Formality for algebroid stacks

Ryszard Nest

University of Copenhagen

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Joint work with

Paul Bressler, Alexander Gorokhovsky, Boris Tsygan

2-groupoids

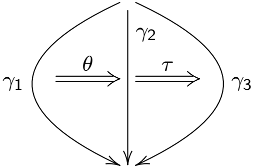
Data:

1. Units G_0 $\bullet^x$
2. Arrows G_1 $\bullet^x \xrightarrow{\gamma} \bullet^y$
   composable when range of one coincides with the source of the next one.
3. Two-morphisms G_2

$$\begin{tikzcd} \bullet^x \arrow[d, bend right=60, "\gamma_1"'] \arrow[d, bend left=60, "\gamma_2"] \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] \\ \bullet^y \end{tikzcd}$$

Two-morphisms have a "natural" composition structure satisfying natural associativity conditions.

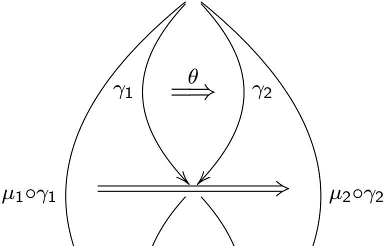
Horizontal



$$\tau \circ \theta$$

and

Vertical



Canonical example - 2-groupoid of categories

1. Objects: Categories
2. Arrows: Functors
3. 2-morphisms: Natural transformations of functors

Simplicial nerve of a 2-groupoid

1. $\mathfrak{N}_0\mathtt{G}$ is the set of objects of $\mathtt{G}$.
2. For $n \geq 1$, $\mathfrak{N}_n\mathtt{G}$ is the set of data:

$$((\mu_i)_{0\leq i\leq n}, (g_{ij})_{0\leq i<j\leq n}, (c_{ijk})_{0\leq i<j<k\leq n}),$$

where

   1. (μ_i) is an n-tuple of objects of $\mathtt{G}$,
   2. (g_{ij}) is a collection of 1-morphisms with $g_{ij}\colon \mu_j \to \mu_i$ and
   3. (c_{ijk}) is a collection of 2-morphisms with $c_{ijk} : g_{ij}g_{jk} \to g_{ik}$ which satisfies $c_{ijl}c_{jkl} = c_{ikl}c_{ijk}$ (in the set of 1-morphisms $g_{ij}g_{jk}g_{kl} \to g_{il}$).

Nerve $\mathfrak{N}_*\mathtt{G}$ of a 2-groupoid is a Kan simplicial set with $\pi_n(\mathfrak{N}_*\mathtt{G}) = 0$ for $n > 2$.

Definition

Two 2-groupoids are equivalent, if their nerves are weakly homotopy equivalent.

A bigroupoid is a similar data as a 2-groupoid, except that the associativity of the composition $G^1(x,y) \times G^1(y,z) \to G^1(x,z)$ is satisfied up to a natural transformation which satisfies the pentagonal identity.

Duskin

There exist indempotent endofunctors

$\Pi_n : \{\text{simplicial Kan sets}\} \to \{\text{simplicial Kan sets}\}, \; \pi_k(\Pi_n(\cdot)) = 0 \text{ for } k > n,$

and the range of Π_2 consist of nerves of bigroupoids (up to weak homotopy equivalence).

As above, two bigroupoids are called equivalent, if their nerves are weakly homotopy equivalent.

Suppose that $\mathfrak{g}$ is a DGLA. A Maurer-Cartan element of $\mathfrak{g}$ is an element $\gamma \in \mathfrak{g}^1$ satisfying

$$d\gamma + \frac{1}{2}[\gamma,\gamma] = 0. \tag{1}$$

2-groupoid of a nilpotent DGLA with $\mathfrak{g}^i = 0$ for $i < -1$

1. $\mathrm{MC}^2(\mathfrak{g})_0$ is the set of Maurer-Cartan elements of $\mathfrak{g}$.
2. $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2) = \exp \mathfrak{g}^0$.
   Here the product in the unipotent group $\exp \mathfrak{g}^0$ is defined by the Hausdorff-Cambell formula and $\exp \mathfrak{g}^0$acts on the set of Maurer-Cartan elements of $\mathfrak{g}$ by $d + \operatorname{ad}\gamma_2 = \operatorname{Ad}\exp X\,(d + \operatorname{ad}\gamma_1)$.
3. Given $\gamma \in \mathrm{MC}^2(\mathfrak{g})_0$, $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y) = \exp_\gamma \mathfrak{g}^{-1}$.
   Here $\mathfrak{g}^{-1}$ has the Lie bracket given by $[a,\, b]_\gamma = [a,\, db + [\gamma,\, b]]$ and $\exp_\gamma t$ acts by $(\exp_\gamma t)\cdot(\exp X) = \exp(dt + [\gamma, t])\exp X$.

A morphism of nilpotent DGLA $\phi : \mathfrak{g} \to \mathfrak{h}$ induces a functor $\phi : \mathrm{MC}^2(\mathfrak{g}) \to \mathrm{MC}^2(\mathfrak{h})$. The following holds:

Suppose that $\phi : \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism (isomorphism on cohomology) of DGLA's and let $\mathfrak{m}$ be a nilpotent commutative ring. Then the induced map $\phi : \mathrm{MC}^2(\mathfrak{g} \otimes \mathfrak{m}) \to \mathrm{MC}^2(\mathfrak{h} \otimes \mathfrak{m})$ is an equivalence of 2-groupoids.

Let $\mathfrak{g}$ be a L_∞-algebra. Recall that an L_∞-algebra is a graded vector space $\mathfrak{g}$ equipped with operations

$$\bigwedge^k \mathfrak{g} \to \mathfrak{g}[2-k] \colon x_1 \wedge \ldots \wedge x_k \mapsto [x_1, \ldots, x_k]$$

defined for $k = 1, 2, \ldots$. which satisfy a sequence of Jacobi identities. It follows from the Jacobi identities that the unary operation $[.] \colon \mathfrak{g} \to \mathfrak{g}[1]$ is a differential, which we will usually denote by d.

An easiest way to visualize an L_∞-algebra is to observe that, if $\mathfrak{g}^*$ is a DGLA, then $d + [,]$ is a transpose of an odd coderivation δ on the tensor coalgebra $T\mathfrak{k}[1]$ satisfying $\delta^2 = 0$. Now allow δ to have higher terms:

1. $d^t : \mathfrak{k} \to \mathfrak{k}$
2. $[,]^t : \mathfrak{k} \to \mathfrak{k}^{\otimes 2}$
3. $m_3^t : \mathfrak{k} \to \mathfrak{k}^{\otimes 3}$
4. and so on.

The *lower central series* of an L_∞-algebra $\mathfrak{g}$ is the canonical decreasing filtration $F^\bullet\mathfrak{g}$ with $F^i\mathfrak{g} = \mathfrak{g}$ for $i \leq 1$ and defined recursively for $i \geq 1$ by

$$F^{i+1}\mathfrak{g} = \sum_{k=2}^{\infty} \sum_{\substack{i=i_1+\cdots+i_k \\ i_k \leqslant i}} [F^{i_1}\mathfrak{g}, \ldots, F^{i_k}\mathfrak{g}].$$

An L_∞-algebra is *nilpotent* if there exists an i such that $F^i\mathfrak{g} = 0$.

Suppose that $\mathfrak{g}$ is a nilpotent L_∞-algebra. An element $\mu \in \mathfrak{g}^1$ is called a *Maurer-Cartan element* (of $\mathfrak{g}$) if it satisfies the condition

$$\mathcal{F}(\mu) := \delta\mu + \sum_{k=2}^{\infty} \frac{1}{k!}[\mu^{\wedge k}] = 0 \quad (\in \mathfrak{g}^2).$$

We will denote by $\mathrm{MC}(\mathfrak{g})$ the set of Maurer Cartan elements of $\mathfrak{g}$.

Hinich, Getzler

$\Sigma(\mathfrak{g})$ is the simplicial set

$$[n] \to \mathrm{MC}(\mathfrak{g}) \otimes \Omega(\Delta_n)),$$

where $\Omega(\Delta_n)$ is the differential graded commutative algebra of differential forms on the standard n-simplex. $\Sigma(\mathfrak{g})$ is a Kan simplicial set and, if $\mathfrak{g}_i = 0$ for $i < -n+1$, $\pi_k(\Sigma(\mathfrak{g})) = 0$ for $k > n$.

Theorem

Let $\mathfrak{g} \to \mathfrak{h}$ be a L_∞ quasiisomorphism of nilpotent L_∞ algebras. The induced map of simplicial sets $\Sigma(\mathfrak{g}) \to \Sigma(\mathfrak{h})$ is a weak homotopy equivalence.

Let $\mathfrak{g}$ be a nilpotent DGLA satisfying $\mathfrak{g}_i = 0$ for $i \leq -2$. By now we have two simplicial Kan sets associated to $\mathfrak{g}$, the nerve of Deligne 2-groupoid $\mathfrak{N}(MC^2(\mathfrak{g}))$ and $\Sigma(\mathfrak{g})$.

Theorem

Let $\mathfrak{g}$ be a nilpotent DGLA satisfying $\mathfrak{g}_i = 0$ for $i \leq -2$. Then $\mathfrak{N}(MC^2(\mathfrak{g}))$ and $\Sigma(\mathfrak{g})$ are weakly homotopy equivalent.

In particular, for nilpotent DGLA's $\mathfrak{g}$ such that $\mathfrak{g}_i = 0$ for $i \leq -2$ the two notions of equivalence coincide, and the following definition is unambiguous.

Definition

The bigroupoid associated to a nilpotent DGLA $\mathfrak{g}$ such that $\mathfrak{g}_i = 0$ for $i \leq -2$ is $\Pi_2(\Sigma(\mathfrak{g}))$

The explicit homotopy equivalence $\Sigma(\mathfrak{g}) \to \mathfrak{N}(MC^2(\mathfrak{g}))$ is given by "non-commutative integration". To be more specific, every element $\mu \in \Sigma_n(\mathfrak{g}) \in (\Omega_n \otimes \mathfrak{g})^1$ is a triple: $\mu = (\mu^{0,1}, \mu^{1,0}, \mu^{2,-1})$ satisfying the Maurer Cartan equations, i. e.

$$\begin{aligned} d\mu^{0,1} + \frac{1}{2}[\mu^{0,1}, \mu^{0,1}] = 0 \\ d_{DR}\mu^{0,1} + d\mu^{1,0} + [\mu^{0,1}, \mu^{1,0}] = 0 \\ d_{DR}\mu^{1,0} + \frac{1}{2}[\mu^{1,0}, \mu^{1,0}] + d\mu^{2,-1} + [\mu^{0,1}, \mu^{2.-1}] = 0 \\ d_{DR}\mu^{2,-1} + [\mu^{1,0}, \mu^{2.-1}] = 0 \end{aligned}$$

1. In particular, 0-simplices of both simplicial sets coincide.
2. 1-simplices of $\Sigma(\mathfrak{g})$ are given by $[0,1] \ni t \to \mu(t) \in MC(\mathfrak{g})$ and $[0,1] \to X_t \in \mathfrak{g}_0$ such that $\frac{d}{dt}\mu_t = [X_t, \mu_t]$, and the corresponding 1-simplex of $\mathfrak{N}(MC^2(\mathfrak{g}))$ is given by the gauge transformation $\mu_0 \to \mu_1$ obtaining by integrating this first order differential equation, i.e. the holonomy transformation along a path.
3. Integration of 2-simplices is the part which refers to "non-abelian integration", as it corresponds to computing holonomy over 2-simplices with values in a field of Lie groups $(\exp(\mathfrak{g}_{-1}, [,]_\mu))$ varying over the simplex.

Suppose that A is a k-algebra with associative product m. The k-vector space $C^n(A)$ of Hochschild cochains of degree $n \geq 0$ is defined by

$$C^n(A, A) := \mathrm{Hom}_k(A^{\otimes n}, A) .$$

The graded vector space $\mathfrak{g}(A) := C^\bullet(A, A)[1]$ has a canonical structure of a DGLA under the Gerstenhaber bracket denoted by $[\ ,\ ]$ and differential $\delta = [m, \cdot]$.

$C^\bullet(A, A)[1]$ is canonically isomorphic to the (graded) Lie algebra of derivations of the free associative co-algebra generated by $A[1]$. For a unital algebra we will work with the subspace of *normalized cochains* $\overline{C}^\bullet(A)$. Suppose in addition that R is a commutative Artin k-algebra with the nilpotent maximal ideal $\mathfrak{m}_R$ The DGLA $\mathfrak{g}(A) \otimes_k \mathfrak{m}_R$ is nilpotent and satisfies $\mathfrak{g}^i(A) \otimes_k \mathfrak{m}_R = 0$ for $i < -1$.

$\mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$ is well defined and

$$R \mapsto \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$$

is a functor on the category of commutative Artin algebras.

Let R be a commutative Artin k-algebra with maximal ideal $\mathfrak{m}_R$. There is a canonical isomorphism $R/\mathfrak{m}_R \cong k$.
A *(R-)star product* on A is an associative R-bilinear product on $A \otimes_k R$ such that the canonical isomorphism of k-vector spaces $(A \otimes_k R) \otimes_R k \cong A$ is an isomorphism of algebras. Thus, a star product is an R-deformation of A.

$\mathrm{Def}(A)(R)$

The 2-category of R-star products on A, denoted $\mathrm{Def}(A)(R)$, is defined as the subcategory of the 2-category Alg^2_R of R-algebras with

- Objects:
  R-star products on A,
- 1-morphisms $\phi : m_1 \to m_2$ between the star products m_i: R-algebra homomorphisms $\phi : (A \otimes_k R, m_1) \to (A \otimes_k R, m_2)$ which reduce to the identity map modulo $\mathfrak{m}_R$.
- 2-morphisms $b : \phi \to \psi$, where $\phi, \psi : m_1 \to m_2$ are two 1-morphisms
  Elements $b \in 1 + A \otimes_k \mathfrak{m}_R \subset A \otimes_k R$ such that
  $m_2(\phi(a), b) = m_2(b, \psi(a))$ for all $a \in A \otimes_k R$.

It follows easily from the above definition and the nilpotency of $\mathfrak{m}_R$ that $\mathrm{Def}(A)(R)$ is a 2-groupoid.

It is clear that the assignment $R \mapsto \mathrm{Def}(A)(R)$ extends to a functor on the category of commutative Artin k-algebras.
Suppose that μ is an R-star product on A. Since $\mu - m = 0 \mod \mathfrak{m}_R$ we have $\mu - m \in \mathfrak{g}^1(A) \otimes_k \mathfrak{m}_R$.

The associativity of μ is equivalent to the fact that $\mu - m$ satisfies the Maurer-Cartan equation, i.e. $\mu - m \in \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)_0$.

It is easy to see that the assignment $\mu \mapsto \mu - m$ extends to a functor

$$\mathrm{Def}(A)(R) \to \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R) . \tag{2}$$

The following proposition is obvious.

Theorem

The functor (2) *is an isomorphism of* 2*-groupoids.*

let X be a smooth compact manifold, $C^\infty(X)$ the sheaf of smooth functions, $A = C^\infty(X) = \Gamma(C^\infty(X))$ the algebra of smooth functions. We will be interested in $\mathrm{Def}(C^\infty(X))$, or rather its twist by a gerbe, but that comes later. First Kontsevich formality
From now on assume

We restrict ourselves to Hochschild cochains which are given by (poly)differential operators. In particular, our DEF - functor refers to local deformations of product, i. e. such that $\mu(f, g)(x)$ depends only on the values of functions f and g and finitely many of their derivatives at x, for all $x \in X$.

The space of polyvectorfields $\Gamma(X, \Lambda^* TX)$ is a DGLA with trivial differential and Schouten bracket.

Theorem (Kontsevich)

$\mathfrak{g}(C^\infty(X))$ and $\Gamma(X, \Lambda^* TX)$ have equivalent Deligne 2-groupoids.

A more precise statement is as follows.

Theorem

There exists an *DGLA* $\mathfrak{k}$ and a quasiisomorphisms of L_∞-algebras

$$\begin{tikzcd} & \mathfrak{k} \arrow[dl] \arrow[dr] & \\ \mathfrak{g}(C^\infty(X)) & & \Gamma(X, \Lambda^* TX) \end{tikzcd}$$

Equivalently, $\mathfrak{g}(C^\infty(X))$ and $\Gamma(X, \Lambda^* TX)$ are quasiisomorphic as L_∞-algebras.

Our goal is a similar result for a stack - which is an object defined by "almost" glueing local data, hence we will need "sheaf-theoretic" constructions. But first, what is a stack.

Let X be a smooth manifold. A stack on X is an equivalence class of the following data:

1. An open cover $X = \cup U_i$;
2. a sheaf of rings $\mathcal{A}_i$ on every U_i;
3. an isomorphism of sheaves of rings $G_{ij} : \mathcal{A}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{A}_i|(U_i \cap U_j)$ for every i, j;
4. an invertible element $c_{ijk} \in \mathcal{A}_i(U_i \cap U_j \cap U_k)$ for every i, j, k satisfying

$$G_{ij} G_{jk} = \mathrm{Ad}(c_{ijk}) G_{ik} \tag{3}$$

   such that, for every i, j, k, l,

$$c_{ijk} c_{ikl} = G_{ij}(c_{jkl}) c_{ijl} \tag{4}$$

There is an "obvious" notion of isomorphism of stacks, that we will not write down here.
A special case of a stack is a gerbe

Definition

A gerbe is a stack for which $\mathcal{A}_i = C^\infty{}_{U_i}$ and $G_{ij} = \mathrm{id}$. In this case c_{ijk} form a two-cocycle in $Z^2(X, C^\infty(X)^*)$.
From now on we will call it a twisted form of $C^\infty(X)$ and denote it by $\mathcal{S}$

A familiar version of a gerbe on a locally compact space X is a locally trivial bundle $\mathcal{E}$ of compact operators over X. Locally,

$$\mathcal{E}|U \simeq U \times \mathcal{K}(H),$$

hence we get transition "functions"

$$U \cap V \to [U_{U,V}] \in Aut(\mathcal{K}(H)) = U(H)/\mathbb{T}.$$

Choosing a concrete unitary representatives $U_{U,V}$ produces a two-cocycle $c_{U,V,W} = U_{U,V} U_{V,W} U_{W,U} \in C(U \cap V \cap W, \mathbb{T})$.

There is a categorical way of describing a stack:

1. A sheaf of categories $\mathcal{C}_i$ on U_i for every i;
2. an invertible functor $G_{ij} : \mathcal{C}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{C}_i|(U_i \cap U_j)$ for every i, j;
3. an invertible natural transformation

$$c_{ijk} : G_{ij} G_{jk}|(U_i \cap U_j \cap U_k) \xrightarrow{\sim} G_{ik}|(U_i \cap U_j \cap U_k)$$

such that, for any i, j, k, l, the two natural transformations from $G_{ij} G_{jk} G_{kl}$ to G_{il} that one can obtain from the c_{ijk}'s are the same on $U_i \cap U_j \cap U_k \cap U_l$.

In this language, a gerbe is a sheaf with fiber the category of (continuous, $*$)-representations of the C*-algebra of compact operators.

We will not formally define the deformation theory of the stack, but the idea is simple - just repeat what we did for an associative algebra. In particular a deformation of a stack is a stack of R-algebras which coincides with the original stack modulo the maximal ideal $\mathfrak{m}_R$.

Let us spell it out in the case of a gerbe.

Definition

Consider a gerbe given by a two-cocycle $c^{(0)}_{\alpha\beta\gamma}$. A *deformation quantization* of this gerbe is a stack such that:

1. $\mathcal{A}_\alpha = \mathcal{O}_{U_\alpha}[[\hbar]]$ as a sheaf of vector spaces, with an associative $\mathbb{C}[[\hbar]]$-linear product structure $*$ of the form

$$f * g = fg + \sum_{m=1}^{\infty} (i\hbar)^m P_m(f, g).$$

   where P_m are bidifferential operators and $1 * f = f * 1 = f$.
2. $G_{\alpha\beta}(f) = f + \sum_{m=1}^{\infty} (i\hbar)^m T_m(f)$ where T_m are differential operators;
3. $c_{\alpha\beta\gamma} = \sum_{m=0}^{\infty} (i\hbar)^m c^{(m)}_{\alpha\beta\gamma}$.

In particular, associated to a stack $\mathcal{S}$ on X, we get the functor $\mathrm{Def}(\mathcal{S})$ from Artinian algebras to 2-groupoids.

Given a Čech cocycle c describing a gerbe, Recall that a gerbe is given by a covering $\{U_i\}_{i\in I}$ and a Čech 2-cocycle c. Construct the algebra

$$\mathcal{A} = \{f = \{f\}_{ij},\ i,j \in I \mid f_{ij} \in C^{\infty}(U_i),\ \mathrm{supp}(f_{ij}) \subset U_i \cap U_j\}$$

and with the product given by $\{f \cdot g\}_{ij} = \sum_k c_{ikj}^{-1} f_{ik} g_{kj}$. $\mathfrak{g}(\mathcal{A})$ is the DGLA controlling deformations of the gerbe, but it has two major disadvantages. It is not easy to describe the "local" cochains and (the support condition) it does not adapt well to the sheaf-theoretic situations like in the case of complex manifold. Instead of it we will use jets.

Let $Diff_X$ denote the sheaf of differential operators on X . It is the subalgebra of $End(C^\infty(X))$ generated by functions and vectorfields, and has a coproduct Δ. We set

[redacted]

We set $\mathcal{J}_X(U) = End_{C^\infty(U)}(Diff_U, C^\infty(U))$.

1. $\mathcal{J}_X$ is a locally free sheaf of algebras with the product defined by
$$l_1 l_2(D) = l_1 \otimes l_2(\Delta D).$$
2. $\mathcal{J}_X$ has a canonical flat connection, given by
$$\nabla_X^{can} l(D) = l(XD) - Xl(D)$$
for $l \in \mathcal{J}_X$.

The shifted normalized Hochschild complex $\overline{C}^\bullet(\mathcal{J}_X)[1]$ is given by locally defined $C^\infty(X)$-linear continuous Hochschild cochains. The $C^\infty(X)$-linearity means that we take the Hochschild cochains in the fiber direction only.

$\overline{C}^\bullet(\mathcal{J}_X)[1]$ is a sheaf of DGLA under the Gerstenhaber bracket and the Hochschild differential δ. The canonical flat connection on $\mathcal{J}_X$ induces one, also denoted ∇^{can}, on $\overline{C}^\bullet(\mathcal{J}_X))[1]$. The flat connection ∇^{can} commutes with the differential δ and acts by derivations of the Gerstenhaber bracket.

$\mathfrak{g}(\mathcal{J}_X)$

The de Rham complex $\mathtt{DR}(\overline{C}^{\bullet}(\mathcal{J}_X))[1] := (\Omega^{\bullet}_X \otimes \overline{C}^{\bullet}(\mathcal{J}_X))[1])$ equipped with the differential $\nabla^{can} + \delta$ and the Lie bracket induced by the Gerstenhaber bracket is a sheaf $\mathfrak{g}(\mathcal{J}_X)$ of DGLA on X

Theorem

$(\mathfrak{g}(\mathcal{J}_X), \nabla^{can} + \delta), [,])$ and $\mathfrak{g}(C^{\infty}(X))$ are quasiisomorphic. In particular, the 2-groupoid of $\mathfrak{g}(\mathcal{J}_X)$ controls deformations of $C^{\infty}(X)$.

Note that Kontsevich formality says, that the fibers of $\mathfrak{g}(\mathcal{J}_X)$ are formal, i. e. quasiisomorphic to fiberwise polyvectorfields.

one should think of a gerbe as a kind of twisting of the (sheaf of) smooth functions on a manifold X. We will need the associated "twisting" of jets. For that we need to introduce some operations on jets.

1. The sheaf of abelian Lie algebras $\mathcal{J}_X/C^\infty(X)$ acts by derivations of degree -1 on the graded Lie algebra $\overline{C}^\bullet(\mathcal{J}_X)[1]$ via the adjoint action (the $\cap$-product).
2. This action commutes with the Hochschild differential. Therefore
3. the (abelian) graded Lie algebra

$$\Omega^\bullet_X \otimes \mathcal{J}_X/C^\infty(X)$$

acts by derivations on the graded Lie algebra $\Omega^\bullet_X \otimes \overline{C}^\bullet(\mathcal{J}_X))[1]$.

4. We denote the action of the form $\omega \in \Omega^\bullet_X \otimes \mathcal{J}_X/C^\infty(X)$ by ι_ω.

Suppose that $\omega \in \Gamma(X; (\Omega^2 \otimes \mathcal{J}_X/C^\infty(X)))$ satisfies $\nabla^{can}\omega = 0$

- ι_ω acts as an odd derivations and commutes with the differential $\nabla^{can} + \delta$

[redacted]

ω-twist $\mathfrak{g}(\mathcal{J}_X)_\omega$ is the DGLA with the same underlying graded Lie algebra structure as $\mathfrak{g}(\mathcal{J}_X)$ and the differential given by $\nabla^{can} + \delta + \iota_\omega$. The isomorphism class of this DGLA depends only on the cohomology class of ω in $H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/C^\infty(X)), \nabla^{can})$.

The twisted form $\mathcal{S}$ of $C^\infty(X)$ is determined up to equivalence by its class in $H^2(X; \mathcal{O}_{\mathcal{X}}{}^\times)$.
The composition

$$(C^\infty)^* \to (C^\infty)^*/\mathbb{C}^* \xrightarrow{\log} C^\infty/\mathbb{C} \xrightarrow{j^\infty} \mathtt{DR}(\mathcal{J}/\mathcal{O})$$

induces the map

$$H^2(X; (C^\infty)^*) \to H^2(X; \mathtt{DR}(\mathcal{J}/\mathcal{C}^\infty)) \cong H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/C^\infty(X)), \nabla^{can})$$

We denote by $[\mathcal{S}] \in H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/C^\infty(X)), \nabla^{can})$ the image of the class of $\mathcal{S}$.

$\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$ is the DGLA associated to the stack $\mathcal{S}$.

Theorem

The DGLA $\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$ controls the deformation theory of the stack $C^\infty(X)(S)$.

Our goal is the result analogous to Kontsevich. This is not quite what we get, instead we get an L_∞-algebra,

First some formulas describing the operation of forms on polyvectorfields. The canonical pairing $\langle\ ,\ \rangle\colon \Omega^1(X)\otimes\Gamma(TX)\to C^\infty(X)$ extends to the pairing

$$\langle\ ,\ \rangle\colon \Omega^1_X\otimes\Gamma(\Lambda^* TX)\to\Gamma(\Lambda^* TX)[-1]$$

For $k\geq 1$, $\omega=\alpha_1\wedge\ldots\wedge\alpha_k$, $\alpha_i\in\Omega^1(X)$, $i=1,\ldots,k$, let

$$\Phi(\omega)\colon\ \mathrm{Sym}^k\,\Gamma(\Lambda^* TX)[2]\to\Gamma(\Lambda^* TX)[k]$$

denote the map given by the formula

$$\Phi(\omega)(\pi_1,\ldots,\pi_k)=(-1)^{(k-1)(|\pi_1|-1)+\ldots+2|(\pi_{k-3}|-1)+(|\pi_{k-2}|-1)}\times \sum_\sigma \mathrm{sgn}(\sigma)\langle\alpha_1,\pi_{\sigma(1)}\rangle\wedge\cdots\wedge\langle\alpha_k,\pi_{\sigma(k)}\rangle,$$

where $|\pi|=l$ for $\pi\in\Gamma(\Lambda^l TX)$. For $\alpha\in C^\infty(X)$ let $\Phi(\alpha)=\alpha\in\Gamma(\Lambda^0 TX)$.

Definition

Let ω be a closed 3-form on X. Then the space $\Gamma(\Lambda^* TX)_\omega$ of polyvectorfields on X is an L_∞-algebra with

- trivial differential
- binary operation given by Schouten bracket
- the ternary operation given by $\Phi(\omega)$
- all other operations equal to zero.

Theorem

Suppose that $\mathcal{S}$ is a twisted form of $C^\infty(X)$. Let ω be a closed 3-form on X which represents $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$ (the Dixmier Douady class of $\mathcal{S}$). For any Artin algebra R with maximal ideal $\mathfrak{m}_R$ there is an equivalence of 2-groupoids

$$\mathrm{MC}^2(\Gamma(\Lambda^* TX)_\omega \otimes \mathfrak{m}_R) \cong \mathrm{Def}(\mathcal{S})(R)$$

natural in R.

Corollary

The isomorphism classes of formal deformations of $\mathcal{S}$ are in a bijective correspondence with equivalence classes of Maurer-Cartan elements of the L_∞-algebra $\Gamma(\Lambda^* TX)_\omega \widehat{\otimes} t \cdot \mathbb{C}[[t]]$.

These are the formal *twisted Poisson structures*, i.e. the formal series $\pi = \sum_{k=1}^{\infty} t^k \pi_k$, $\pi_k \in \Gamma(X; \bigwedge^2 \mathcal{T}_X)$, satisfying the equation

$$[\pi, \pi] = \Phi(\omega)(\pi, \pi, \pi).$$