Hopf-cyclic cohomology of quantized enveloping algebras

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Cotor-groups

For a coalgebra C and C-comodules of opposite parity V, V', the Cotor-groups $\mathrm{Cotor}_C(V, V')$ are computed by

$$\mathbf{CB}(V, C, V') = \bigoplus_{n \geq 0} V \otimes C^{\otimes\, n} \otimes V',$$

$$d(v \otimes c^1 \otimes \ldots \otimes c^n \otimes v') =$$

$$v_{<0>} \otimes v_{<1>} \otimes c^1 \otimes \ldots \otimes c^n \otimes v' +$$

$$\sum_{j=1}^{n} (-1)^j\, c^1 \otimes \ldots \otimes \Delta(c^j) \otimes \ldots \otimes c^n \otimes v'$$

$$+ (-1)^{n+1}\, v \otimes c^1 \otimes \ldots \otimes c^n \otimes v'_{<-1>} \otimes v'_{<0>}.$$

SAYD modules over Hopf algebras

Let $\mathcal{H}$ be a Hopf algebra.

A right module - left comodule M over $\mathcal{H}$ is called a right-left SAYD module over $\mathcal{H}$ if

- M is a right-left AYD module:
  $\blacktriangledown(m \cdot h) = S(h_{(3)}) m_{<-1>} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}$,
- M is stable: $m_{<0>} \cdot m_{<-1>} = m$.

One dimensional SAYD modules, ${}^{\sigma}k_{\delta}$

The conditions for k to be a SAYD module over $\mathcal{H}$:

- k is a right $\mathcal{H}$-module via a character $\delta : \mathcal{H} \to k$,

$$1_k \cdot h := \delta(h),$$

- k is a left $\mathcal{H}$-comodule via a group-like $\sigma \in \mathcal{H}$,

$$\blacktriangledown : k \to \mathcal{H} \otimes k, \quad \blacktriangledown(\mathbf{1}) = \sigma \otimes \mathbf{1},$$

- k is a right-left AYD module if and only if

$$S_{\delta}^{2} = \mathrm{Ad}_{\sigma}, \quad S_{\delta}(h) := \delta(h_{(1)})S(h_{(2)}),$$

- k is stable if and only if

$$\delta(\sigma) = 1.$$

Such a pair (δ, σ) is called a modular pair in involution (MPI).

Hopf-cyclic cohomology of Hopf algebras

Let M be a right-left SAYD module over a Hopf algebra $\mathcal{H}$.

Then we have the graded space

$$C(\mathcal{H}, M) := \oplus_{n\geq 0} C^n(\mathcal{H}, M), \quad C^n(\mathcal{H}, M) := M \otimes \mathcal{H}^{\otimes n}$$

with the face operators

$$d_i : C^n(\mathcal{H}, M) \to C^{n+1}(\mathcal{H}, M), \quad 0 \leq i \leq n+1$$
$$d_0(m \otimes h^1 \otimes \ldots \otimes h^n) = m \otimes 1 \otimes h^1 \otimes \ldots \otimes h^n,$$
$$d_i(m \otimes h^1 \otimes \ldots \otimes h^n) = m \otimes h^1 \otimes \ldots \otimes h^i{}_{(1)} \otimes h^i{}_{(2)} \otimes \ldots \otimes h^n,$$
$$d_{n+1}(m \otimes h^1 \otimes \ldots \otimes h^n) = m_{<0>} \otimes h^1 \otimes \ldots \otimes h^n \otimes m_{<-1>},$$

Hopf-cyclic cohomology of Hopf algebras

the degeneracy operators

$$s_j : C^n(H, M) \to C^{n-1}(H, M), \quad 0 \leq j \leq n-1$$
$$s_j(m \otimes h^1 \otimes \ldots \otimes h^n) = m \otimes h^1 \otimes \ldots \otimes \varepsilon(h^{j+1}) \otimes \ldots \otimes h^n,$$

and the cyclic operator

$$t : C^n(H, M) \to C^n(H, M),$$
$$t(m \otimes h^1 \otimes \ldots \otimes h^n) = m_{<0>} \cdot {h^1}_{(1)} \otimes S({h^1}_{(2)}) \cdot (h^2 \otimes \ldots \otimes h^n \otimes m_{<-1>}).$$

Hopf-cyclic cohomology of Hopf algebras

The Hopf-cyclic cohomology of H with coefficients in M is calculated by

$$(\mathrm{Tot}(\mathcal{H}, M), b+B), \quad \mathrm{Tot}^n(\mathcal{H}, M) := C^n(\mathcal{H}, M) \oplus C^{n-2}(\mathcal{H}, M) \oplus \dots$$

with the Hochschild coboundary

$$b : C^n(\mathcal{H}, M) \to C^{n+1}(\mathcal{H}, M), \qquad b := \sum_{i=0}^{n+1} (-1)^i d_i.$$

and the Connes boundary operator

$$B : C^n(\mathcal{H}, M) \to C^{n-1}(\mathcal{H}, M), \qquad B := \left(\sum_{i=0}^{n} (-1)^{ni} t^i \right) s_{n-1} t.$$

Hopf-cyclic cohomology of Hopf algebras, $HC(\mathcal{H}, M)$

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
M \otimes \mathcal{H}^{\otimes 4} & \xrightarrow{B} & M \otimes \mathcal{H}^{\otimes 3} & \xrightarrow{B} & M \otimes \mathcal{H}^{\otimes 2} & \xrightarrow{B} & M \otimes \mathcal{H} \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
M \otimes \mathcal{H}^{\otimes 3} & \xrightarrow{B} & M \otimes \mathcal{H}^{\otimes 2} & \xrightarrow{B} & M \otimes \mathcal{H} & & \\
\uparrow b & & \uparrow b & & & & \\
M \otimes \mathcal{H}^{\otimes 2} & \xrightarrow{B} & M \otimes \mathcal{H} & & & & \\
\uparrow b & & & & & & \\
M \otimes \mathcal{H} & & & & & & \\
\uparrow & & & & & & \\
M & & & & & &
\end{array}
$$

Cotor-groups to Hochschild cohomology

There are isomorphisms:

Theorem (Crainic, 02)

Let $\mathcal{H}$ be a Hopf algebra, and (δ, σ) an MPI on $\mathcal{H}$. Then

$$HH^n(\mathcal{H}, {}^{\sigma}k_{\delta}) \cong \mathrm{Cotor}^n_{\mathcal{H}}(k, {}^{\sigma}k), \qquad n \geq 0.$$

connecting the Cotor-groups to the Hochschild cohomology...

Hopf-cyclic cohomology of Hopf algebras

and there is a long exact sequence,

$$\cdots \longrightarrow HH^n(\mathcal{H}, M) \xrightarrow{B} HC^{n-1}(\mathcal{H}, M) \xrightarrow{S} HC^{n+1}(\mathcal{H}, M) \xrightarrow{I} HH^{n+1}(\mathcal{H}, M) \xrightarrow{B} \cdots$$

connecting the Hochschild cohomology to the Hopf-cyclic cohomology.

The cohomological machinery

Let $\pi : C \longrightarrow D$ be a coalgebra projection, and $Z := C \oplus D$ with the coalgebra structure

$$\Delta(y) := y_{(1)} \otimes y_{(2)}, \quad \forall\, y \in D$$

$$\Delta(x) := x_{(1)} \otimes x_{(2)} + \pi(x_{(1)}) \otimes x_{(2)} + x_{(1)} \otimes \pi(x_{(2)}), \quad \forall\, x \in C$$

$$\varepsilon(x \oplus y) := \varepsilon(y).$$

Also, let M be a C-bicomodule as well as a Z-bicomodule.

The cohomological machinery

We introduce the following decreasing filtration:

$$G_p^{p+q} = \bigoplus_{n_0+\cdots+n_p=q} M \otimes Z^{\otimes n_0} \otimes D \otimes Z^{\otimes n_1} \otimes \ldots \otimes D \otimes Z^{\otimes n_p},$$

for $p \geq 0$, and

$$G_p^{p+q} = 0$$

for $p < 0$. On the associated spectral sequence we get

$$E_0^{i,j} = G_i^{i+j}/G_{i+1}^{i+j} = \bigoplus_{n_0+\cdots+n_i=j} M \otimes C^{\otimes n_0} \otimes D \otimes C^{\otimes n_1} \otimes \ldots \otimes D \otimes C^{\otimes n_i}.$$

As a result,

$$E_1^{0,j} = HH^j(C, M), \quad \text{and} \quad E_2^{i,j} = 0, \; i > 0,$$

and hence

$$HH^n(Z, M) \cong HH^n(C, M), \qquad n \geq 0.$$

The cohomological machinery

Next consider the filtration

$$F_p^{p+q} = \bigoplus_{n_0+\cdots+n_p=q} M \otimes Z^{\otimes n_0} \otimes C \otimes Z^{\otimes n_1} \otimes \ldots \otimes C \otimes Z^{\otimes n_p}$$

for $p \geq 0$, and $F_p^{p+q} = 0$ for $p < 0$.

In the associated spectral sequence we will have

$$E_0^{i,j} = F_i^{i+j}/F_{i+1}^{i+j} = \bigoplus_{n_0+\cdots+n_i=j} M \otimes D^{\otimes n_0} \otimes C \otimes D^{\otimes n_1} \otimes \ldots \otimes C \otimes D^{\otimes n_i}$$

Assuming C is left and right D-coflat, *i.e.*

$$\mathrm{Cotor}_D(X, C) = 0, \quad \text{for any right } C\text{-comodule } X,$$
$$\mathrm{Cotor}_D(C, Y) = 0, \quad \text{for any left } C\text{-comodule } Y,$$

The cohomological machinery

Therefore,

Theorem

Let $\pi : C \longrightarrow D$ be a coalgebra projection, M a C-bicomodule, and C be coflat both as a left and a right D-comodule.

Then there is a spectral sequence, whose E_1-term is

$$E_1^{i,j} = HH^j(D, \underbrace{C \,\square_D \cdots \square_D C}_{i \text{ many}} \,\square_D M),$$

converging to $HH^{i+j}(C, M)$.

The cohomological machinery

In terms of Cotor-groups

Theorem

Let $\pi : C \longrightarrow D$ be a coalgebra projection, and $M = M' \otimes M''$ a C-bicomodule such that the left C-comodule structure is given by M' and the right C-comodule structure is given by M''. Let also C be coflat both as a left and a right D-comodule.

Then there is a spectral sequence, whose E_1-term is of the form

$$E_1^{i,j} = \mathrm{Cotor}_D^j(M'', \underbrace{C \,\square_D \cdots \square_D\, C}_{i \text{ many}} \,\square_D\, M'),$$

converging to $\mathrm{Cotor}_C^{i+j}(k, M)$.

Faithfully coflat H-Galois coextensions

In order to satisfy the hypothesis of the theorem we recall:

Theorem (Schneider, 90)

Let H be a Hopf algebra with a bijective antipode, C a left H-module coalgebra, and $D := C/H^+C$.

Then,

(a) *C is a projective left H-module,*

(b) $\mathrm{can} : H \otimes C \longrightarrow C \square_D C$, $h \otimes c \mapsto h \cdot c_{(1)} \otimes c_{(2)}$ *is injective,*

if and only if

(a) *C is faithfully coflat left (and right) D-comodule,*

(b) $\mathrm{can} : H \otimes C \longrightarrow C \square_D C$ *is an isomorphism.*

Quantized enveloping algebras

Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ is the algebra with 4ℓ generators E_i, F_i, K_i, K_i^{-1}, $1 \leq i \leq \ell$, and relations

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad i \neq j.$$

Quantized enveloping algebras

Its Hopf algebra structure is given by:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_j) = F_j \otimes 1 + K_j^{-1} \otimes F_j$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i.$$

Computation - $C = U_q(\mathfrak{g})$

Let

$$H := \mathrm{Span}\left\{K_1^{q_1} \dots K_\ell^{q_\ell} F_1^{r_1} \dots F_\ell^{r_\ell} \mid r_1, \dots r_\ell \geq 0,\ q_1, \dots, q_\ell \in \mathbb{Z}\right\}.$$

Then,

$$D = C/CH^+ = \mathrm{Span}\left\{E_1^{p_1} \dots E_\ell^{p_\ell} K_1^{q_1} \dots K_\ell^{q_\ell} \mid p_1, \dots p_\ell \geq 0,\ q_1, \dots, q_\ell \in \mathbb{Z}\right\}.$$

Computation - MPI

Let $\alpha_1, \ldots, \alpha_\ell$ be the simple roots of $\mathfrak{g}$, and define

$$K_\lambda := K_1^{m_1} \ldots K_\ell^{m_\ell}, \qquad \lambda = \sum_{i=1}^{\ell} m_i \alpha_i.$$

Theorem (Klimyk-Schmudgen)

Let ρ be the half-sum of the positive roots of $\mathfrak{g}$. Then,

$$S^2(a) = K_{2\rho} a K_{2\rho}^{-1}, \qquad a \in U_q(\mathfrak{g}).$$

Computation - $HH(U_q(\mathfrak{g}), {}^{\sigma}k)$

For $\mu = K_1^{p_1} \dots K_\ell^{p_\ell}$, using the projection

$$D \longrightarrow W := \mathrm{Span}\{K_1^{m_1} \dots K_\ell^{m_\ell} \mid m_1, \dots, m_\ell \in \mathbb{Z}\},$$

we first observe that

Lemma

$$\mathrm{Cotor}_D^n(k, {}^{\mu}k) = \begin{cases} k^{\oplus \frac{(p_1+\dots+p_\ell)\,!}{p_1\,!\dots\, p_\ell\,!}} & \textit{if } n = p_1 + p_2 + \dots + p_\ell, \\ 0 & \textit{if } n \neq p_1 + p_2 + \dots + p_\ell. \end{cases}$$

Computation - $HH(U_q(\mathfrak{g}), {}^{\sigma}k)$

On the second move, for $\sigma = K_{2\rho}$ we obtain

Lemma

$$\mathrm{Cotor}^n_D(k, \underbrace{C \,\Box_D \, \ldots \,\Box_D \, C}_{i \text{ many}} \,\Box_D \, {}^{\sigma}k) = \begin{cases} k^{\oplus \binom{\ell}{i}} & \text{if } n = \ell - i, \\ 0 & \text{if } n \neq \ell - i. \end{cases}$$

Computation - $HH(U_q(\mathfrak{g}), {}^{\sigma}k)$

Collecting these results,

Theorem

Let $C := U_q(\mathfrak{g})$ and $\sigma := K_{2\rho}$. Then we have

$$HH^n(U_q(\mathfrak{g}), {}^{\sigma}k) \cong \mathrm{Cotor}^n_C(k, {}^{\sigma}k) = \begin{cases} k^{\oplus 2^{\ell}} & n = \ell \\ 0 & n \neq \ell. \end{cases}$$

Computation - $HC(U_q(\mathfrak{g}), {}^{\sigma}k)$

Finally we apply the SBI-sequence...

Theorem

For $\sigma := K_{2\rho}$, *and* $\epsilon \equiv 0 \pmod 2$, *we have*

$$HP^{\epsilon}(C, {}^{\sigma}k) = k^{\oplus 2^{\ell}}, \qquad HP^{1-\epsilon}(C, {}^{\sigma}k) = 0.$$

The case $\mathfrak{g} = s\ell_2$

$C = U_q(s\ell_2)$ is the algebra generated by E, F, K, K^{-1} with the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

It is a Hopf algebra by

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0,$$
$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

The case $\mathfrak{g} = s\ell_2$

In this case,

$$\mathcal{H} = \mathrm{Span}\{K^q F^r \mid r \geq 0,\ q \in \mathbb{Z}\},$$
$$D = \mathrm{Span}\{E^p K^q \mid p \geq 0,\ q \in \mathbb{Z}\}$$

and $\sigma = K$. Hence,

$$\underbrace{C \,\square_D \,\ldots\, \square_D\, C}_{i\ many} \,\square_D\, {}^{\sigma}k =$$

$$\mathrm{Span}\left\{ K^{1-q_1-\ldots-q_{s-1}} F^{q_s} \otimes \ldots \otimes \underbrace{K^{1-q_1} \otimes \ldots \otimes K^{1-q_1}}_{i_2\ many} \otimes \right.$$

$$\left. KF^{q_1} \otimes \underbrace{K \otimes \ldots \otimes K}_{i_1\ many} \otimes \mathbf{1} \mid q_1, \ldots, q_s \geq 0 \right\}.$$

The case $\mathfrak{g} = s\ell_2$

As a result,

$$E_1^{i,j} = H\left(E_0^{i,j}, d_0\right) = \begin{matrix} 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ k & k & k & & k \\ 0 & k & k^{\oplus 2} & & k^{\oplus i} \end{matrix}$$

The case $\mathfrak{g} = s\ell_2$

Therefore,

$$E_2^{i,j} = H\left(E_1^{i,j}, d_1\right) = \begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots \\ k & 0 & 0 & \cdots & 0 & \cdots \\ 0 & k & 0 & \cdots & 0 & \cdots \end{array}$$

The case $\mathfrak{g} = s\ell_2$

where

$$E_2^{0,1} = \langle \mathbf{1} \otimes E \otimes \mathbf{1} \rangle \,, \qquad E_2^{1,0} = \langle \mathbf{1} \otimes KF \otimes \mathbf{1} \rangle \,,$$

proving

$$HH^n(U_q(s\ell_2), {}^{\sigma}k) = \mathrm{Cotor}^n_{U_q(s\ell_2)}(k, {}^{\sigma}k) = \begin{cases} k \oplus k & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$
$$= \Big\langle \mathbf{1} \otimes E, \mathbf{1} \otimes KF \Big\rangle.$$

So we recover

Theorem (Crainic, 02)

$$HP^0(U_q(s\ell_2), {}^{\sigma}k) = 0, \qquad HP^1(U_q(s\ell_2), {}^{\sigma}k) = k \oplus k.$$

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

For the crossed product algebra $A = C_c^\infty(F\mathbb{R}) \rtimes \mathrm{Diff}(\mathbb{R})$, the differential operators

$$Y(f \rtimes \varphi) := y\frac{\partial}{\partial y}(f) \rtimes \varphi$$
$$X(f \rtimes \varphi) := y\frac{\partial}{\partial x}(f) \rtimes \varphi$$
$$\delta_n(f \rtimes \varphi) := y^n\frac{\partial^n}{\partial x^n}(\log \frac{\partial \varphi}{\partial x})f \rtimes \varphi, \qquad n \geq 1$$

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

form a Hopf algebra by

$$
\begin{aligned}
&[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \qquad [\delta_p, \delta_q] = 0,\\
&\Delta(Y) = Y \otimes 1 + 1 \otimes Y,\\
&\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1,\\
&\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,\\
&\varepsilon(X) = \varepsilon(Y) = \varepsilon(\delta_n) = 0,\\
&S(X) = -X + \delta_1 Y, \quad S(Y) = -Y, \quad S(\delta_1) = -\delta_1.
\end{aligned}
$$

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

For $C = \mathcal{H}_1$, we have

$$(\delta, \sigma) = (\varepsilon, 1),$$

and we choose

$$\mathcal{H} = \mathcal{F}, \qquad \text{to get} \qquad D = \mathcal{U}.$$

As a result,

$$\underbrace{C \,\square_D \,\ldots, \square_D \, C}_{i \text{ many}} \,\square_D \, k = \mathcal{F}^{\otimes\, i} \otimes k.$$

Therefore,

$$E_1^{i,j} = \mathrm{Cotor}^j_{\mathcal{U}}(k, \mathcal{F}^{\otimes\, i} \otimes k),$$
$$E_2^{i,j} = \mathrm{Cotor}^j_{\mathcal{U}}(k, k) \otimes \mathrm{Cotor}^i_{\mathcal{F}}(k, k) \Rightarrow \mathrm{Cotor}^{i+j}_{\mathcal{H}_1}(\mathcal{H}_1, {}^{\sigma}k).$$

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

Hence,

Theorem (Moscovici-Rangipour, 07)

There is a spectral sequence whose E_1-term is

$$E_1^{i,0} = \mathcal{F}^{\otimes\, i}, \qquad E_1^{i,1} = (kX \otimes \mathcal{F}^{\otimes\, i}) \oplus (kY \otimes \mathcal{F}^{\otimes\, i})$$
$$E_1^{i,0} = k(X \wedge Y) \otimes \mathcal{F}^{\otimes\, i}, \qquad E_1^{i,j} = 0, \quad j \geq 3$$

converging to $HH^{i+j}(\mathcal{H}_1, {}^{\sigma}k)$, the total complex of the bicomplex

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{U}^{\otimes 2} & \longrightarrow & \mathcal{F} \otimes \mathcal{U}^{\otimes 2} & \longrightarrow & \mathcal{F}^{\otimes 2} \otimes \mathcal{U}^{\otimes 2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{U} & \longrightarrow & \mathcal{F} \otimes \mathcal{U} & \longrightarrow & \mathcal{F}^{\otimes 2} \otimes \mathcal{U} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
k & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}^{\otimes 2} & \longrightarrow & \cdots
\end{array}$$

filtered by

$$F_p = \bigoplus_{i \geq p} \bigoplus_{j \geq 0} \mathcal{F}^{\otimes i} \otimes \mathcal{U}^{\otimes j}.$$

Connes-Moscovici Hopf algebra $\mathcal{H}_1$

As a result of the "Cartan homotopy formula for $\mathcal{H}_1$"

$$\mathrm{Id} - \mathrm{ad}(Y) = [E_Y + e_Y, b + B],$$

Theorem (Moscovici-Rangipour, 07)

$$HP^0(\mathcal{H}_1, k) = k, \qquad HP^1(\mathcal{H}_1, k) = k.$$