \otimes_\Psi id$

Quantum groups deformation

$\mathbb{G} = (A, \Delta)$ and $(A, \rho) \in \mathrm{Obj}\, \mathcal{C}^*_\Gamma$ and

$$\Delta \in \mathrm{Mor}_\Gamma (A, A \otimes A)$$

$$\Delta \circ \rho_\gamma = (\rho_\gamma \otimes \rho_\gamma) \circ \Delta$$

group of autom. of $\mathbb{G}$

Applying RD^Ψ we get (A^Ψ, Δ^Ψ) where $A^\Psi \in \mathrm{Obj}(\mathcal{C}^*_\Gamma)$ and $\Delta^\Psi \in \mathrm{Mor}_\Gamma(A^\Psi, A^\Psi \otimes A^\Psi)$

Properties

(a) Δ^Ψ is coassociative

$$(\Delta^\Psi \otimes_\Psi \mathrm{id}) \circ \Delta^\Psi = (\mathrm{id} \otimes_\Psi \Delta^\Psi) \circ \Delta^\Psi$$

(b) Δ^Ψ sat. cancellation law

$$A^\Psi \underset{\Psi}{\otimes} A^\Psi = \{ \Delta^\Psi(a)(\mathbb{1} \otimes b) : a, b \in A^\Psi \}^{\mathrm{cls}}$$

$$= \{ \Delta^\Psi(a)(b \otimes \mathbb{1}) : a, b \in A^\Psi \}^{\mathrm{cls}}$$