Quantization by categorification.

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Categorification of geometry. History

- Grothendieck (toposes, Grothendieck categories of quasicoherent sheaves),
- Gabriel-Rosenberg (reconstruction of quasi-compact quasi-separated schemes from their Grothendieck categories of quasicoherent sheaves),
- Balmer, Lurie, Brandenburg-Chirvasitu (reconstruction theorems from monoidal categories).

Theorem (Corollary from Balmer's and Murfet's theorems)

Every quasi-compact semi-separated scheme with an ample family of invertible sheaves can be reconstructed uniquely up to isomorphism from its monoidal category Qcoh_X*:*

$$X = \mathrm{Spec}^{\otimes}(\mathrm{D}^{cpct}(\mathrm{Qcoh}_X)).$$

Theorem (Brandenburg-Chirvasitu)

For a quasi-compact quasi-separated scheme X *and an arbitrary scheme* Y *the pullback construction* $f \mapsto f^*$ *implements an equivalence between the discrete category of morphisms* $X \to Y$ *and the category of cocontinuous strong opmonoidal functors* $\mathrm{Qcoh}_Y \to \mathrm{Qcoh}_X$*.*

Monoidal schemes

Corollary

There is a fully faithful 2-functor from the 2-category (with discrete categories of 1-morphisms) of quasi-compact quasi-separated schemes with an ample family of invertible sheaves to the 2-category of abelian monoidal categories with doctrinal additive adjunctions and doctrinal (in the sense of Max Kelly) natural transformations

$$\mathrm{Sch} \longrightarrow \mathit{AbMonCat},$$

$$X \quad \vdash\!\!\rightsquigarrow \quad \mathrm{Qcoh}_X,$$

$$\begin{array}{ccc} X & & \mathrm{Qcoh}_X \\ \downarrow f & \vdash\!\!\rightsquigarrow & f^* \uparrow \dashv \downarrow f_* \\ Y & & \mathrm{Qcoh}_Y. \end{array}$$

We call a monoid *R* in a monoidal abelian category Qcoh_Y *ring over Y* if $R \otimes (-)$ and $(-) \otimes R$ are additive right exact and for any two R-bimodules M_1, M_2 in Qcoh_Y the canonical coequalizer defining the R-balanced tensor product

$$M_1 \otimes R \otimes M_2 \rightrightarrows M_1 \otimes M_2 \longrightarrow M_1 \otimes_R M_2,$$

remains a coequalizer after tensoring in Qcoh_Y from any side by an arbitrary R-bimodule.

In Ab, Vect_k or Mod_K it is satisfied automatically, in general it is sufficient to conclude that the category of R-bimodules is canonically monoidal abelian.

Coordinate algebras and their spectra

If R is a ring over a monoidal scheme Y and a^* is an additive opmonoidal monad on Bim_R we call the pair $A = (R, a^*)$ *coordinate algebra over Y*. By $\mathrm{Spec}_Y(A)$ we mean a monoidal scheme such that

$$\mathrm{Qcoh}_{\mathrm{Spec}_Y(A)} = \mathrm{Bim}_R^{a^*},$$

the monoidal Eilenberg-Moore category of the opmonoidal monad a^* on the monoidal category Bim_R.

One has a canonical morphism of monoidal schemes

$$\mathrm{Spec}_Y(A) \xrightarrow{p^A} Y,$$

being an opmonoidal $\dashv$ monoidal adjunction $p_A^* \dashv p_*^A$, where $p_*^A : \mathrm{Qcoh}_{\mathrm{Spec}_Y(A)} \to \mathrm{Qcoh}_Y$ is the forgetful functor and its left adjoint is the free construction functor of the form $p_A^* \mathscr{G} = a^*(R \otimes \mathscr{G} \otimes R)$.

Affine morphisms

We call a morphism $f : X \to Y$ of monoidal schemes *affine* if f_* is faithful and exact, the monoid $f_*\mathscr{O}_X$ is a ring over Y and the natural (in $(\mathscr{F}_1, \mathscr{F}_2)$) transformation

$$f_*\mathscr{F}_1 \otimes_{f_*\mathscr{O}_X} f_*\mathscr{F}_2 \longrightarrow f_*(\mathscr{F}_1 \otimes \mathscr{F}_2)$$

is an isomorphism.

Stein factorization

The following theorem is a monoidal analog of Grothendieck's characterization of affine morphisms among quasi-compact quasi-separated ones in terms of Stein factorization from EGA II § 1.

Theorem

A morphism $f : X \to Y$ of monoidal schemes is affine if and only if there is a coordinate algebra A over Y and a Stein factorization

$$\begin{array}{ccc} X & \xrightarrow{f^A} & \mathrm{Spec}_Y(A) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p^A} & \\ & Y & \end{array}$$

such that f^A is an isomorphism.

Affine monoidal schemes

$\mathrm{Qcoh}_{\mathrm{Spec}(\mathbb{Z})} := \mathrm{Ab}.$

Theorem

A monoidal scheme X is affine if and only if Qcoh_X has an Ab-copowered projective generator P being a comonoid such that the map

$$\mathrm{Qcoh}_X(P, \mathscr{F}_1) \otimes_{\mathrm{Qcoh}_X(P, \mathscr{O}_X)} \mathrm{Qcoh}_X(P, \mathscr{F}_2) \to \mathrm{Qcoh}_X(P, \mathscr{F}_1 \otimes \mathscr{F}_2),$$

natural in $(\mathscr{F}_1, \mathscr{F}_2)$ is an isomorphism.

Examples of affine monoidal schemes

1. $\mathrm{Spec}(\mathbb{Z})$. $\mathrm{Qcoh}_{\mathrm{Spec}(\mathbb{Z})} := \mathrm{Ab}$, $P := \mathbb{Z}$, $\mathcal{A} = (\mathbb{Z}, 1)$.
2. $\mathrm{Spec}(A)$ for a commutative ring A. $\mathrm{Qcoh}_{\mathrm{Spec}(A)} := \mathrm{Mod}_A$, $P = A$, $\mathcal{A} = (R, a^*)$ where the ring R in Ab is the commutative ring A itself and $a^* = (-)_{A \otimes A} A = (-)/[A, -])$ is an opmonoidal (idempotent) monad of symmetrization on the category of A-bimodules, which makes sense because A is commutative.
3. $\mathrm{Spec}(A)$ for a ring A. $\mathrm{Qcoh}_{\mathrm{Spec}(A)} := \mathrm{Bim}_A$, $P = A \otimes A =$ the Sweedler comonoid, $\mathcal{A} = (R, a^*)$, $R = A$ and a^* is the identity opmonoidal monad on Bim_A.

3. 4. $\mathrm{Spec}(A)$ for a bialgebroid A. $\mathrm{Qcoh}_{\mathrm{Spec}(A)} := \mathrm{Mod}_A$, $P = A =$ the underlying comonoid, $\mathcal{A} = (R, a^*)$, $R = A$, an opmonoidal monad a^* on the category of R-bimodules is defined as tensoring an R-bimodule over the enveloping ring $R^e = R^{op} \otimes R$ by a bialgebroid A.

Due to Szlachanyi, every additive opmonoidal monad a^* on Bim_R admitting a right adjoint is of that form.

Corings on monoidal schemes

We call a comonoid C in a monoidal abelian category Qcoh_X *coring on X* if $C \otimes (-)$ and $(-) \otimes C$ are additive right exact and for any two C-bicomodules M_1, M_2 in Qcoh_X the canonical equalizer defining the C-co-balanced cotensor product

$$M_1 \Box^C M_2 \longrightarrow M_1 \otimes M_2 \rightrightarrows M_1 \otimes C \otimes M_2$$

remains an equalizer after tensoring in Qcoh_X from any side by an arbitrary C-bicomodule.

In Ab, Vect_k or Mod_K it is not satisfied automatically, imposing flatness and purity conditions. In general it is sufficient to conclude that the category of C-bicomodules is canonically monoidal abelian.

Gluings and their quotients

If C is a coring over a monoidal scheme X and g_* is an additive monoidal comonad on Bim_R we call the pair $G = (C, g_*)$ *gluing in X*. By X/G we mean a monoidal scheme such that

$$\mathrm{Qcoh}_{X/G} = \mathrm{Bic}^{C}_{a^*},$$

the monoidal Eilenberg-Moore category of the monoidal comonad g_* on the monoidal category Bic^C.
One has a canonical morphism of monoidal schemes

$$X \xrightarrow{q^G} X/G,$$

being an opmonoidal $\dashv$ monoidal adjunction $q^*_G \dashv q^G_*$, where $q^*_G : \mathrm{Qcoh}_{X/G} \to \mathrm{Qcoh}_X$ is the coforgetful functor and its right adjoint is the cofree construction functor of the form
$q^G_* \mathscr{F} = g_*(C \otimes \mathscr{F} \otimes C)$.

(Faithfully) flat morphisms

We call a morphism $f : X \to Y$ of monoidal schemes *(faithfully) flat* if f_* is (faithful and) exact, the comonoid $f^*\mathscr{O}_Y$ is a coring on X and the natural (in $(\mathscr{G}_1, \mathscr{G}_2)$) transformation

$$f^*(\mathscr{G}_1 \otimes \mathscr{G}_2) \longrightarrow f^*\mathscr{G}_1 \square^{f^*\mathscr{O}_Y} f^*\mathscr{G}_2$$

is an isomorphism.

Universal property of the quotient aka faithfully flat descent

Theorem

A morphism $f : X \to Y$ of monoidal schemes is faithfully flat if and only if there is a gluing G in X and a unique factorization

$$\begin{array}{ccc} X & \xrightarrow{\quad q^G \quad} & X/G \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^G} & \\ & Y & \end{array}$$

such that f^G is an isomorphism.

Examples of gluings

1. Finite open covering of a scheme Y, $Y = \bigcup_i V_i$, $f : X \coprod_i V_i \to \bigcup_i V_i = Y$, Qcoh_X quasicoherent sheaves on X, $C := \mathscr{O}_X$, $g_* := p_{0*}p_1^*$, $\mathrm{Qcoh}_{X/G}$:= gluing data.
2. Group action $a : X \times G \to X$ of an affine group scheme G on a scheme X, $p : X \times G \to X$ projection, $C := \mathscr{O}_X$, $g_* := p_* a^*$, $\mathrm{Qcoh}_{X/G}$:= G-equivariant quasicoherent sheaves on X, X/G "homotopy quotient".
3. Corepresentations of a bicoalgebroid. k a commutative ring, C a pure coalgebra over k, C_e its co-enveloping coalgebra, A C-bicoalgebroid, $g_* := A\Box^{C_e}(-)$. $\mathrm{Qcoh}_X = \mathrm{Mod}_k$, $\mathrm{Qcoh}_{X/G}$:= corepresentations of the bicoalgebroid (C, A)

Base change

Base change context: Cartesian square in schemes

$$\begin{array}{ccc} X & \xrightarrow{p} & S \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{r} & T, \end{array}$$

with q affine, faithfully flat.
Then canonical transformations

$$r^* q_* \to f_* p^*, \quad q^* r_* \to p_* f^*$$

are isomorphisms.
In monoidal geometry we take them as a substitute of a good cartesian square.

Universal G-bundle

Let us consider a canonical monoidal quotient map q of a one point space over a field k by the canonical action of an affine group scheme G.

Theorem

The (monoidal) quotient map $q : \star \to \star/G$ is an affine faithfully flat G-principal fibration.

Classifying maps for G-bundles

Theorem

Any affine faithfully flat principal G-fibration $f : X \to Y$ with schemes X and Y affine over a field k (faithfully flat H-Galois extension of commutative k-algebras $B = A^{coH} \subset A$) is a base change of the universal principal G-fibration, i.e. there is a base change diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & \star \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{r} & \star/G \end{array}$$

The classifying map r is defined as the associated vector bundle construction

$$r^* V = A \square^H V.$$

All that means that...

Grothendieck's idea works also in Noncommutative Geometry!
Forget about spaces (groups, algebras,...) go abelian monoidal (tensor triangulated, tensor A_∞, ...) categories and (co)monads.

Monoidal deformation quantization of classical geometry

Classical schemes lead to symmetric monoidal categories, classical inverse image functors are strong (op)monoidal.We are to deform it.

We define a *formal deformation of the monoidal structure* (η, μ) as a sequence $(\mu_0, \mu_1, \ldots)$ of natural transformations

$$\mu_i^{\mathscr{F}_1, \mathscr{F}_2} : f_* \mathscr{F}_1 \otimes f_* \mathscr{F}_2 \to f_* \left(\mathscr{F}_1 \otimes \mathscr{F}_2 \right)$$

such that $\mu_0 = \mu$ and for $k > 0$ the following identities hold

$$\sum_{i+j=k} \mu_i^{\mathscr{F}_1, \mathscr{F}_2 \otimes \mathscr{F}_3} \left(Id^{f_* \mathscr{F}_1} \otimes \mu_j^{\mathscr{F}_2, \mathscr{F}_3} \right) = \sum_{i+j=k} \mu_i^{\mathscr{F}_1 \otimes \mathscr{F}_2, \mathscr{F}_3} \left(\mu_j^{\mathscr{F}_1, \mathscr{F}_2} \otimes Id^{f_* \mathscr{F}_3} \right)$$

Taking sequences of length $n+1$ $(\mu_0, \dots, \mu_n)$ such that $\mu_0 = \mu$ and the identities hold for $k = 1, \dots, n$ we obtain the definition of an *n-th infinitesimal deformation of the monoidal structure* (η, μ). We define the group of sequences $\varphi = (\varphi_0, \varphi_1, \dots)$ of natural transformations

$$\varphi_i^{\mathscr{F}} : f_*\mathscr{F} \to f_*\mathscr{F}$$

such that $\varphi_0^{\mathscr{F}} = Id^{f_*\mathscr{F}_3}$, with the neutral element $(Id, 0, 0, \dots)$ and the composition

$$(\varphi\widetilde{\varphi})_k^{\mathscr{F}} = \sum_{i+j=k} \varphi_i^{\mathscr{F}} \widetilde{\varphi}_j^{\mathscr{F}}.$$

We say that formal deformations $(\mu_0, \mu_1, \ldots)$ and $(\widetilde{\mu}_0, \widetilde{\mu}_1, \ldots)$ are *equivalent* if there exists a sequence $(\varphi_0, \varphi_1, \ldots)$ such that for $k > 0$

$$\sum_{i+j=k} \varphi_i^{\mathscr{F}_1 \otimes \mathscr{F}_2} \mu_j^{\mathscr{F}_1, \mathscr{F}_2} = \sum_{i+j_1+j_2=k} \widetilde{\mu}_i^{\mathscr{F}_1, \mathscr{F}_2} \left(\varphi_{j_1}^{\mathscr{F}_1} \otimes \varphi_{j_2}^{\mathscr{F}_2} \right).$$

Taking sequences of length $n+1$ $(\varphi_0, \ldots, \varphi_n)$ and n-th infinitesimal deformations $(\mu_0, \ldots, \mu_n)$ and $(\widetilde{\mu}_0, \ldots, \widetilde{\mu}_n)$ such that $\varphi_0 = Id$ and hold for $k = 1$ we obtain the definition of *equivalence of n-th infinitesimal deformations*.

Hochschild complex

We define abelian groups of k-cochains as follows.
For $k = 0$ it consists of morphisms

$$c^0 : \mathscr{O}_Y \to f_* \mathscr{O}_X. \tag{1}$$

For $k > 0$ it consists of natural transformations c^k of multifunctors

$$c^{\mathscr{F}_1, \ldots, \mathscr{F}_k} : f_* \mathscr{F}_1 \otimes \cdots \otimes f_* \mathscr{F}_k \to f_* \left(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_k \right). \tag{2}$$

In the element-wise convention

$$1 \mapsto c^0(1) \tag{3}$$

and for $k > 0$

$$n_1 \otimes \cdots \otimes n_k \mapsto c^{\mathscr{F}_1, \ldots, \mathscr{F}_k}(n_1, \ldots, n_k). \tag{4}$$

Differential for $k = 0$

We equip them with the Hochschild type differential as follows.

$$(\mathrm{d}c^0)^{\mathscr{F}_1}(n_1) := \mu^{\mathscr{F}_1,\mathscr{O}_X}(n_1, c^0(1)) - \mu^{\mathscr{O}_X,\mathscr{F}_1}(c^0(1), n_1)$$

Differential for $k > 0$

$$
\begin{aligned}
&(\mathrm{d}c^k)^{\mathscr{F}_1,\dots,\mathscr{F}_{k+1}}(n_1,\dots,n_{k+1}) \\
&:= (-1)^{k-1}\mu^{\mathscr{F}_1,\mathscr{F}_2\otimes\cdots\otimes\mathscr{F}_k}(n_1, c^{\mathscr{F}_2,\dots,\mathscr{F}_{k+1}}(n_2,\dots,n_{k+1})) \\
&+\sum_{i=1}^{k}(-1)^{i+k-1}c^{\mathscr{F}_1,\dots,\mathscr{F}_i\otimes\mathscr{F}_{i+1},\dots,\mathscr{F}_{k+1}}(n_1,\dots,\mu^{\mathscr{F}_i,\mathscr{F}_{i+1}}(n_i,n_{i+1}),\dots,n_{k+1}) \\
&+\mu^{\mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_k,\mathscr{F}_{k+1}}(c^{\mathscr{F}_1,\dots,\mathscr{F}_k}(n_1,\dots,n_k),n_{k+1}).
\end{aligned}
$$

By a routine checking we see that $d^2 = 0$. We call the resulting cohomology *Hochschild cohomology of a monoidal functor*.

Gerstenhaber algebra

Theorem

The Hochschild complex of a monoidal functor with the above product and substitutions is a strong homotopy Gerstenhaber algebra.

Hochschild cohomology and deformations

Theorem

There is a one-to-one correspondence between the group of equivalence classes of 1-st infinitesimal deformations of a given monoidal functor and its second Hochschild cohomology group.

Moreover, succesive lifting of equivalences "modulo t^2, t^3, ..." of formal deformations obviously makes perfect sense also for monoidal functors and the following theorem still holds.

Theorem

If the second Hochschild cohomology of a given monoidal functor vanishes then its all formal deformations are equivalent.

Symmetric monoidal categories

Let categories Qcoh_X, Qcoh_Y be symmetric with symmetries

$$\sigma_X^{\mathscr{F}_1,\mathscr{F}_2} : \mathscr{F}_1 \otimes \mathscr{F}_2 \to \mathscr{F}_2 \otimes \mathscr{F}_1, \quad \sigma_Y^{\mathscr{G}_1,\mathscr{G}_2} : \mathscr{G}_1 \otimes \mathscr{G}_2 \to \mathscr{G}_2 \otimes \mathscr{G}_1 \quad (5)$$

The symmetry of the monoidal functor of $f_* : \mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$ is described by the identity

$$\mu^{\mathscr{F}_1,\mathscr{F}_2} = f_* \left(\sigma_X^{\mathscr{F}_2,\mathscr{F}_1}\right) \mu^{\mathscr{F}_2,\mathscr{F}_1} \sigma_Y^{f_*\mathscr{F}_1, f_*\mathscr{F}_2}. \quad (6)$$

Note that if the monoidal functor f_* is symmetric then the monoid $f_*\mathscr{O}_X$ is commutative.

Poisson functors

We say that a symmetric monoidal functor f_* is *Poisson* if there is given a natural transformation π of bifunctors

$$\pi^{\mathscr{F}_1,\mathscr{F}_2} : f_*\mathscr{F}_1 \otimes f_*\mathscr{F}_2 \to f_*\left(\mathscr{F}_1 \otimes \mathscr{F}_2\right) \tag{7}$$

satisfying the following identities:

(skew symmetry)

$$\pi^{\mathscr{F}_1,\mathscr{F}_2} = -f_* \left(\sigma_X^{\mathscr{F}_2,\mathscr{F}_1}\right) \pi^{\mathscr{F}_2,\mathscr{F}_1} \sigma_Y^{f_*\mathscr{F}_1, f_*\mathscr{F}_2}, \tag{8}$$

(Jacobi identity)

$$\pi^{\mathscr{F}_1,\mathscr{F}_2\otimes\mathscr{F}_3}\left(Id^{f_*\mathscr{F}_1}\otimes\pi^{\mathscr{F}_2,\mathscr{F}_3}\right)-\pi^{\mathscr{F}_1\otimes\mathscr{F}_2,\mathscr{F}_3}\left(\pi^{\mathscr{F}_1,\mathscr{F}_2}\otimes Id^{f_*\mathscr{F}_3}\right.$$

$$(9)$$

$$=f_*\left(\sigma_X^{\mathscr{F}_2,\mathscr{F}_1}\otimes Id^{\mathscr{F}_3}\right)\pi^{\mathscr{F}_2,\mathscr{F}_1\otimes\mathscr{F}_3}\left(Id^{f_*\mathscr{F}_2}\otimes\pi^{\mathscr{F}_1,\mathscr{F}_3}\right)\left(\sigma_Y^{f_*\mathscr{F}_1,f_*\mathscr{F}_2}\otimes Id^{f_*\mathscr{F}_3}\right)$$

(derivation identity)

$$\pi^{\mathscr{F}_1,\mathscr{F}_2\otimes\mathscr{F}_3}\left(Id^{f_*\mathscr{F}_1}\otimes\mu^{\mathscr{F}_2,\mathscr{F}_3}\right)-\mu^{\mathscr{F}_1\otimes\mathscr{F}_2,\mathscr{F}_3}\left(\pi^{\mathscr{F}_1,\mathscr{F}_2}\otimes Id^{f_*\mathscr{F}_3}\right) \tag{10}$$

$$= f_*\left(\sigma_X^{\mathscr{F}_2,\mathscr{F}_1}\otimes Id^{\mathscr{F}_3}\right)\mu^{\mathscr{F}_2,\mathscr{F}_1\otimes\mathscr{F}_3}\left(Id^{f_*\mathscr{F}_2}\otimes\pi^{\mathscr{F}_1,\mathscr{F}_3}\right)\left(\sigma_Y^{f_*\mathscr{F}_1,f_*\mathscr{F}_2}\otimes Id^{f_*\mathscr{F}_3}\right)$$

Theorem

If the symmetric monoidal functor f_* is Poisson then the commutative monoid $f_*\mathscr{O}_X$ is Poisson as well, with the Poisson bracket

$$\pi^{\mathscr{O}_X,\mathscr{O}_X}: \ f_*\mathscr{O}_X \otimes f_*\mathscr{O}_X \to f_*\left(\mathscr{O}_X \otimes \mathscr{O}_X\right) = f_*\mathscr{O}_X. \qquad (11)$$

Quasi-classical limit

Theorem

Given a formal deformation $(\mu_0, \mu_1, \ldots)$ *of a symmetric monoidal structure* (η, μ) *on* f_* *the natural transformation* π *of bifunctors*

$$\pi^{\mathscr{F}_1,\mathscr{F}_2} := \mu_1^{\mathscr{F}_1,\mathscr{F}_2} - f_*\left(\sigma_X^{\mathscr{F}_2,\mathscr{F}_1}\right)\mu_1^{\mathscr{F}_2,\mathscr{F}_1}\sigma_Y^{f_*\mathscr{F}_1,f_*\mathscr{F}_2} \qquad (12)$$

makes f_* *a Poisson functor.*

Cyclic schemes

Definition

A *cyclic scheme* X is a *monoidal abelian category* $(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X)$ equipped with a *cyclic functor* $\Gamma_X : \mathrm{Qcoh}_X \to \mathrm{Ab}$, *i.e.* an additive functor equipped with a natural isomorphism

$$\gamma_{\mathscr{F}_0,\mathscr{F}_1} : \Gamma_X(\mathscr{F}_0 \otimes \mathscr{F}_1) \to \Gamma_X(\mathscr{F}_1 \otimes \mathscr{F}_0)$$

satisfying the following identities

$$\gamma_{\mathscr{F}_1,\mathscr{F}_2\otimes\mathscr{F}_0} \circ \gamma_{\mathscr{F}_0,\mathscr{F}_1\otimes\mathscr{F}_2} = \gamma_{\mathscr{F}_0\otimes\mathscr{F}_1,\mathscr{F}_2},$$

$$\gamma_{\mathscr{O}_X,\mathscr{F}} = \gamma_{\mathscr{F},\mathscr{O}_X} = \mathrm{Id}_{\tau_X(\mathscr{F})},$$

$$\gamma_{\mathscr{F}_1,\mathscr{F}_0} = \gamma^{-1}_{\mathscr{F}_0,\mathscr{F}_1}$$

.

Cyclic symmetry

Lemma

$$\gamma_{\mathscr{F}_n,\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_{n-1}} \circ \gamma_{\mathscr{F}_{n-1},\mathscr{F}_n\otimes\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_{n-2}} \circ \cdots \circ \gamma_{\mathscr{F}_0,\mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_n}$$

$$= \mathrm{Id}_{\tau_X(\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_n)}.$$

Example. Commutative schemes

- With every classical commutative scheme (quasi-compact, quasi-separated) X one can associate an abelian monoidal category $(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X)$ of quasi-coherent sheaves.
  It is equipped with a canonical cyclic functor of sections

$$\Gamma_X := \Gamma(X, -) : \mathrm{Qcoh}_X \to \mathrm{Ab}$$

  where the cyclic structure comes from the symmetry of the monoidal structure.
- For an affine scheme $X = \mathrm{Spec}(A)$, A being a commutative ring there is a strong monoidal equivalence

$$(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X) \xrightarrow{\sim} (\mathrm{Mod}_A, \otimes_A, A),$$

  and the cyclic functor forgets the A-module structure.

Example: Cyclic spectra of associative rings

Let R be a unital associative ring. We define a cyclic scheme X so that is the monoidal abelian category of R-bimodules

$$(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X) := (\mathrm{Bim}_R, \otimes_R, R)$$

with the tensor product balanced over R.
If $\mathscr{F} = M$ is an R-bimodule, we have a canonical cyclic functor

$$\Gamma_X(\mathscr{F}) = \Gamma_R(M) := M \otimes_{R^o \otimes R} R$$

obtained by tensoring balanced over the enveloping ring $R^o \otimes R$.
The natural transformation γ is the flip

$$(M_0 \otimes_R M_1) \otimes_{R^o \otimes R} R \to (M_1 \otimes_R M_0) \otimes_{R^o \otimes R} R,$$

$$(m_0 \otimes m_1) \otimes r \mapsto (m_1 \otimes m_0) \otimes r,$$

well defined and satisfying axioms of a cyclic functor thanks to balancing over $R^o \otimes R$.
We call this cyclic scheme the cyclic spectrum of an associative ring R.

Traces

We want to unravel the natural origin of traces. First, we want to understand the *character*

$$S/[S,S] \to R/[R,R]. \tag{13}$$

of a representation $S \to \mathrm{End}_R(P)$ of the ring S on a finitely generated projective right R-module P.
The point is that in general it *is not* induced by any ring homomorphism $S \to R$, but merely by some *mild correspondence* from S to R.

Mild correspondences

Basic principles of *mild correspondences* we derive from classical algebraic geometry.There a correspondence f from a scheme X to a scheme Y is a diagram of (quasi-compact and quasi-separated) schemes

$$\begin{array}{ccc} \widetilde{X} & \stackrel{\widetilde{f}}{\longrightarrow} & Y \\ \pi \downarrow & & \\ X & & \end{array}$$

and we call it *mild* if its domain projection π is finite and flat.

Induced adjunction

Although a correspondence f is not a honest morphism of schemes $f : X \to Y$, it still defines a monoidal functor of a direct image $f_* := \widetilde{f}_* \pi^* : \mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$ between categories of quasi-coherent sheaves. It is monoidal because $\widetilde{f}_*$ is monoidal and π^* is strong opmonoidal, hence monoidal as well.
If in addition f is mild f_* has a left adjoint (hence canonically opmonoidal) functor $f^* \dashv f_*$

Coalgebra

Moreover, there exist an $\mathscr{O}_X$-coalgebra D equipped with a structure of an $\pi_*\mathscr{O}_{\widetilde{X}}$-module s.t.

$$f^* := \pi_* \widetilde{f}^*(-) \otimes_{\pi_* \mathscr{O}_{\widetilde{X}}} D : \mathrm{Qcoh}_Y \to \mathrm{Qcoh}_X,$$

$$f_* = \widetilde{f}_*(\mathscr{H}om_X(D, -)^{\sim}) : \mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$$

where $(-)^{\sim}$ denotes sheafifying by localisation of a $\pi_*\mathscr{O}_{\widetilde{X}}$-module to obtain a quasi-coherent sheaf on $\widetilde{X} = \mathrm{Spec}_X(\pi_*\mathscr{O}_{\widetilde{X}})$, the relative spectrum of a commutative quasi-coherent $\mathscr{O}_X$-algebra $\pi_*\mathscr{O}_{\widetilde{X}}$.

Mild correspondences of affine schemes

Thus for affine schemes $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ a mild correspondence f from X to Y can be written as a homomorphism of commutative rings

$$S \to \mathrm{Hom}_R(D, R),\ s \mapsto (d \mapsto s(d))$$

where the ring on the right hand side is a convolution ring dual to some cocommutative R-coalgebra D, *i.e.* its unit is a counit $\varepsilon : D \to R$ and multiplication comes from the comultiplication $D \to D \otimes_R D$, $d \mapsto d_{(1)} \otimes d_{(2)}$ (Heyneman-Sweedler notation) via dualization, *i.e.*

$$\mathrm{Hom}_R(D, R) \otimes \mathrm{Hom}_R(D, R) \to \mathrm{Hom}_R(D, R),$$

$$\rho_1 \otimes \rho_2 \mapsto (d \mapsto \rho_1(d_{(1)})\rho_2(d_{(2)})).$$

Adjunction for affine schemes

The corresponding adjunction between monoidal categories of modules $\mathrm{Qcoh}_X = \mathrm{Mod}_R$ and $\mathrm{Qcoh}_Y = \mathrm{Mod}_S$ is given as follows

$$f_* M = \mathrm{Hom}_R(D, M),$$

$$f^* N = (N \otimes_S \mathrm{Hom}_R(D, R)) \otimes_{\mathrm{Hom}_R(D,R)} D = N \otimes_S D.$$

A monoidal structure of f_* (or equivalently, an opmonoidal structure of f^*) is related to the coalgebra structure of D as follows.

The morphism $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is defined as $S \to \mathrm{Hom}_R(D, R),\ s \mapsto (d \mapsto s(d))$, with respect to which the image of the unit of S is equal to the counit of D, and the natural transformation $f_* \mathscr{F}_0 \otimes f_* \mathscr{F}_1 \to f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)$ is defined by means of the comultiplication of D as

$$\mathrm{Hom}_R(D, M_1) \otimes_S \mathrm{Hom}_R(D, M_2) \to \mathrm{Hom}_R(D, M_1 \otimes_R M_2),$$

$$\mu_1 \otimes \mu_2 \mapsto (d \mapsto \mu_1(d_{(1)}) \otimes \mu_2(d_{(2)})).$$

Mild correspondences of noncommutative rings

This can be easily extended to noncommutative rings by noticing that, for R being commutative, R itself and any coalgebra D over R are symmetric R-bimodules, hence

$$\mathrm{Hom}_R(D, R) = \mathrm{Hom}_{R^o \otimes R}(D, R)$$

where on the right hand side we have homomorphisms of R-bimodules regarded as right modules over the enveloping ring $R^o \otimes R$. This still makes sense if one takes noncommutative rings R and S, and an arbitrary R-coring D instead of a cocommutative R-coalgebra over a commutative ring R.

Mild correspondences of rings. Definition

Then we say that a *mild correspondence from a ring S to a ring R* is given if there is given a ring homomorphism

$$S \to \mathrm{Hom}_{R^o \otimes R}(D, R), \ s \mapsto (d \mapsto s(d))$$

where the structure of the convolution ring on $\mathrm{Hom}_{R^o \otimes R}(D, R)$ is induced from the R-coring structure of D.

Adjunction for noncommutative rings

A mild correspondence $S \to \mathrm{Hom}_{R^o \otimes R}(D, R)$ from a ring S to a ring R defines an adjunction between monoidal categories of bimodules $\mathrm{Qcoh}_X = \mathrm{Bim}_R$ and $\mathrm{Qcoh}_Y = \mathrm{Bim}_S$ as follows

$$f_* M = \mathrm{Hom}_{R^o \otimes R}(D, M), \;\; f^* N = N \otimes_{S^o \otimes S} D.$$

A monoidal structure of f_* (or equivalently, an opmonoidal structure of f^*) generalizes the structure of the convolution ring.

What mild correspondences have to do with traces?

$\mathrm{End}_R(P)$ is a convolution ring $\mathrm{Hom}_{R^\circ \otimes R}(D, R)$ of an R-coring $D = P^* \otimes P$ whose canonical counit $\varepsilon : D \to R$ is the evaluation of elements of $P^* = \mathrm{Hom}_R(P, R)$ on elements of P,

$$P^* \otimes P \to R,$$

$$p^* \otimes p \to p^*(p),$$

its canonical comultiplication $D \to D \otimes_R D$, $d \mapsto d_{(1)} \otimes d_{(2)}$ can be written in terms of any dual basis $(p_i, p_i^*)_{i \in I}$ for P as

$$P^* \otimes P \to (P^* \otimes P) \otimes_R (P^* \otimes P),$$

$$p^* \otimes p \mapsto \sum_{i \in I} (p^* \otimes p_i) \otimes (p_i^* \otimes p),$$

What is the corresponding adjunction?

The morphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is defined as above and the natural transformation $f_*\mathscr{F}_1 \otimes f_*\mathscr{F}_2 \to f_*(\mathscr{F}_1 \otimes \mathscr{F}_2)$ is defined by means of the comultiplication of D as

$$\mathrm{Hom}_{R^o \otimes R}(D, M_1) \otimes_S \mathrm{Hom}_{R^o \otimes R}(D, M_2) \to \mathrm{Hom}_{R^o \otimes R}(D, M_1 \otimes_R M_2),$$

$$\mu_1 \otimes \mu_2 \mapsto (d \mapsto \mu_1(d_{(1)}) \otimes \mu_2(d_{(2)})).$$

What is the character from this categorical perspective?

It is an R-component of a natural isomorphism of additive functors $\mathrm{Bim}_R \to \mathrm{Ab}$ whose M-component is

$$\mathrm{Hom}_{R^o \otimes R}(D, M) \otimes_{S^o \otimes S} S \to M \otimes_{R^o \otimes R} R,$$

$$\mu \otimes s \mapsto (\mu \otimes R)(\delta(1))$$

where $\delta \in \mathrm{Hom}_{S^o \otimes S}(S, D \otimes_{R^o \otimes R} R)$ is a canonical element which can be written in terms of any dual basis as

$$S \to (P^* \otimes P) \otimes_{R^o \otimes R} R,$$

$$s \mapsto \sum_{i \in I} (p_i^* \otimes s \cdot p_i) \otimes 1.$$

Character as a natural transformation

Finally, the character of the above representation can be written as a natural transformation

$$\Gamma_Y f_* \to \Gamma_X$$

where X and Y are cyclic spectra of rings R and S, respectively.

The trace property

It is easy to check that the trace property is equivalent to commutativity of all natural diagrams

$$\begin{array}{ccccc}
\Gamma_Y(f_*\mathscr{F}_0 \otimes f_*\mathscr{F}_1) & \longrightarrow & \Gamma_Y(f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)) & \longrightarrow & \Gamma_X(\mathscr{F}_0 \otimes \mathscr{F}_1) \\
\downarrow{\scriptstyle \gamma_{f_*\mathscr{F}_0, f_*\mathscr{F}_1}} & & & & \downarrow{\scriptstyle \gamma_{\mathscr{F}_0, \mathscr{F}_1}} \\
\Gamma_Y(f_*\mathscr{F}_1 \otimes f_*\mathscr{F}_0) & \longrightarrow & \Gamma_Y(f_*(\mathscr{F}_1 \otimes \mathscr{F}_0)) & \longrightarrow & \Gamma_X(\mathscr{F}_1\mathscr{F}_0),
\end{array}$$

Categorical back-bone of cyclic (co)homology

Motivated by this we consider now (large) abelian groups of natural transformations

$$c^{\mathscr{F}_0,\cdots,\mathscr{F}_n} : \Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \longrightarrow \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n) ,$$

$$c_{\mathscr{G}_0,\cdots,\mathscr{G}_n} : \Gamma_Y(\mathscr{G}_0 \otimes \cdots \otimes \mathscr{G}_n) \longrightarrow \Gamma_X(f^*\mathscr{G}_0 \otimes \cdots \otimes f^*\mathscr{G}_n) .$$

All this collection of abelian of natural transformations groups forms a cocyclic object.

- Cofaces come from the composition with natural transformations $f_*\mathscr{F}_0 \otimes f_*\mathscr{F}_1 \to f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)$ defining the monoidal structure of f_*,
- codegeneracies come from the structural morphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$,
- cyclic operators come from the natural transformations γ of the cyclic functors.

Example: Cyclic cohomology of an algebra

For an algebra A over a field k we prepare the following categorical environment.

$$\mathrm{Qcoh}_X = \mathrm{Vect}^{op}, \Gamma_X(V) = V^*, \ \mathrm{Qcoh}_Y = \mathrm{Vect}, \Gamma_Y(V) = V,$$

$$f_* V = Hom(V, A).$$

The cocyclic object of natural transformations:

$$\Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \to \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n)$$

reads as

$$Hom(V_0, A) \otimes \cdots \otimes Hom(V_n, A) \to Hom(V_0 \otimes \cdots \otimes V_n, k)$$

whose component corresponding to $V_0 = \cdots = V_n = k$ is

$$A \otimes \cdots \otimes A \to k,$$

the classical cocyclic object $A^\natural$ of Connes.

A cyclic Eilenberg-Moore construction

Let R be a ring in a monoidal category Qcoh_Y, and Bim_R be its monoidal category of bimodules equipped with an opmonoidal monad a^*. For any opmonoidal monad a^* on the monoidal category Bim_R of R-bimodules over a ring R in a monoidal category Qcoh_Y, with structural natural transformations

$$\mu_{a^*}^M : a^*a^*M \to a^*M, \ \eta_{a^*}^M : M \to a^*M,$$
$$\delta_{a^*}^{M_0,M_1} : a^*(M_0 \otimes_R M_1) \to a^*M_0 \otimes_R a^*M_1,$$

and a structural morphism

$$\varepsilon : a^*R \to R,$$

one defines a natural transformation of *right fusion*

$$\varphi_{a^*}^{M_0,M_1} : a^*(M_0 \otimes_R a^*M_1) \to a^*M_0 \otimes_R a^*M_1$$

as a composition

$$a^*(M_0 \otimes_R a^*M_1) \xrightarrow{\delta_{a^*}^{M_0,a^*M_1}} a^*M_0 \otimes_R a^*a^*M_1 \xrightarrow{a^*M_0 \otimes_R \mu_{a^*}^{M_1}} a^*M_0 \otimes_R a^*M_1 \ .$$

Monoidal Eilenberg-Moore construction for Hopf monads on bimodule categories

The Eilenberg-Moore category $(\mathrm{Bim}_R)^{a^*}$ of a^* consists of objects M equipped with with morphisms

$$\alpha_M : a^* M \to M,$$

satisfying some properties (commutative diagrams). What is important, they form a monoidal category as follows.

$$\alpha_{M_0 \otimes_R M_1} : a^*(M_0 \otimes_R M_1) \to M_0 \otimes_R M_1$$

$$a^*(M_0 \otimes_R M_1) \xrightarrow{\delta_{a^*}^{M_0, M_1}} a^* M_0 \otimes_R a^* M_1 \xrightarrow{\alpha_{M_0} \otimes_R \alpha_{M_1}} M_0 \otimes_R M_1,$$

We will denote by A the pair (R, a^*), and by $\mathrm{Spec}_Y(A)$ the Eileberg-Moore category $(\mathrm{Bim}_R)^{a^*}$.

Stable anti-Yetter-Drinfeld conditions for twisted cyclic functors

We say that a functor $\tau_R : \mathrm{Bim}_R \to \mathrm{Ab}$ is a *twisted cyclic functor*, if it is equipped with two natural transformations, the *twisted transposition*

$$\tau_R(M_0 \otimes_R M_1) \xrightarrow{t_R^{M_0,M_1}} \tau_R(M_1 \otimes_R a^* M_0)$$

and the *right action* of the opmonoidal monad a^*

$$\tau_R a^* \xrightarrow{\alpha_{\tau_R}} \tau_R$$

satisfying the following conditions.

SAYD-type conditions. Preparation

First, for the composition $\tau_R a^*$ we define an analogical twisted transposition, a natural transformation

$$\tau_R a^*(M_0 \otimes_R M_1) \xrightarrow{t_{R,a^*}^{M_0,M_1}} \tau_R a^*(M_1 \otimes_R a^* M_0)$$

being a composition

$$\begin{array}{ccccc}
\tau_R a^*(M_0 \otimes_R M_1) & & \tau_R(a^* M_1 \otimes_R a^* M_0) & \xrightarrow{\tau_R(\varphi_{a^*}^{M_1,M_0})^{-1}} & \tau_R a^*(M_1 \otimes_R a^* M_0) \\
\downarrow \tau_R(\delta^{M_0,M_1}) & & \uparrow \tau_R(M_0 \otimes \mu_{a^*}(M_1)) & & \\
\tau_R(a^* M_0 \otimes_R a^* M_1) & \xrightarrow{t_R^{a^* M_0, a^* M_1}} & \tau_R(a^* M_1 \otimes_R a^* a^* M_0)) & &
\end{array}$$

Anti-Yetter-Drinfeld condition

The first condition for τ_R to be a twisted cyclic functor consists in commutativity of the following diagram

$$\begin{array}{ccc} \tau_R a^*(M_0 \otimes_R M_1) & \xrightarrow{t_{R,a^*}^{M_0,M_1}} & \tau_R a^*(M_1 \otimes_R a^* M_0) \\ {\scriptstyle \alpha_{\tau_R}^{M_0 \otimes_R M_1}} \downarrow & & \downarrow {\scriptstyle \alpha_{\tau_R}^{M_0 \otimes_R a^* M_1}} \\ \tau_R(M_0 \otimes_R M_1) & \xrightarrow{t_R^{M_0,M_1}} & \tau_R(M_1 \otimes_R a^* M_0). \end{array}$$

which means that $t_{R,a^*}^{M_0,M_1}$ lifts $t_R^{M_0,M_1}$ along the Hopf monad a^* action α_{τ_R} on τ_R.

Stability condition

The second condition for τ_R to be a twisted cyclic functor consists in commutativity of the following diagram

$$\begin{array}{ccc} \tau_R(M) & \xrightarrow{t_R^{M,R}} & \tau_R a^*(M) \\ & \diagdown\!\!\diagdown & \Big\downarrow \alpha^M_{\tau_R} \\ & & \tau_R(M) \end{array}$$

where the horizontal arrow utilizes identifications via tensoring by the monoidal unit R as follows

$$\tau_R(M) = \tau_R(M \otimes_R R) \xrightarrow{t_R^{M,R}} \tau_R(R \otimes_R a^* M) = \tau_R a^*(M).$$

This means that the Hopf monad a^* action α_{τ_R} on τ_R neutralizes the twisted transposition with the monoidal unit R.

Cyclic functor on the monoidal Eilenberg-Moore category from SAYD conditions

The following coequalizer diagram

$$\tau_R a^* M \underset{\tau_R(\alpha_M)}{\overset{\alpha^M_{\tau_R}}{\rightrightarrows}} \tau_R M \longrightarrow \tau_A M$$

defines an additive functor $\tau_A : \mathrm{Qcoh}_{\mathrm{Spec}_Y(A)} \to \mathrm{Ab}$.

Theorem

τ_A makes $\mathrm{Spec}_Y(A)$ a cyclic scheme.

Example: Hopf-cyclic cohomology of an algebra

For a left H-module algebra A over a Hopf algebra H over a field k and a right-left stable anti-Yetter-Drinfeld H-module Γ we can consider the Hopf bialgebroid or $B = (k, b^* = H \otimes (-))$ and

$$\mathrm{Qcoh}_X = \mathrm{Vect}^{op}, \Gamma_X(V) = Hom(V, k),$$

$$\mathrm{Qcoh}_{\mathrm{Spec}(B)} = H - \mathrm{Mod}, \Gamma_{\mathrm{Spec}(B)}(V) = \Gamma \otimes_H V,$$

$$f_* V = Hom(V, A),\ \ f_* M = \ _H Hom(M, A).$$

The cocyclic object of natural transformations:

$$\Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \to \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n)$$

reads as

$$\Gamma \otimes_H (Hom(V_0, A) \otimes \cdots \otimes Hom(V_n, A)) \to Hom(V_0 \otimes \cdots \otimes V_n, k)$$

whose component corresponding to $V_0 = \cdots = V_n = k$ is

$$\Gamma \otimes_H (A \otimes \cdots \otimes A) \to k,$$

the cocyclic object of Hajac-Khalkhali-Rangipour-Sommerhäuser.