Metric, torsion and minimal operators on noncommutative tori.

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1. METRIC IN NONCOMMUTATIVE GEOMETRY
    - A proposition: spectral geometry a la Connes
    - Measuring geometry through spectral data
    - The ubiquitous heat kernel
    - What is the metric ?
2. THE NONCOMMUTATIVE TORI
    - The usual Dirac and its conformal rescaling
    - The asymmetric noncommutative torus
    - Higher dimensional cases and minimality
    - Dirac operators on principal $U(1)$ bundles
3. CONCLUSIONS

Spectral triples

Definition: the spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact
2. even ST: $\exists \gamma \in \mathcal{A}' : \gamma^2 = 1, \gamma = \gamma^\dagger, \gamma D + D\gamma = 0,$
3. $\exists\, J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
   $J\gamma = \pm\gamma J, JD = \pm DJ, [J\pi(a)J, \pi(b)] = 0,$
4. $[[D, a], J\pi(b)J] = 0$ (D: first order differential operator)
5. ...+ conditions of „analysis" type

Theorem [Connes]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

EXAMPLES OF SPECTRAL GEOMETRIES

- The Noncommutative Torus: $UV = e^{2\pi i\theta} VU$
  Dirac operator the same as on the torus [Connes]
- Finite matrix algebras ($M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \oplus \cdots$
  Dirac operator is a finite hermitian matrix [AS & Paschke, Krajewski]
- Quantum spaces (q-deformations of spheres)
  Interesting Dirac operators [Dabrowski, AS, Landi, Varilly, et al]
- Moyal deformation $[x^\mu, x^\nu] = \theta^{\mu\nu}$
  The usual Dirac [Gracia-Bondia et al]
- κ-deformation $[x^0, x^i] = \frac{1}{\kappa} x^i$
  Doubly Special Relativity [Matassa]

HOW TO CONSTRUCT THEM?

There is so far no general method.
There are very few examples.

Geometry from spectral data (1)

The spectral properties of the Dirac operator

Classically it is known that the spectrum of the Dirac is discrete (separate eigenvalues), with finite multiplicities and no point of convergence apart form ∞. Roughly speaking - spectrum of Dirac squared is like the spectrum of Laplace operator. The properties do not change if we modify the operator (change the metric, add connections, torsion etc etc.)

So one can compute some spectral functionals

$$S(\mathcal{D}, f) = \mathrm{tr}\left(f(\mathcal{D}^2)\right),$$

where f is a suitable function such that the expression makes sense and tr is the usual trace on the Hilbert space.

Geometry from spectral data (2)

The exotic trace

Although the usual trace does not extend to the operators like Dirac (or its powers) there are some *exotic traces* which might be interpreted as regularized traces (something of the form of ζ-function regularization).

The spectral functional (2)

Using this exotic traces we can postulate the spectral functional to be (for example)

$$S(\mathcal{D}, \Lambda) = \sum_n \Lambda^n \fint \mathcal{D}^{-n},$$

where $\fint$ is that exotic trace. For most n that would be 0 but some terms would be nonzero.

Heat-kernel asymptotics

Theorem (Gilkey)

For a Riemannian n-dimensional manifold M, with a metric compatible connection, ∇, on a vector bundle E over M and a Laplace-type operator is of the form:

$$L = -g^{ab}\nabla_a\nabla_b - Z,$$

we have:

$$tr\, e^{-tL} = \sum_{k=0}^{n} \left(\frac{1}{4\pi t}\right)^{\frac{k-n}{2}} \int_M a_{[k]}(x) + o(t).$$

where $a_{[k]}(x)$ are functions on M (De Witt–Seeley–Gilkey).

Heat-kernel asymptotics

$$a_{[0]} = \mathrm{rank}(E), \quad a_{[1]} = 0, \quad a_{[2]} = \mathrm{tr}(-\frac{1}{6}R + Z).$$

where R is the scalar curvature. In case the *genuine* or *minimal* scalar Laplace operator we have:

$$a_{[2]} = -\frac{1}{6}R,$$

so from the first two terms we recover volume and the (integrated) curvature.

Remark

First term gives nothing else but the so-called Weyl's theorem about the growth of eigenvalues of the Laplace operator.

So, what is the metric ?

Where is the metric hidden ?

The Dirac operator encodes the metric:

$$d(p,q) = \sup_{||[D,f]||\leq 1} |f(p) - f(q)|,$$

WARNING: easy but not computable !

What we can compute ?

We can compute the volume, the volume functional, the norms of various objects:

$$f \mapsto \fint f|D|^{-d}, \quad (\int_M \sqrt{g} f)$$

We can compute the curvature and the curvature funcional:

$$f \mapsto \fint f|D|^{-d+2}, \quad (\int_M \sqrt{g} R(g) f)$$

The question!

All Diracs ?

What is the space of all possible Dirac operators ?

- so far, there is no definitive answer
- generalize what is first order differential operator
- $[D, a]$ bounded is not enough ?

Differential and pseudodifferential operators

The metric is equivalently given by the principal symbol of the Laplace operator but it is a second order operator. Its naive square root is a *first order* pseudodifferential operator only and $[\sqrt{\Delta}, a]$ is still bounded.

The problem:

If we have a family of Dirac operators...

Then:

- how can we identify the metric ?
- how can we compute *some* geometric quantities (like curvature) ?
- how do we identify *minimal* operators (like the *true* Laplace operator ?

For example:

Classically, all of the operators written below, for a function, $h > 0$, give the same metric:

$$\frac{1}{2}(h^2 D + Dh^2)^2, \quad (hDh)^2, \quad h^2 D^2 h^2, \quad \frac{1}{2}(h^4 D^2 + D^2 h^4).$$

but which one is *the genuine* (spinor) Laplace operator ??

The noncommutative problem:

How to create a family of Laplace-type operators ?

- Take a real spectral triple with a Dirac operator D
- take $h > 0$ in the commutant of the algebra: $h \in J\mathcal{A}J$,
- consider operators:

$$\frac{1}{2}(h^2 D + Dh^2)^2, \quad (hDh)^2, \quad h^2 D^2 h^2, \quad \frac{1}{2}(h^4 D^2 + D^2 h^4).$$

- compute the *spectral functionals* using the generalization of Wodzicki residue:

$$⨍ P = \mathrm{Res}_{s=0}(\mathrm{tr}\, P|D|^s).$$

- try to identify *the geometric meaning* of them

The problem

There is no way to identify *torsion* or *torsion-like* objects through some other means.

A perfect example: NC Torus

- Noncommutative Torus (Tori): $C^\infty(\mathbb{T}^n_\Theta)$, is an algebra, which is generated by two (n) unitary operators, which commute up to a scalar phase:

$$U_i\, U_l = e^{2\pi i \theta_{kl}}\, U_l\, U_k ,$$

where θ_{kl} is a real antisymmetric matrix (for $n = 2$ $i, l = 1, 2$).
- Let $\mathfrak{t}$ be the trace on $C^\infty(\mathbb{T}^n_\Theta)$, $\mathfrak{t}\big(\sum_{K\in\mathbb{Z}^n} a_K\, U^k\big) := a_0$ and $\mathcal{H}_\mathfrak{t}$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^n_\Theta)$ with respect of the norm induced by the scalar product $\langle a, b\rangle := \mathfrak{t}(a^* b)$. On $\mathcal{H}_\mathfrak{t}$ we consider the left regular representation of $C^\infty(\mathbb{T}^n_\Theta)$ by bounded operators.
- Let δ_μ, $\mu \in \{1, \ldots, n\}$, be the n (pairwise commuting) canonical derivations, defined by

$$\delta_\mu(U^K) := K_\mu U^K .$$

SPECTRAL TRIPLE ON NC TORI (N-DIMENSIONAL)

- Let $\mathcal{A}_\Theta := C^\infty(\mathbb{T}^n_\Theta)$ acting on $\mathcal{H} := \mathcal{H}_\mathfrak{t} \otimes \mathbb{C}^{2^m}$ with $n = 2m$ or $n = 2m + 1$,
  Each element of $\mathcal{A}_\Theta$ is represented on $\mathcal{H}$ as $L(a) \otimes 1_{2^m}$ where L (resp. R) is the left (resp. right) multiplication.
- The Tomita conjugation $J_0(a) := a^*$ satisfies $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$ where C_0 is an operator on $\mathbb{C}^{2^m}$.
- The Dirac operator is given by

$$\mathcal{D} := -i\, \delta_\mu \otimes \gamma^\mu,$$

- And it has been shown that this is (basically) the unique equivariant Dirac operator on the noncommutative torus.
- What about nonequivariant Dirac operators ?

Theorem

There exists a family of conformally rescaled Dirac operators on the noncommutative 2-torus for which the Gauss-Bonnet formula holds, that is $\zeta_D(0) = 0$, where $\zeta_D(z) = Tr(|D|^z)$. Classically this means that: $\int_{\mathbb{T}^2} \sqrt{g} R(g) = 0$.

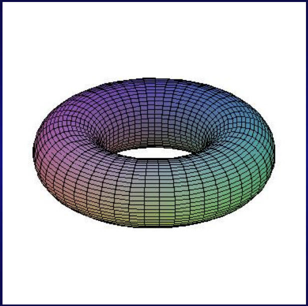
- First family of operators of the type (and conformally rescaled Laplace operators)

$$D_h = hDh, \qquad h^2 D^2 h^2,$$

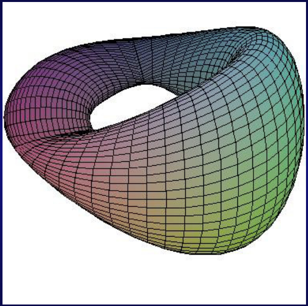
  where $h \in JC^\infty(\mathbb{T}^2_\Theta)J$, so it is in the commutant, $h > 0$, was introduced by Connes and Tretkoff.
- 4-dimensional version was studied by Fatzizadeh + Khalkhali and AS.
- n-dimensional version is possible

Asymmetric torus

Take a torus with the metric $dx^2 + k^{-2}(x, y)dy^2$ (that is, for instance the usual „round" torus embedded in $\mathbb{R}^3$ which has $k^{-1} = c + \cos y$).



Torus embedded in $\mathbb{R}^3$



Asymmetric torus in $\mathbb{R}^3$

The scalar curvature of the torus with such metric reads

$$R = 2k^{-1}\partial_x^2(k) - 4k^{-2}(\partial_x(k))^2.$$

Asymmetric torus

The Dirac operator is:

$$D = -i\sigma^1 \delta_1 - i\sigma^2 \left(k\,\delta_2 + \tfrac{1}{2}\delta_2(k)\right),$$

Theorem (L.Dabrowski+AS)

The scalar curvature functional for the asymmetric torus is:

$$\begin{aligned}\sqrt{g}R = & F_{11}(\delta_1(k),\delta_1(k)) + F'_{11}(\delta_1(k)^2) \\ & + F_{22}(\delta_2(k),\delta_2(k)) + F'_{22}(\delta_2(k)^2) \\ & + F_1(\delta_{11}(k)) + F_2(\delta_{22}(k)),\end{aligned}$$

where

$$F_{11}(s,t) = -\frac{2\pi}{3k^3}\frac{(2s^2+4st+4s+3+8t+3t^2)}{(t+1)^3(s+1)(s+t)},$$

$$F'_{11}(s) = \frac{4\pi}{3k^3}\frac{1}{(s+1)^3},$$

$$F_{22}(s,t) = \frac{\pi}{2k}\frac{(t^2-6t+1)}{(t+1)^3},$$

$$F'_2(s) = -\frac{\pi}{2k}\frac{(s^2-6s+1)}{(s+1)^3},$$

and

$$F_1(s) = \frac{2\pi}{3k^2}\frac{1}{(s+1)^2}.$$

$$F_2(s) = 0.$$

and its trace vanishes.

First, the square of D reads

$$
\begin{aligned}
D^2 = & \left((\delta_1)^2 + k^2(\delta_1)^2\right) \\
& + \left(\tfrac{3}{2}k\delta_2(k) + \tfrac{1}{2}\delta_2(k)k + i\sigma^3\delta_1(k)\right)\delta_2 \\
& + \left(\tfrac{1}{4}(\delta_2(k))^2 + \tfrac{1}{2}i\sigma^3\delta_{12}(k) + \tfrac{1}{2}k\delta_{22}(k)\right).
\end{aligned}
$$

and its symbol is

$$\sigma(D^2) = a_0 + a_1 + a_2,$$

where

$$
\begin{aligned}
a_0 &= \left(\xi_1^2 + k^2\xi_2^2\right) \\
a_1 &= \left(\tfrac{3}{2}k\delta_2(k) + \tfrac{1}{2}\delta_2(k)k + i\sigma^3\delta_1(k)\right)\xi_2 \\
a_2 &= \left(\tfrac{1}{4}(\delta_2(k))^2 + \tfrac{1}{2}i\sigma^3\delta_{12}(k) + \tfrac{1}{2}k\delta_{22}(k)\right).
\end{aligned}
$$

$$\zeta(0) = -\int \mathfrak{t}(b_2(\xi))\, d\xi,$$

where $b_2(\xi)$ is a symbol of order -4 of the pseudodifferential operator $(D^2+1)^{-1}$. It can be computed by pseudodifferential calculus of symbols from the symbol $a_2(\xi) + a_1(\xi) + a_0(\xi)$ of D^2:

$$\begin{aligned} b_2 = -\,(&b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 \\ &+ \partial_1(b_1)\delta_1(a_2)b_0 + \partial_2(b_1)\delta_2(a_2)b_0 + \frac{1}{2}\partial_{11}(b_0)\delta_1^2(a_2)b_0 \\ &+ \frac{1}{2}\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0), \end{aligned}$$

where

$$\begin{aligned} b_1 &= -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0), \\ b_0 &= (a_2+1)^{-1}, \end{aligned}$$

$$b_2 = A + B + C,$$

where

$$\begin{aligned}
A = &- 2kb_0^2\delta_1(k)kb_0\delta_1(k)b_0\xi_2^4 + 4kb_0^2\delta_1(k)kb_0^2\delta_1(k)b_0\xi_1^2\xi_2^4 - 2kb_0^2\delta_1(k)b_0\delta_1(k)kb_0\xi_2^4 \\
&+ 4kb_0^2\delta_1(k)b_0^2\delta_1(k)kb_0\xi_1^2\xi_2^4 + 8kb_0^3\delta_1(k)kb_0\delta_1(k)b_0\xi_1^2\xi_2^4 + 8kb_0^3\delta_1(k)b_0\delta_1(k)kb_0\xi_1^2\xi_2^4 \\
&- b_0\delta_1(k)b_0\delta_1(k)b_0\xi_2^2 + 2b_0^2\delta_1(k)\delta_1(k)b_0\xi_2^2 - 2b_0^2\delta_1(k)kb_0\delta_1(k)kb_0\xi_2^4 \\
&+ 4b_0^2\delta_1(k)kb_0^2\delta_1(k)kb_0\xi_1^2\xi_2^4 - 2b_0^2\delta_1(k)k^2b_0\delta_1(k)b_0\xi_2^4 + 4b_0^2\delta_1(k)k^2b_0^2\delta_1(k)b_0\xi_1^2\xi_2^4 \\
&- 8b_0^3\delta_1(k)\delta_1(k)b_0\xi_1^2\xi_2^2 + 8b_0^3\delta_1(k)kb_0\delta_1(k)kb_0\xi_1^2\xi_2^4 + 8b_0^3\delta_1(k)k^2b_0\delta_1(k)b_0\xi_1^2\xi_2^4,
\end{aligned}$$

$$\begin{aligned}
B = &\tfrac{15}{4}kb_0\delta_2(k)kb_0\delta_2(k)b_0\xi_2^2 - 3kb_0\delta_2(k)k^2b_0^2\delta_2(k)kb_0\xi_2^4 - 3kb_0\delta_2(k)k^3b_0^2\delta_2(k)b_0\xi_2^4 \\
&+ \tfrac{9}{4}kb_0\delta_2(k)b_0\delta_2(k)kb_0\xi_2^2 + 6k^2b_0^2\delta_2(k)\delta_2(k)b_0\xi_2^2 - 8k^2b_0^2\delta_2(k)kb_0\delta_2(k)kb_0\xi_2^4 \\
&- 10k^2b_0^2\delta_2(k)k^2b_0\delta_2(k)b_0\xi_2^4 + 4k^2b_0^2\delta_2(k)k^3b_0^2\delta_2(k)kb_0\xi_2^6 + 4k^2b_0^2\delta_2(k)k^4b_0^2\delta_2(k)b_0\xi_2^6 \\
&- 12k^3b_0^2\delta_2(k)kb_0\delta_2(k)b_0\xi_2^4 + 4k^3b_0^2\delta_2(k)k^2b_0^2\delta_2(k)kb_0\xi_2^6 + 4k^3b_0^2\delta_2(k)k^3b_0^2\delta_2(k)b_0\xi_2^6 \\
&- 10k^3b_0^2\delta_2(k)b_0\delta_2(k)kb_0\xi_2^4 - 8k^4b_0^3\delta_2(k)\delta_2(k)b_0\xi_2^4 + 8k^4b_0^3\delta_2(k)kb_0\delta_2(k)kb_0\xi_2^6 \\
&+ 8k^4b_0^3\delta_2(k)k^2b_0\delta_2(k)b_0\xi_2^6 + 8k^5b_0^3\delta_2(k)kb_0\delta_2(k)b_0\xi_2^6 + 8k^5b_0^3\delta_2(k)b_0\delta_2(k)kb_0\xi_2^6 \\
&- \tfrac{1}{4}b_0\delta_2(k)\delta_2(k)b_0 + \tfrac{3}{4}b_0\delta_2(k)kb_0\delta_2(k)kb_0\xi_2^2 + \tfrac{5}{4}b_0\delta_2(k)k^2b_0\delta_2(k)b_0\xi_2^2 \\
&- b_0\delta_2(k)k^3b_0^2\delta_2(k)kb_0\xi_2^4 - b_0\delta_2(k)k^4b_0^2\delta_2(k)b_0\xi_2^4;
\end{aligned}$$

and

$$\begin{aligned}
C = &+ kb_0^2\delta_{11}(k)b_0\xi_2^2 - 4kb_0^3\delta_{11}(k)b_0\xi_1^2\xi_2^2 + b_0^2\delta_{11}(k)kb_0\xi_2^2 - 4b_0^3\delta_{11}(k)kb_0\xi_1^2\xi_2^2 \\
&- \tfrac{1}{2}kb_0\delta_{22}(k)b_0 + 2k^2b_0^2\delta_{22}(k)kb_0\xi_2^2 + 4k^3b_0^2\delta_{22}(k)b_0\xi_2^2 - 4k^4b_0^3\delta_{22}(k)kb_0\xi_2^4 - 4k^5b_0^3\delta_{22}(k)b_0\xi_2^4,
\end{aligned}$$

The remaining part of the proof follows the idea of computations by Lesch (rearrangement lemma):

$$\int_0^\infty f_0(uk^2)\cdot b_1 \cdot f_1(uk^2)\cdot b_2 \cdots b_p \cdot f_p(uk^2)du =$$

$$= k^{-2} F(\Delta_2^{(1)}, \Delta_2^{(1)}\Delta_2^{(2)}, \ldots, \Delta_2^{(1)}\cdots\Delta_2^{(p)})(b_1 \cdot b_2 \cdots b_p),$$

where the function $F(s_1, \ldots, s_p)$ is

$$F(s) = \int_0^\infty f_0(u) f_1(us_1)\cdots f_p(us_p)du$$

and $\Delta_2^{(j)}$; signifies the square of the modular operator $\Delta_2 = \Delta^2$, acting on the j-th component of the product. Here we shall rather use $\Delta = k^{-1}\cdot k$ instead of its square.
In our case we need to adapt the formula to a slightly different setting, when we integrate over two variables ξ_1 and ξ_2

$$\mathcal{J} = \int d\xi_1 \int d\xi_2 \; k^{n_1} b_0^{m_1}(\xi_1,\xi_2)\, X\, k^{n_2} b_0^{m_2}(\xi_1,\xi_2)\; Y\, k^{n_3} b_0^{m_3}(\xi_1,\xi_2)\xi_1^{2k_1}\xi_2^{2k_2},$$

where X, Y are derivations of k and $b_0(\xi_1,\xi_2) = \frac{1}{1+\xi_1^2+k^2\xi^2}$.

WHY GAUSS-BONET HOLDS ?

First of all, observe that

$$F_{22}(s,1) + F'_{22}(1) = 0, \qquad F_2(1) = 0,$$

so all terms containing $\delta_2(k)$ and $\delta_{22}(k)$ vanish.
For the terms containing $\delta_1(k)$ we have:

$$F_{11}(s,1) + F'_{11}(1) = -\frac{\pi}{3k^3}\frac{s+3}{(s+1)^2},$$

then using the identity:

$$\mathfrak{t}\left(k^{-2}\delta_{11}(k)\right) = 2\mathfrak{t}\left(k^{-2}\delta_1(k)k^{-1}\delta_1(k)\right) = 2\mathfrak{t}\left(k^{-3}\Delta^{-1}(\delta_1(k))\delta_1(k)\right),$$

which follows directly from the Leibniz rule and the fact that the trace is closed, we can rewrite:

$$\mathfrak{t}\left(F_{11}(\delta_1(k),\delta_1(k)) + F'_{11}(\delta_1(k)^2) + F_1(\delta_{11}(k))\right) = \mathfrak{t}\left(k^{-3}H(\Delta)(\delta_1(k))\delta_1(k)\right),$$

(CONTINUED)

where

$$H(s) = \frac{\pi}{3k^3} \frac{1-s}{s(s+1)^2}.$$

Next, we observe that for any A and B and an entire function H:

$$\mathfrak{t}\left(k^{-3} H(\Delta)(A) B\right) = \mathfrak{t}\left(H(\Delta)(\Delta^3(A)) k^{-3} B\right) = \mathfrak{t}\left(k^{-3} B H(\Delta)(\Delta^3(A))\right),$$

and

$$\mathfrak{t}\left(k^{-3} H(\Delta)(A) B\right) = \mathfrak{t}\left(k^{-3} A H(\Delta^{-1})(B)\right).$$

Now if $A = B$ then both expressions on the right-hand side are identical. In our case, however:

$$H(s)s^3 = \frac{\pi}{3k^3} \frac{s^2(1-s)}{(s+1)^2}, \quad \text{and} \quad H(s^{-1}) = \frac{\pi}{3k^3} \frac{s^2(s-1)}{(s+1)^2},$$

and therefore since

$$H(s)s^3 = -H(s^{-1}),$$

the trace of the above expression must vanish, hence, the Gauss-Bonnet theorem holds.

Arbitrary Dirac

Theorem (L.Dabrowski + AS)

Let

$$D = \sum_{j,\mu=1}^{2} \left(\sigma^j e_j^\mu \delta_\mu + \frac{1}{2} \sigma^j \delta_\mu (e_j^\mu) \right),$$

be a general Dirac operator on the NC Torus, with elements e_j^μ *from the commutant, then for* $e_j^\mu = \delta_j^\mu + \varepsilon h_j^\mu$*, we have up to* ε^2*:*

$$\begin{aligned}
\sqrt{g}R =& 2\varepsilon \left(+\delta_1\delta_1(h_2^2) + \delta_2\delta_2(h_1^1) - \delta_1\delta_2(h_1^2) - \delta_2\delta_1(h_2^1) \right) \\
&+\varepsilon^2 \Big(\left[h_1^1, \delta_1\delta_2(h_2^1) + (\delta_1)^2(h_2^2) - 2(\delta_2)^2(h_1^1) \right]_+ + \left[h_2^2, \delta_1\delta_2(h_1^2) + (\delta_2)^2(h_1^1) - 2(\delta_1)^2(h_2^2) \right]_+ \\
&\quad + \left[h_1^2, 2\delta_1\delta_2(h_2^2) + \delta_1\delta_2(h_1^1) - (\delta_2)^2(h_1^2) - (\delta_1)^2(h_2^1) - (\delta_2)^2(h_1^2) \right]_+ \\
&\quad + \left[h_2^1, 2\delta_1\delta_2(h_1^1) + \delta_1\delta_2(h_2^2) - (\delta_2)^2(h_1^2) - (\delta_1)^2(h_1^2) - (\delta_1)^2(h_2^1) \right]_+ \\
&\quad + \left[\delta_2(h_1^1), 2\delta_1(h_2^1) + \delta_1(h_1^2) \right]_+ + \left[\delta_1(h_2^2), 2\delta_1(h_1^2) + \delta_2(h_1^2) \right]_+ \\
&\quad + \left[\delta_1(h_1^1), \delta_1(h_2^2) + \delta_2(h_2^1) \right]_+ + \left[\delta_2(h_2^2), \delta_2(h_1^1) + \delta_1(h_1^2) \right]_+ - 2\left[\delta_2(h_1^2), \delta_2(h_2^1) + \delta_1(h_2^1) \right]_+ \\
&\quad -2\left(\delta_2(h_1^2)\right)^2 - 2\left(\delta_1(h_2^1)\right)^2 - 4\left(\delta_2(h_1^1)\right)^2 - 4\left(\delta_1(h_2^2)\right)^2 \Big) + O(\varepsilon^3)
\end{aligned}$$

What is the lesson ?

First problem:

The *assymetric* noncommutatie torus is a *good* candidate for a nontrivial metric. Is it (if so, in what sense) equivalent to the conformal rescaling of the *flat torus* ?

Second problem:

Can we use it to characterize *all* possible Dirac operators ?

Third problem:

Is there a way to extract *curvature* alone ? Is there an algebraic way to compute it ?

References:

L. Dabrowski, A. Sitarz, Curved noncommutative torus and Gauss-Bonnet, J.Math.Phys. 54, 013518 (2013)

L. Dabrowski, A. Sitarz, Asymmetric noncommutative torus, arXiv:1406.4645

INTO 4 DIMENSIONS: MINIMALITY

THEOREM (AS)

In 4 dimensions the conformally rescaled Laplace operator (as proposed by Khalkhali and Fatzizadeh) is not minimal:

$$\Delta_h = \sum_{a=1}^{4} h^{-2}\delta_a(h^2\delta_a)h^{-2},$$

does not minimize the functional: $\Phi(\Delta_h) = Wres(\Delta_h^{-1})$.

PROOF.

Using the calculus of pseudodifferential operators on the noncommutative torus and the formula for Wodzicki residue expressed in terms of symbol or order -4 of Δ_h^{-1}:

$$\Delta_h = h^{-2}\Big(\sum_a \delta_a^2\Big) + \sum_a Y^a\delta_a,$$

then we look whether the functional has a minimum for some Y. $\square$

Using:

$$Y_a = \delta_a(h^{-2}) + T_a,$$

we have

Theorem

The Wodzicki residue of Δ^{-2} depends only on h:

$$Wres(\Delta^{-2}) = 2\pi^2\, \mathfrak{t}(h^4).$$

whereas for Δ^{-1}:

$$Wres(\Delta^{-1}) = \frac{\pi^2}{2}\Big(\mathfrak{t}(h^2 T_a h^2 T_a h^2) + \mathfrak{t}(h^2 [T_a, \delta_a(h^2)]) \\ - \mathfrak{t}(\delta_a(h^2) h^{-2} \delta_a(h^2))\Big)\,.$$

What is the lesson here?

First problem:

What are *minimal operators* in NCG ?

Second problem:

What is *metric* and *torsion* ?

Third problem:

Can one distinguish *curvature* from *torsion* ?

Fourth problem:

Is there an *algebraic way* to determine torsion or torsion like objects ?

References:

Andrzej Sitarz, Wodzicki residue and minimal operators on a noncommutative 4-dimensional torus,

J. Pseudo-Differ. Oper. Appl. DOI 10.1007/s11868-014-0097-1

Dirac from $U(1)$ connection

Dirac operators on $U(1)$ bundles

Consider a $U(1)$ principal fibre bundle $M \to N$, assume that the metric is compatible with the bundle structure. Since the metric on M completely determines the metric on N, connection 1-form ω and the length of the fibres ℓ, as a consequence the Dirac operator D_M on M can be expressed in terms of D_N, ω, and ℓ.

Hopf-Galois extensions

There is a good notion of noncommutative principal fibre bundles and connections – Hopf-Galois extensions and strong connections. So, posing the same question as in the classical case we have established a way to construct a Dirac operator over an algebra A on which $C(U(1))$ coacts from a strong connection and a spectral triple over its algebra of coinvariants.

References:

L. Dąbrowski, AS, Noncommutative circle bundles and new Dirac operators, Comm.Math.Phys, 318, 1, 111 (2013)

L. Dąbrowski, AS, A.Zucca, Dirac operator on noncommutative principal circle bundles, IJGMP, 11, 1 (2014),

THE $\mathbb{T}^3_\theta \to \mathbb{T}^2_\theta$ BUNDLE

Let us consider a $U(1)$ principal bundle $\mathbb{T}^3_\theta \to \mathbb{T}^2_\theta$, given by a natural $U(1)$ action:

$$z \cdot (U_1^{\alpha_1} U_2^{\alpha_2} U_3^{\alpha_3}) = z^{\alpha_3} U_1^{\alpha_1} U_2^{\alpha_2} U_3^{\alpha_3},$$

For this $U(1)$ noncommutative principal bundle we can (starting with the standard Dirac over $\mathbb{T}^3_\theta$ construct a connection one-form ω and then lift the standard Dirac operator over $\mathbb{T}^2_\theta$:

$$D = \sigma^1 \delta_1 + \sigma^2 \delta_2,$$

to a ω-dependent Dirac operator over $\mathbb{T}^3_\theta$.

THE DIRAC OPERATOR ON $\mathbb{T}^3_\theta$ FROM U(1)-CONNECTION

$$D_\omega = \sum_{i=1}^{3} \sigma^i \delta_i - J\omega J^{-1} \delta_3,$$

which is:

$$D_\omega = \sum_{i=1}^{3} \sigma^i \delta_i + (\sigma^2 \omega_2^o + \sigma^3 \omega_3^o) \delta_3,$$

or, in more generality, we should consider

$$D'_\omega = D_\omega + Z,$$

where Z is a bounded part, which needs to be fixed. Classically the Z-part (0-order term comes from the requirement that the Dirac operator comes from a lift of Levi-Civita connection (no torsion). For a noncommutative torus - minimizing a functional.

THE DIRAC OPERATOR $D\!\!\!/_{\omega}$

So, we compute the relevant part (leading term of the heat kernel expansion) for this operator - again using the approximation that ω is small and expanding $Z = Z_0 + Z_1\epsilon + \cdots$. What we obtain ?

- at ϵ^0: $Z_0 = 0$,
- at ϵ^2: $Z_1 = -\frac{1}{4}(\delta_2\omega_3 - \delta_3\omega_2)$,

which is exactly the classical term (compare Bär, Amman) !

CLAIM

In the case of the $U(1)$ bundle $\mathbb{T}^3_\theta \to T^2_\theta$ the minimality condition fixes the Dirac operator compatible with the connection ω.

Conclusions

Remark 1

There are *MANY* interesting (curved) operators out there !

Remark 2

Generally, this is possible for *ANY* reasonable real spectral triple !

Remark 3

Once we have a family of them one can ask the questions about their freedom and *MINIMALITY* - to identify natural geometric objects (like curvature, torsion...)

Remark 4

Computationally - it is a very tough job ! - but we are here just at the beginning.

Conclusions

Remark 5

There are many interesting questions:

- what is the distance on the space of states they define ?
- how can we identify the metric (in general) ?
- what are the fluctuations of such Diracs ?
- what are the most general conditions they satisfy ?

Remark 6

Can one really use them to track some (other) topological invariants ?

Remark 7

For q-deformations and the Dirac on the Standard Podleś Sphere - it is a completely different story.

Conclusions

Remark 8

Some abstract problems: take D - a Dirac type operator, $h > 0$, what can you say about spectral properties of hDh ? Where are the poles and what are the residues of:

$$\zeta_h(z) = \mathrm{tr}\, |hDh|^z,$$

Remark 9

Extending the results to q-spheres, Moyal and other examples: *work in progress*.

THANK YOU !