Integration on and duality of algebraic quantum groupoids

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20th of August 2014

Plan and background

I would like to discuss

1. What is a quantum groupoid in the algebraic setup?
2. Integration on algebraic quantum groupoids
3. Pontrjagin duality for algebraic quantum groupoids
4. The passage to operator-algebraic quantum groupoids

following

- T.T. *Integration on and duality of algebraic quantum groupoids.* (arxiv:1403.5282, submitted)

and generalising the theory of multiplier Hopf algebras [Van Daele] and

- the finite-dimensional case [Böhm-Nill-Szlachányi; Nikshych-Vainerman; . . .]
- partial integration and duality in the fiber-wise finite case [Böhm-Szlachányi]
- the case of weak multiplier Hopf algebras (w.i.p) [Van Daele-Wang]

What is a quantum groupoid? First idea and main examples

Idea A *quantum groupoid* consists of a *total algebra* A, a *base algebra* B, *target* and *source maps* $B, B^{op} \to A$ and a *comultiplication* $\Delta\colon A \to A \underset{B}{*} A$ subject to conditions that depend on the setting

Example 1 The *function algebra* of a finite groupoid $X \overset{s}{\underset{t}{\leftleftarrows}} G \overset{m}{\leftarrow} G_s{\times}_t G$

- $A = C(G)$ and $B = C(X)$
- $s^*, t^*\colon C(X) \hookrightarrow C(G)$
- $\Delta = m^*\colon C(G) \to C(G_s{\times}_t G)$ given by $\delta_\gamma \mapsto \sum_{\gamma'\gamma''=\gamma} \delta_{\gamma'} \otimes \delta_{\gamma''}$

Example 2 The *convolution algebra* of a finite groupoid as above

- $A = C(G)$ and $B = C(X)$
- $B = B^{op} \hookrightarrow A$ given by extending functions by 0 outside X
- $\Delta\colon C(G) \to C(G_{(s,t)}{\times}_{(s,t)} G)$ the diagonal map $\delta_\gamma \mapsto \delta_\gamma \otimes \delta_\gamma$

Example 3 *Deformations* of **1** and **2** for *Poisson-Lie groupoids* G

Further examples of quantum groupoids

Example 4 Assume that $\mathbb{G}_1$ and $\mathbb{G}_2$ are compact quantum groups. Then *monoidal equivalences* $\mathrm{Rep}(\mathbb{G}_1) \sim \mathrm{Rep}(\mathbb{G}_2)$ correspond with *linking quantum groupoids* [De Commer], where

- $B = \mathbb{C}^2$ and $A = \bigoplus_{i,j=1,2} C(\mathbb{G}_{ij})$ with $\mathbb{G}_{ii} = \mathbb{G}_i$
- Δ has components $\Delta_{ijk}\colon C(\mathbb{G}_{ij}) \mapsto C(\mathbb{G}_{ik}) \otimes C(\mathbb{G}_{kj})$; in particular, $\mathbb{G}_i \circlearrowright \mathbb{G}_{ij} \circlearrowleft \mathbb{G}_j$

Example 5 Extending *Woronowicz-Tannaka-Krein duality*, assume

- $\mathcal{C}$ is a semi-simple rigid C^*-tensor category
- ${}_{\mathbb{Z}}\mathrm{Hilb}_{\mathbb{Z}}$ is the category of $\mathbb{Z}$-bigraded Hilbert spaces

Then *fiber functors* $F\colon \mathcal{C} \to {}_{\mathbb{Z}}\mathrm{Hilb}_{\mathbb{Z}}$ correspond with *partial compact quantum groups* [De Commer+T.], where the dual is given by

- $B = C_c(\mathbb{Z})$, $A = \bigoplus \mathrm{Nat}(F_{ln}, F_{km})$ and $\Delta(\tau) \approx (\tau_{X\otimes Y})_{X,Y\in\mathcal{C}}$

Towards the definition of algebraic quantum groupoids

Definition A *bialgebroid* consists of

- a unital algebra A and commuting unital subalgebras $B, C \subseteq A$
- anti-isomorphisms $S: B \to C$ and $S: C \to B$ (possibly $S^2 \neq \mathrm{id}$)
- a *left comultiplication* and a *right comultiplication*
$$\Delta_B: A \to {}_BA \otimes {}_{S(B)}A \quad \text{and} \quad \Delta_C: A \to A_{S(C)} \otimes A_C$$
satisfying
  - $\Delta_B(a)(b \otimes 1) = \Delta_B(a)(1 \otimes S(b))$ for all a, b and multiplicativity
  - $\Delta_B(cb) = (c \otimes b)$ for all b, c and coassociativity
  - corresponding conditions for Δ_C
  - joint coassociativity relating Δ_B and Δ_C
- a *left counit* ${}_B\varepsilon: A \to B$ and a *right counit* $\varepsilon_C: A \to C$

Remark The inclusions $B \overset{\mathrm{id}}{\underset{S}{\rightrightarrows}} A$ correspond to a functor ${}_A\mathrm{Mod} \to {}_B\mathrm{Mod}_B$ and Δ_B and ${}_B\varepsilon$ correspond to compatible monoidal structures on ${}_A\mathrm{Mod}$

A general definition of algebraic quantum groupoids

Definition A *multiplier bialgebroid* consists of

- an algebra A and commuting subalgebras $B, C \subseteq M(A)$ (possibly non-unital but with suitable regularity properties)
- anti-isomorphisms $S: B \to C$ and $S: C \to B$
- a *left comultiplication* and a *right comultiplication* Δ_B and Δ_C taking values in a left and a right multiplier algebra such that
  1. $\Delta_B(a)(1 \otimes a')$ and $\Delta_B(a)(a' \otimes 1)$ lie in ${}_BA \otimes {}_{S(B)}A$
  2. $(a \otimes 1)\Delta_C(a')$ and $(1 \otimes a)\Delta_C(a')$ lie in $A_{S(C)} \otimes A_C$
  3. Δ_B, Δ_C are co-associative, multiplicative, jointly co-associative

Theorem+Definition [T.-Van Daele] TFAE:

- There exist a left and a right counit and an antipode
- the four maps sending $a \otimes a' \in A \otimes A$ to each of the products in 1. and 2. induce bijections $A \underset{B}{\otimes} A \to {}_BA \otimes {}_{S(B)}A, \ \ldots, \ \ldots, \ \ldots$

If these conditions hold, we call A a *regular multiplier Hopf algebroid*

Why consider integration on algebraic quantum groupoids?

Definition A *left integral* on a (multiplier) Hopf algebra A is a functional $\phi\colon A \to \mathbb{C}$ satisfying $(\mathrm{id} \otimes \phi)(\Delta(a)) = \phi(a)$ for all $a \in A$. Likewise, one defines *right integrals*.

Significance Integrals on (multiplier) Hopf algebras are the key to

1. extending *Pontrjagin duality* [Van Daele]
   - $\dim A < \infty$: $(A \otimes A)' = A' \otimes A'$, so A' becomes a Hopf algebra
   - $\dim A = \infty$: $\hat{A} = \{\phi(-a) : a \in A\} \subseteq A'$ is a multiplier Hopf algebra
2. developing the *structure theory of CQGs* [Woronowicz]
   - averaging inner products and morphisms, find that every representation is equivalent to a unitary and splits into irreducibles
3. passing to *completions* in the form of operator algebras [Kustermans-Van Daele]
   - the GNS-construction $\pi_\phi\colon A \to \mathcal{B}(H_\phi)$ yields the C^*-algebra $\overline{\pi_\phi(A)}$ and the von Neumann algebra $\pi_\phi(A)''$ of a LCQG

What do we need for integration — heuristics

Ansatz For integration on a regular multiplier Hopf algebroid with total algebra A and base algebras $B, C \subseteq M(A)$, we need

- a map ${}_C\phi_C\colon A \to C$ that is *left-invariant*: for all $a, a' \in A$, $c \in C$,
  1. ${}_C\phi_C(ac) = {}_C\phi_C(a)c$ and $(\mathrm{id} \underset{C}{\otimes} {}_C\phi_C)((a \otimes 1)\Delta_C(a')) = a\,{}_C\phi_C(a')$
  2. ${}_C\phi_C(ca) = c\,{}_C\phi_C(a)$ and $(\mathrm{id} \underset{B}{\otimes} {}_C\phi_C)(\Delta_B(a)(a' \otimes 1)) = {}_C\phi_C(a)a'$
- a map ${}_B\psi_B\colon A \to B$ that is *right-invariant*
- functionals μ_B, μ_C on B, C that are *relatively invariant*:

  $$\phi\colon A \xrightarrow{{}_C\phi_C} C \xrightarrow{\mu_C} \mathbb{C} \quad \text{and} \quad \psi\colon A \xrightarrow{{}_B\psi_B} B \xrightarrow{\mu_B} \mathbb{C}$$

  are related by invertible multipliers δ, δ' s.t. $\psi = \phi(\delta -) = \phi(-\delta')$

Example 1 For the function algebra of an étale groupoid $X \underset{t}{\overset{s}{\leftleftarrows}} G$, let

- ${}_C\phi_C, {}_B\psi_B\colon C_c(G) \to C_c(X)$ be summation along the fibers of t or s
- $\mu_B = \mu_C$ on $C_c(X)$ be integration w.r.t. a quasi-invariant measure

Further examples of quantum groupoids

Example 2 For the convolution algebra of an étale groupoid G, let

- ${}_C\phi_C = {}_B\psi_B : C_c(G) \to C_c(X)$ be the restriction of functions to $X \subseteq G$
- $\mu_B = \mu_C$ on $C_c(X)$ be integration w.r.t. a quasi-invariant measure

Example 4 Assume that $\mathbb{G}_1$ and $\mathbb{G}_2$ are compact quantum groups with a monoidal equivalence $\mathrm{Rep}(\mathbb{G}_1) \sim \mathrm{Rep}(\mathbb{G}_2)$ and associated *linking quantum groupoid* $B = \mathbb{C}^2$ and $A = \bigoplus_{i,j=1,2} C(\mathbb{G}_{ij})$

- have Haar states $h_i = h_{ii}$ on $C(\mathbb{G}_i) = C(\mathbb{G}_{ii})$ and unique states h_{ij} on $C(\mathbb{G}_{ij})$ invariant for $\mathbb{G}_i \circlearrowright \mathbb{G}_{ij} \circlearrowleft \mathbb{G}_j$
- ${}_C\phi_C(a) = \sum_j h_{ij}(a_{ij})$ and ${}_B\psi_B(a) = \sum_i h_{ij}(a_{ij})$

Example 5 Given a fiber functor $F : \mathcal{C} \to_{\mathbb{Z}} \mathrm{Hilb}_{\mathbb{Z}}$ with associated partial CQG $B = C_c(\mathbb{Z})$ and $A = \bigoplus \mathrm{Nat}(F_{ln}, F_{km})'$,

- ${}_C\phi_C$ and ${}_B\psi_B$ come from evaluating a $\tau \in \mathrm{Nat}(F_{ln}, F_{km})$ at $1_{\mathcal{C}}$

What do we need for integration — formal definition

Definition Consider a regular multiplier Hopf algebroid as above.

- A *base weight* consists of functionals μ_B, μ_C on B, C subject to
  1. faithfulness, i.e., if $\mu_B(bB) = 0$ or $\mu_B(Bb) = 0$, then $b \neq 0$
  2. $\mu_B \circ S = \mu_C = \mu_B \circ S^{-1}$ and 3. $\mu_B \circ {}_B\varepsilon = \mu_C \circ \varepsilon_C$
- Call a functional $\omega : A \to \mathbb{C}$ *adapted* (to μ_B, μ_C) if one can write
$$\omega = \mu_B \circ {}_B\omega = \mu_B \circ \omega_B = \mu_C \circ {}_C\omega = \mu_C \circ \omega_C$$
with ${}_B\omega \in \mathrm{Hom}({}_BA, {}_BB)$, $\omega_B \in \mathrm{Hom}(A_B, B_B)$, . . .
- A *left integral* is an adapted functional ϕ s.t. ${}_C\phi = \phi_C =: {}_C\phi_C$ is left-invariant. We call ϕ *full* if ${}_B\phi$ and ϕ_B are surjective. Similarly, we define *(full) right integrals*.

Key observation For adapted functionals υ, ω, we can define $\upsilon \odot \mathrm{id}$, $\mathrm{id} \odot \omega$ and $\upsilon \odot \omega$ on all kinds of *balanced tensor products* $A \odot A$, e.g.,

$$\upsilon \underset{B}{\otimes} \omega : A \underset{B}{\otimes} A \to \mathbb{C},\ a \otimes b \mapsto \mu_B(\upsilon_B(a)\,{}_B\omega(b)) = \upsilon(a\,{}_B\omega(b)) = \omega(\upsilon_B(a)b)$$

The main results on integrals

Theorem [T.] Let A be a regular multiplier Hopf algebroid with base weight (μ_B, μ_C) and full left integral ϕ.

1. If ${}_BA, A_B, {}_CA, A_C$ are projective, then ϕ is faithful.

Assume that ${}_BA, A_B, {}_CA, A_C$ are flat and that ϕ is faithful.

2. There exists an *automorphism* σ^ϕ s.t. (A, σ^ϕ) is a twisted trace. Moreover, $\sigma^\phi(c) = S^2(c)$ for $c \in C$, and $\sigma^\phi(M(B)) = M(B)$.
3. Every left integral has the form $\phi(b-)$ with $b \in M(B)$.
4. Every right integral has the form $\phi(\delta-)$ with $\delta \in M(A)$.
5. There exist invertible *modular elements* $\delta, \delta^\dagger \in M(A)$ such that $\phi \circ S^{-1} = \phi(\delta-)$ and $\phi \circ S = \phi(-\delta^\dagger)$. These elements satisfy

$$\Delta_C(\delta) = \delta \otimes \delta, \quad \Delta_B(\delta) = \delta^\dagger \otimes \delta, \quad \Delta_B(\delta^\dagger) = \delta^\dagger \otimes \delta^\dagger, \quad \Delta_C(\delta^\dagger) = \delta \otimes \delta^\dagger$$
$$S(\delta^\dagger) = \delta^{-1}, \quad \varepsilon(\delta a) = \varepsilon(a) = \varepsilon(a\delta^\dagger), \quad \text{and (in the *-case) } \delta^\dagger = \delta^*.$$

An example coming from quantum group actions

Example Assume that

- H is a regular (multiplier) Hopf algebra with integrals ϕ_H, ψ_H,
- B is an algebra with a right action of H, written $x \triangleleft h$
- μ_B is a faithful H-invariant trace on B

Then $C = B^{\mathrm{op}}$ carries a left H-action and an H-invariant trace μ_C s.t.

$$h \triangleright x^{\mathrm{op}} = (x \triangleleft S_H^{-1}(h))^{\mathrm{op}} \quad \text{and} \quad \mu_C(x^{\mathrm{op}}) = \mu_B(x)$$

We obtain a regular multiplier Hopf algebroid with integrals, where

- $A = C \rtimes H \ltimes B$ is the space $C \otimes H \otimes B$ with the multiplication
$$(y \otimes h \otimes x)(y' \otimes h' \otimes x') = y(h_{(1)} \triangleright y') \otimes h_{(2)}h'_{(1)} \otimes (x \triangleleft h'_{(2)})x'$$
- the left and right comultiplication Δ_B and Δ_C are given by
$$\Delta_B(y \otimes h \otimes x)(a \otimes b) = yh_{(1)}a \otimes h_{(2)}xb$$
$$(a \otimes b)\Delta_C(y \otimes h \otimes x) = ayh_{(1)} \otimes bh_{(2)}x$$
- $\phi(y \otimes h \otimes x) = \mu_C(y)\phi_H(h)\mu_B(x), \quad \psi(y \otimes h \otimes x) = \mu_C(y)\psi_H(h)\mu_B(x)$

The dual algebra of a measured multiplier Hopf algebroid

Definition We call a regular multiplier Hopf algebroid *measured* if it is equipped with a base weight and full and faithful left and a right integrals and if the modules ${}_BA, A_B, {}_CA, A_C$ are flat.

Lemma Consider the space $\hat{A} := \{\phi(a-) : a \in A\} \subseteq A'$.

1. $\hat{A} = \{\phi(-a) : a \in A\} = \{\psi(a-) : a \in A\} = \{\psi(-a) : a \in A\}$.
2. Let $\upsilon, \omega \in \hat{A}$. Then the compositions
$$\upsilon *_B \omega := (\upsilon \otimes \omega) \circ \Delta_B \quad \text{and} \quad \upsilon *_C \omega := (\upsilon \otimes \omega) \circ \Delta_C$$
(a) are well-defined, (b) belong to $\hat{A}$ and (c) coincide.
3. $\hat{A}$ is a non-degenerate, idempotent algebra w.r.t. $(\upsilon, \omega) \mapsto \upsilon * \omega$.

Proof of assertion 2.(c):

- coassociativity $\Rightarrow (\upsilon *_B \theta) *_C \omega = \upsilon *_B (\theta *_C \omega)$ for all μ-adapted θ
- counit property $\Rightarrow \upsilon *_B \varepsilon = \upsilon$ and $\varepsilon *_C \omega = \omega$
- relations 1.+2. $\Rightarrow \upsilon *_B \omega = \upsilon *_B (\varepsilon *_C \omega) = (\upsilon *_B \varepsilon) *_C \omega = \upsilon *_C \omega$

The duality of measured regular multiplier Hopf algebroids

Theorem [T.] Let (A, μ, ϕ, ψ) form a MRMHAd. Then there exists a *dual MRMHAd* $(\hat{A}, \hat{\mu}, \hat{\phi}, \hat{\psi})$, where $\hat{A}$ was defined above and

- $\hat{B} = C$ and $\hat{C} = B$ are embedded in $M(\hat{A})$ such that
$$c\omega = \omega(-c), \quad \omega c = \omega(-S^{-1}(c)), \quad b\omega = \omega(S^{-1}(b)-), \quad \omega b = \omega(b-)$$
for all $c \in C$, $b \in B$, $\omega \in \hat{A}$
- the left and the right comultiplication $\hat{\Delta}_{\hat{B}}$ and $\hat{\Delta}_{\hat{C}}$ of $\hat{A}$ satisfy
$$(\hat{\Delta}_{\hat{B}}(\upsilon)(1 \otimes \omega)|a \otimes a') = (u \otimes \omega|(a \otimes 1)\Delta_C(a'))$$
$$((\upsilon \otimes 1)\hat{\Delta}_{\hat{C}}(\omega)|a \otimes a') = (u \otimes \omega|\Delta_B(a)(1 \otimes a'))$$
for all $a, a' \in A$, $\upsilon, \omega \in \hat{A}$
- the dual counit $\hat{\varepsilon}$, antipode $\hat{S}$ and integrals $\hat{\phi}$ and $\hat{\psi}$ are given by
$$\hat{\varepsilon}(\phi(-a)) = \phi(a), \quad \hat{S}(\omega) = \omega \circ S, \quad \hat{\phi}(\psi(a-)) = \varepsilon(a) = \hat{\psi}(\phi(-a))$$

In the $*$-case, $\omega^* = \omega \circ * \circ S$ and $\hat{\psi}(\phi(-a)^*\phi(-a)) = \phi(a^*a)$.

Theorem [T.] Every m.r.m.H.a. is naturally isomorphic to its bidual.

Outline of the construction of the dual comultiplications

By [T.-Van Daele], an RMHAd is determined by the algebras A, $B, C \subseteq M(A)$, the anti-automorphisms $B \leftrightarrows C$, and the bijections

$$T_1 : A \underset{B}{\otimes} A \to A \underset{l}{\otimes} A, \quad a \otimes a' \mapsto \Delta_B(a)(1 \otimes a')$$
$$T_2 : A \underset{C}{\otimes} A \to A \underset{r}{\otimes} A, \quad a \otimes a' \mapsto (a \otimes 1)\Delta_C(a').$$

Starting from these maps, we obtain

- dual bijections $(T_1)^\vee$ and $(T_2)^\vee$, taking transposes
- various embeddings $\hat{A} \otimes \hat{A} \to (A \otimes A)^\vee$, using the fact that elements of $\hat{A}$ are adapted functionals and forming balanced tensor products
- bijections $\hat{T}_1$, $\hat{T}_2$, which then define the structure of an RMHAd on $\hat{A}$

$$\begin{array}{ccc} (A \underset{r}{\otimes} A)^\vee & \xrightarrow{(T_2)^\vee} & (A \underset{C}{\otimes} A)^\vee \\ \uparrow & & \uparrow \\ \hat{A} \underset{\hat{B}}{\otimes} \hat{A} & \xrightarrow[\hat{T}_1]{} & \hat{A} \underset{\hat{l}}{\otimes} \hat{A} \end{array} \qquad \begin{array}{ccc} (A \underset{l}{\otimes} A)^\vee & \xrightarrow{(T_1)^\vee} & (A \underset{B}{\otimes} A)^\vee \\ \uparrow & & \uparrow \\ \hat{A} \underset{\hat{C}}{\otimes} \hat{A} & \xrightarrow[\hat{T}_2]{} & \hat{A} \underset{\hat{r}}{\otimes} \hat{A} \end{array}$$

To do: examples from braided-commutative YD-algebras

Theorem [Lu '96; Brzeziński, Militaru '01] Let B be a *braided-commutative Yetter-Drinfeld algebra* over a Hopf algebra H. Then the crossed product $A = B \rtimes H$ for the action is a Hopf algebroid.

Theorem [Neshveyev-Yamashita '13] Let H be a compact quantum group.Then there exists an equivalence between

- unital braided-commutative Y.D.-algebras over H and
- unitary tensor functors from $\mathrm{Rep}(H)$ to C^*-tensor categories.

In the case when

- H is a regular multiplier Hopf algebra with integrals
- B carries a faithful quasi-invariant entire twisted trace

we expect $B \rtimes H$ and $B^{\mathrm{op}} \rtimes \hat{H}^{\mathrm{co}}$ to form mutually dual MRMHAds.

To do: passage to the setting of operator algebras

Let (A, μ, ϕ, ψ) be an MMH-$*$-Ad.

Aim To construct completions on the level of von Neumann algebras, to get a *measured quantum groupoid* [Enock, Lesieur, Vallin], and of C^*-algebras, where a full theory does not exist yet.

We will need additional assumptions, e.g.,

- μ_B and μ_C have associated GNS-representations $B, C \to \mathcal{L}(H_\mu)$
- the modular automorphisms of ϕ and ψ commute
- (the modular element δ relating ϕ and ψ has a square root $\delta^{1/2}$)

The key steps will be to show that

1. ϕ and ψ admit a *bounded* GNS-representation $A \to \mathcal{L}(H)$
2. Δ_B extends to a comultiplication on $A'' \subseteq \mathcal{B}(H)$ rel. to $B'' \subseteq \mathcal{B}(H_\mu)$
3. ϕ and ψ induce left- and right-invariant n.s.f. weights $A'' \to B'', C''$

This was done for *measured proper dynamical quantum groups* [T.] and *partial compact quantum groups* [De Commer-T.]

Steps for the passage to the setting of operator algebras

Theorem [T.] Let (A, μ, ϕ, ψ) be a MMH-$*$-Ad, where μ, ϕ, ψ are positive and μ_B, μ_C admit bounded GNS-representations. Then:

1. ϕ and ψ admit *bounded* GNS-representations $\pi_\phi \colon A \to \mathcal{B}(H_\phi)$ and $\pi_\psi \colon A \to \mathcal{B}(H_\psi)$
2. Δ_B extends to comultiplications on $\pi_\phi(A)'' \subseteq \mathcal{B}(H_\phi)$ and $\pi_\psi(A)'' \subseteq \mathcal{B}(H_\psi)$ relative to $B'' \subseteq \mathcal{B}(H_\mu)$ so that
   - $\pi_\phi(A)''$ and $\pi_\psi(A)''$ become *Hopf-von Neumann bimodules*
   - $\overline{\pi_\phi(A)}$ and $\overline{\pi_\psi(A)}$ become *concrete Hopf C^*-bimodules*
3. $\Lambda_\phi(A) \subseteq H_\phi$ and $\Lambda_\psi(A) \subseteq H_\psi$ are Hilbert algebras so that ϕ and ψ extend to n.s.f. weights on $\pi_\phi(A)''$ and $\pi_\psi(A)''$

Idea of proof: use (C^*)pseudo-multiplicative unitaries [Vallin, T]:

- the map $a \otimes a' \mapsto \Delta_B(a')(a \otimes 1)$ induces a unitary on suitable completions of the domain and range
- identify these completions with certain Connes' fusions of H_ϕ over B''
- show that U^* is a pseudo-multiplicative unitary