Let $\mathcal{C}$ be a monoidal category. $k = \bar{k}$ a field of char. 0
we can talk about algebras in $\mathcal{C}$:
this will be an object $A \in ob\mathcal{C}$, together
w. a morphism $m: A \otimes A \to A$

if $F: \mathcal{C} \to Vec_k$ is a monoidal functor, then
$F(A)$ is an algebra w. multiplication $F(m)$

we think of such a functor as an interpretation
of objects of $\mathcal{C}$ as vector spaces.
of course, we may have more than one interpretation

We give the following definition:

Def^n: A monoidal cat. $\mathcal{C}$ is called:
k-linear if $Hom_{\mathcal{C}}(x,y)$ is a k-vec.sp.
$\forall x,y \in Ob\mathcal{C}$, and composition is k-bilinear
rigid if $\forall x \in ob\mathcal{C}$ $\exists X^* \in Ob\mathcal{C}$ w. morphisms $\mathbb{1} \to X \otimes X^*$,
$X^* \otimes X \to \mathbb{1}$, and similarly, an obj $^*X$ w. morphisms
$\mathbb{1} \to {}^*X \otimes X$, $X \otimes {}^*X \to \mathbb{1}$ (these morphisms satisfy certain conditions)
We say that two algebras A_1, A_2 in Vec_k are
monoidally-equivalent if $\exists$ an abelian k-linear
rigid monoidal category $\mathcal{C}$, an algebra $A \in Ob\mathcal{C}$,
and two exact monoidal functors $F_1, F_2: \mathcal{C} \to Vec_k$
such that $F_i(A) \cong A_i$

An exact functor here will automatically be
faithful. So, for example, A_1 is associative
iff A is associative iff A_2 is associative.

We can talk also on other algebraic structur
for example: a braided vector space
is a pair (V, c) of a fin. dim. vector sp. V
and $c: V\otimes V \xrightarrow{\cong} V\otimes V$ which satisfies the
YB-equation:

$$c_{12}c_{23}c_{12} = c_{23}c_{12}c_{23} : V\otimes V\otimes V \to V\otimes V\otimes V.$$

Example (of monoidally eq. algebras and braided vec sp.)

Let G be a group, $\alpha \in H^2(G, K^\times)$
Then $\mathcal{C} = Vec_G$ is the monoidal cat. of
G-graded vec. sp. We have a tensor
auto. eq. $F_\alpha : \mathcal{C} \to \mathcal{C}$ $F_\alpha(V) = V$

$$F_\alpha(V\otimes W) \to F_\alpha(V) \otimes F_\alpha(W)$$
$$v\otimes w \longrightarrow \alpha^{-1}(|v|, |w|)\, v\otimes w$$

and a forgetful functor $F: \mathcal{C} \to Vec_K$

Then $A = KG$ is an algebra in $\mathcal{C}$, and
the functors F, FF_α shows that KG and
$K^\alpha G$ are monoidally equivalent
so for example, if $G = C_n \times C_n = \langle x, y\rangle$,
$\alpha(x^iy^j, x^ky^l) = \xi^{jk}$; $\xi = \sqrt[n]{1}$, then we get
that K^{n^2}, $M_n(K)$ are monoidally equivalent

If $G = \mathbb{Z}\times\mathbb{Z} = \langle x, y\rangle$, let $V = \mathrm{span}\{v_1, v_2\}$, and
let $c: V\otimes V \to V\otimes V$ be the usual flip.
$$v\otimes w \to w\otimes v$$

$V \in Vec_G$ by writing $\deg v_1 = x$
$\deg v_2 = y$

for $\lambda \in k^\times$, let $\alpha_\lambda(x^i y^j, x^k y^l) = \lambda^{jk}$, $\alpha \in H^2(G, k^\times)$

then we see that (V, c) is monoidally equivalent to $(V, \tilde{c})$, where

$$\tilde{c}: v_i \otimes v_i \to v_i \otimes v_i \quad i = 1, 2$$
$$v_1 \otimes v_2 \to \lambda v_2 \otimes v_1$$
$$v_2 \otimes v_1 \to \lambda^{-1} v_1 \otimes v_2 .$$

The fact that we can get non-isomorphic structures which are monoidally-equivalent is a non-commutative phenomenon.
Indeed, by a theorem of Deligne, if we had changed ~~symm~~ "monoidal category/functor" to "symmetric monoidal" then symmetric monoidal eq $\Leftrightarrow$ isomorphism.

If H is a Hopf alg. then a YD-module (Yetter Drinfeld) over H is a pair (X, Φ) where X is an H-module and $\Phi_Y: X \otimes Y \xrightarrow{\sim} Y \otimes X$ is a family of isomorphisms, natural in Y, which satisfies $\Phi_{Y \otimes Z} = (1_Y \otimes \Phi_Z)(\Phi_Y \otimes 1_Z)$.
In particular, $\Phi_X: X \otimes X \to X \otimes X$ gives a braiding on X.

One of the central questions arising in the classification of fin. dim. Hopf algs regards the "Nichols algebra" of a braided vec. sp.
If (V, c) is a braided vec. sp. then $T(V)$ is a braided Hopf alg:

$\Delta v = v \otimes 1 + 1 \otimes v$, $\Delta(vw) = vw \otimes 1 + 1 \otimes vw + v \otimes w + c(v \otimes w)$ etc

The Nichols algebra $B(V)$ is the biggest (Hopf) quotient of $T(V)$ where all primitives are of degree 1

For example. If c is the flip, $B(V) = S(V)$.

If $c = -$flip, $B(V) = \Lambda(V)$ (exterior algebra)

<u>Question</u>: when is $B(V)$ fin. dim?

<u>Claim</u>: If $(V_1, c_1) \underset{\text{mono-idally eq}}{\sim} (V_2, c_2)$ then

$B(V_1) \underset{\text{mono-idally eq}}{\sim} B(V_2)$. In particular: $B(V_1)$ is fin. dim iff $B(V_2)$ is fin. dim.

The reason is that $B(V)$ can be constructed in $\mathcal{C}$ already.

<u>Example</u>: If $G = S_4$ there are exactly 3 irred. YD H-modules V_1, V_2, V_3 $(H = kG)$ s.t. $B(V_i)$ is fin. dim. (Andruskiewitsch-Heckenberger-Schneider). Moreover, if $B(V)$ is fin. dim then $V = V_i$ for $i = 1, 2$ or 3 (Also AHS)

$\dim B(V_i) = 576$ for $i = 1, 2, 3$

$i = 1$ - Fomin & Kirillov

$i = 2$ - Milinski & Schneider

$i = 3$ - Andruskiewitsch & Graña

irred. YD modules correspond to conj class $[g]$ and a rep of $C_G(g)$.

for V_1, V_2 $g = (12)$ V_3 $g = (1234)$

We can show that $V_1 \sim V_2 \sim V_3$, and deduce the dimension of $B(V_2)$ $B(V_3)$ from $B(V_1)$.

If (V, c) is braided, $c^2 = Id$, then if also $c^*: V^* \otimes V \to V \otimes V^*$ is invertible, we can use Deligne's Theory to prove that (V, c) is monoidally equivalent to some super vector space.

Remarks: - This relation was also studied for C^*-algebras in RepH by Neshveyev & Tuset

- It is problematic to talk about monoidally equivalent Lie/Hopf algebras, because the definition requires to use the flip.