Odd-dimensional multi-pullback quantum spheres

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From Poisson Brackets to Universal Quantum Symmetries

Odd-Dimensional Spheres From Solid Tori

$$S^{2N+1} := \left\{(z_0, \ldots, z_N) \in \mathbb{C}^{N+1} \mid |z_0|^2 + \cdots + |z_N|^2 = 1\right\}.$$

Presenting S^{2N+1} as a gluing of $N+1$ closed solid tori.

Define $V_i := \left\{(z_0, \ldots, z_N) \in S^{2N+1} \mid |z_i| = \max\{|z_0|, \ldots, |z_N|\}\right\}$.

Homeomorphism implementing $V_i \cong D^{\times^i} \times S^1 \times D^{\times^{N-i}}$

$$\phi_i : V_i \to D^{\times^i} \times S^1 \times D^{\times^{N-i}}, (z_0, \ldots, z_N) \mapsto \left(\frac{z_0}{|z_i|}, \ldots, \frac{z_N}{|z_i|}\right),$$

$$\begin{aligned}\phi_i^{-1} : (d_0, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_N)& \\ \mapsto \frac{1}{\sqrt{1 + \sum_{j \neq i} |d_j|^2}}&(d_0, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_N).\end{aligned}$$

S^{2N+1} AS A MULTI-PUSHOUT OF CLOSED SOLID TORI

Assume that $i < j$.

$$
\begin{array}{ccccccc}
 & & & S^{2N+1} & & & \\
 & \nearrow & \nearrow & & \nwarrow & \nwarrow & \\
D^i S^1 D^{N-i} & \xleftarrow{\phi_i} & V_i & & V_j & \xrightarrow{\phi_j} & D^j S^1 D^{N-j} \\
\uparrow & & \nwarrow & & \nearrow & & \uparrow \\
D^i S^1 D^{j-i-1} S^1 D^{N-j} & \xleftarrow{\phi_{ij}} & & V_i \cap V_j & & \xrightarrow{\phi_{ji}} & D^i S^1 D^{j-i-1} S^1 D^{N-j}
\end{array}
$$

Note that $\phi_{ji} \circ \phi_{ij}^{-1} = \mathrm{id}_{D^i S^1 D^{j-i-1} S^1 D^{N-j}}$.

S^{2N+1} is homeomorphic to $\coprod_{0 \leqslant i \leqslant N} D^{\times i} \times S^1 \times D^{\times N-i}$ divided by the identifications prescribed by the diagrams, $(0 \leqslant i < j \leqslant N)$:

$$D^i S^1 D^{N-i} \longleftarrow D^i S^1 D^{j-i-1} S^1 D^{N-j} \hookrightarrow D^j S^1 D^{N-j}$$

The multi-pullback algebra A^π of a finite family

$\{\pi^i_j : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j\in J,\, i\neq j}$ of algebra morphisms is defined as

$$A^\pi := \left\{ (a_i)_{i\in J} \in \prod_{i\in J} A_i \;\middle|\; \pi^i_j(a_i) = \pi^j_i(a_j),\ \forall\, i,j \in J,\ i \neq j \right\}.$$

$C(S^{2N+1})$ is isomorphic as a C*-algebra to a subalgebra of $\prod_{0\leqslant i\leqslant N} C(D)^{\otimes i} \otimes C(S^1) \otimes C(D)^{\otimes N-i}$ defined by the compatibility conditions ($0 \leqslant i < j \leqslant N$, $\otimes$-sign suppressed)

$$\begin{array}{ccc} C(D)^i C(S^1) C(D)^{N-i} & & C(D)^j C(S^1) C(D)^{N-j} \\ & \searrow^{\pi^i_j} \quad {}^{\pi^j_i}\swarrow & \\ & C(D)^i C(S^1) C(D)^{j-i-1} C(S^1) C(D)^{N-j} & \end{array}$$

S^{2N+1} as a $U(1)$-principal bundle

The diagonal action of $U(1)$ on S^{2N+1}

$$S^{2N+1} \times U(1) \ni ((z_0, \dots, z_N), \lambda) \mapsto (z_0\lambda, \dots, z_N\lambda) \in S^{2N+1}.$$

carries componentwise to the pushout presentation, e.g.,

$$D^{\times i} \times S^1 \times D^{\times N-i} \times S^1 \ni ((d_0, \dots, d_{i-1}, s, d_{i+1}, \dots d_N), \lambda)$$
$$\mapsto (d_0\lambda, \dots, d_{i-1}\lambda, s\lambda, d_{i+1}\lambda, \dots d_N\lambda) \in D^{\times i} \times S^1 \times D^{\times N-i}.$$

Hence we can determine a multi-pushout structure of $\mathbb{P}^N(\mathbb{C}) = S^{2N+1}/U(1)$ using the multi-pushout presentation of S^{2N+1}. However, to determine quotients by diagonal actions, we need to gauge them to actions on the rightmost components. This will yield an alternative multi-pushout presentation of S^{2N+1}.

From the Diagonal to the Rightmost Action

Let G be a group, and let X be a G-space.

- $(X \times G)^R$ is $X \times G$ understood as a G-space with G-action $(X \times G) \times G \ni ((x,g),h) \mapsto (x,gh) \in X \times G$.
- $(X \times G)^D$ is $X \times G$ understood as a G-space with G-action $(X \times G) \times G \ni ((x,g),h) \mapsto (xh,gh) \in X \times G$.

G-space isomorphisms

$$\kappa : (X \times G)^R \ni (x,g) \mapsto (xg,g) \in (X \times G)^D,$$
$$\kappa^{-1} : (X \times G)^D \ni (x,g) \mapsto (xg^{-1},g) \in (X \times G)^R.$$

Gauged Multi-pushout Presentation of S^{2N+1}

S^{2N+1} is homeomorphic to $\coprod_{0\leqslant i\leqslant N} D^{\times N}\times S^1$ divided by the identifications prescribed by the diagrams, $0\leqslant i<j\leqslant N$,

$$\begin{array}{ccc} i & & j \\ D^{\times N}\times S^1 & & D^{\times N}\times S^1 \\ \uparrow & & \uparrow \\ D^{\times j-1}\times S^1\times D^{\times N-j}\times S^1 & \xrightarrow{\chi_{ij}} & D^{\times i}\times S^1\times D^{\times N-i-1}\times S^1 \end{array},$$

$$\chi_{ij}:(d_1,\ldots,d_{j-1},t,d_{j+1},\ldots,d_N,s)\mapsto$$
$$(t^{-1}d_1,\ldots,t^{-1}d_i,t^{-1},t^{-1}d_{i+1},\ldots,t^{-1}d_{j-1},t^{-1}d_{j+1},\ldots,t^{-1}d_N,st).$$

The diagrams are $U(1)$-equivariant with respect to actions on the rightmost components, whence they yield a multi-pushout presentation of $S^{2N+1}/U(1)$.

Complex projective spaces

$\mathbb{P}^N(\mathbb{C}) := (\mathbb{C}^{N+1}\backslash\{0\})/\sim$

Here $(x_i)_{0\leqslant i\leqslant N} \sim (y_i)_{0\leqslant i\leqslant N}$ iff $\forall i\ :\ x_i = \alpha y_i$, for some $\alpha \in \mathbb{C}\backslash\{0\}$.
We denote by $[x_0 : \ldots : x_N]$ the class of $(x_0, \ldots, x_N)$ in $\mathbb{P}^N(\mathbb{C})$.

Affine open covering of $\mathbb{P}^N(\mathbb{C})$

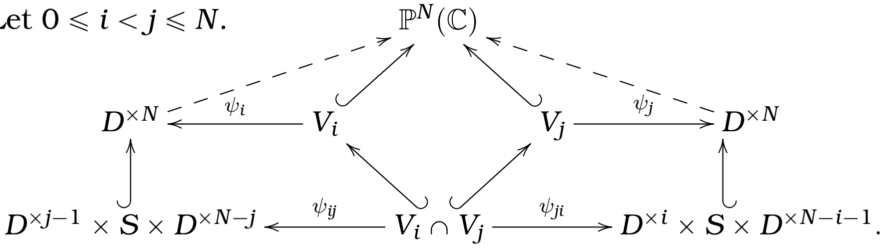
Put $U_i := \{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \mid x_i \neq 0\}$.
Then $U_i \cong \mathbb{C}^N$, $[x_0 : \ldots : x_N] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i})$.

Affine closed covering of $\mathbb{P}^N(\mathbb{C})$

Put $V_i := \{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \mid |x_i| = \max\{|x_0|, \ldots, |x_N|\}\}$.
Then $V_i \cong D^{\times N}$, where the isomorphism is given by:
$\psi_i : V_i \ni [x_0 : \ldots : x_N] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}) \in D^{\times N}$,
$\psi_i^{-1} : (d_1, \ldots, d_N) \mapsto [d_1 : \ldots : d_i : 1 : d_{i+1} : \ldots : d_N]$.

Complex Projective Spaces as Multi-Pushouts of Multi-Disks

Let $0 \leqslant i < j \leqslant N$.



$$\Upsilon_{ij} := \psi_{ji} \circ \psi_{ij}^{-1} : D^{\times j-1} \times S \times D^{\times N-j} \longrightarrow D^{\times i} \times S \times D^{\times N-i-1}$$
$$\Upsilon_{ij}(d_1, \ldots, d_{j-1}, s, d_{j+1}, \ldots, d_N) =$$
$$(s^{-1}d_1, \ldots, s^{-1}d_i, s^{-1}, s^{-1}d_{i+1}, \ldots, s^{-1}d_{j-1}, s^{-1}d_{j+1}, \ldots, s^{-1}d_N).$$

Since Υ_{ij} coincides with χ_{ij} with the rightmost component deleted, we infer that the quotient multi-pushout structure of $S^{2N+1}/U(1)$ agrees with the above multi-pushout presentation.

$C(\mathbb{P}^N(\mathbb{C}))$ as a Multi-Pullback

The C*-algebra $C(\mathbb{P}^N(\mathbb{C}))$ is isomorphic with the subalgebra of $\prod_{i=0}^{N} C(D)^{\otimes N}$ defined by the compatibility conditions:

$$\begin{array}{ccc} i & & j \\ C(D)^{\otimes N} & & C(D)^{\otimes N} \\ \downarrow & & \downarrow \\ C(D)^{\otimes j-1} \otimes C(S) \otimes C(D)^{\otimes N-j} & \xleftarrow{\Upsilon^*_{ij}} & C(D)^{\otimes i} \otimes C(S) \otimes C(D)^{\otimes N-i-1}. \end{array}$$

Toeplitz Algebras and Quantizations of S^{2N+1} and $\mathbb{P}^N(\mathbb{C})$

Take classical pushback diagrams and replace $C(D)$ by $\mathcal{T}$.

Toeplitz algebra is the universal C*-algebra generated

by z and z^* satisfying $z^*z = 1$. We have a short exact sequence of $U(1)$-equivariant C*-homomorphisms:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

Here σ — symbol map , u — the unitary generator u of $C(S^1)$, $\sigma(z) = u$, $\mathcal{K}$ — the ideal of compact operators.

Coaction of $C(U(1))$ on $\mathcal{T}$

Hopf structure on $C(U(1))$

The comultiplication: $\Delta(u) = u \otimes u$, the antipode: $S(u) = u^{-1}$, the counit: $\varepsilon(u) = 1$.

The coaction of $C(U(1))$ on $\mathcal{T}$

comes from the gauge action of $U(1)$ on $\mathcal{T}$ that rescales z by the elements of $U(1)$, i.e., $z \mapsto \lambda z$. Explicitly, we have:

$$\rho : \mathcal{T} \longrightarrow \mathcal{T} \otimes C(U(1)) \cong C(U(1), \mathcal{T}),$$
$$\rho(z) := z \otimes u, \quad \rho(z)(\lambda) = \lambda z.$$

We use the Heyneman-Sweedler notation $\rho(t) =: t_{(0)} \otimes t_{(1)}$.

Quantum Sphere S_H^{2N+1}

By definition, $C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leqslant i < j \leqslant N$, $\otimes$-supressed):

$$\mathcal{T}^i C(S^1) \mathcal{T}^{N-i} \xrightarrow{\sigma_j} \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j} \xleftarrow{\sigma_i} \mathcal{T}^j C(S^1) \mathcal{T}^{N-j}$$

where $\sigma_j := \mathrm{id}^j \otimes \sigma \otimes \mathrm{id}^{n-j}$.

We equip all the algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Gaugeing Coactions

Let H be a commutative Hopf algebra, and P be an H-comodule algebra. Let

- $(P \otimes H)^D$ be the H-comodule algebra $P \otimes H$ with the diagonal coaction $p \otimes h \mapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)} h_{(2)}$
- $(P \otimes H)^R$ be the H-comodule algebra $P \otimes H$ with the coaction on the rightmost factor $p \otimes h \mapsto p \otimes h_{(1)} \otimes h_{(2)}$.

H-comodule algebra isomorphisms

$$g : (P \otimes H)^D \to (P \otimes H)^R, \quad p \otimes h \mapsto p_{(0)} \otimes p_{(1)} h,$$
$$g^{-1} : (P \otimes H)^R \to (P \otimes H)^D, \quad p \otimes h \mapsto p_{(0)} \otimes S(p_{(1)}) h.$$

$C(S_H^{2N+1})$ as a Gauged Multi-Pullback

The following diagrams ($0 \leqslant i < j \leqslant N$) are $U(1)$-equivariant with respect to $U(1)$-actions on the rightmost factors.

$$\begin{array}{ccc} i \quad \mathcal{T}^N C(S^1) & & \mathcal{T}^N C(S^1) \quad j \\ \downarrow \sigma_{j-1} & & \downarrow \sigma_i \\ \mathcal{T}^{j-1} C(S^1) \mathcal{T}^{N-j} C(S^1) & \xleftarrow{\tilde{\Psi}_{ij}} & \mathcal{T}^i C(S^1) \mathcal{T}^{N-i-1} C(S^1) \end{array}$$

$$\tilde{\Psi}_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes v \otimes \bigotimes_{\substack{l=i+1 \\ l \neq j}}^{N} t_l \otimes s$$

$$\mapsto \bigotimes_{\substack{k=0 \\ k \neq i}}^{j-1} t_{k(0)} \otimes S\left(\prod_{\substack{m=0 \\ m \neq i,j}}^{N} t_{m(1)}\right) S(v) s_{(1)} \otimes \bigotimes_{l=j+1}^{N} t_{l(0)} \otimes s_{(2)}.$$

$C(S_H^{2N+1})$ is isomorphic as a $U(1)$-C*-algebra to the multi-pullback $U(1)$-C*-algebra of the above diagrams.

Quantum Complex Projective Spaces from Toeplitz Cubes

$C(\mathbb{P}^N(\mathcal{T}))$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes N}$ defined by the compatibility conditions prescribed by the diagrams $(0 \leqslant i < j \leqslant N)$:

$$
\begin{array}{ccccc}
i & \mathcal{T}^{\otimes N} & & \mathcal{T}^{\otimes N} & j \\
& \Big\downarrow{\scriptstyle \sigma_j} & & \Big\downarrow{\scriptstyle \sigma_{i+1}} & \\
& \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} & \xleftarrow{\Psi_{ij}} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1} &
\end{array}
$$

$$
\Psi_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes h \otimes \bigotimes_{l=i+1}^{N-1} t_l \mapsto \bigotimes_{\substack{k=0 \\ k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\Big(\prod_{\substack{m=0 \\ m\neq i}}^{N-1} t_{m(1)} \Big) h \right) \otimes \bigotimes_{l=j}^{N-1} t_{l(0)}.
$$

Clearly, $C(\mathbb{P}^N(\mathcal{T})) \cong C(S_H^{2N+1})^{U(1)}$.

Generators of $C(S_H^{2N+1})$

Let us define the following elements of $C(S_H^{2N+1})$:

$$a_i := \left(\bigotimes_{l=0}^{i-1} 1 \otimes \left(\begin{cases} z & \text{if } i \neq k \\ u & \text{if } i = k \end{cases} \right) \otimes \bigotimes_{m=i+1}^{n} 1 \right)_{k=0}^{n}$$

It straightforward to check that, for all $i, j \in \{0, \ldots, n\}$, they satisfy:

$$a_i a_j = a_j a_i, \quad a_i a_j^* = a_j^* a_i, \quad \text{when } i \neq j,$$
$$a_i^* a_i = 1, \quad \prod_{i=0}^{n} (1 - a_i a_i^*) = 0.$$

One can show that a_i's generate $C(S_H^{2N+1})$.

Principal Comodule algebras

Let H — a Hopf algebra with bijective antipode, P — a right H-comodule algebra. Then P is a principal comodule algebra iff there exists a strong connection on P.

A strong connection

A linear map $\ell : H \to P \otimes P$, $\quad \ell(h) =: \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}$ s.t.

$$\ell(1_H) = 1_P \otimes 1_P$$
$$\ell(h)^{\langle 1 \rangle}\ell(h)^{\langle 2 \rangle} = \epsilon(h),$$
$$\ell(h_{(1)})^{\langle 1 \rangle} \otimes \ell(h_{(1)})^{\langle 2 \rangle} \otimes h_{(2)} = \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}{}_{(1)},$$
$$S(h_{(1)}) \otimes \ell(h_{(2)})^{\langle 1 \rangle} \otimes \ell(h_{(2)})^{\langle 2 \rangle} = \ell(h)^{\langle 1 \rangle}{}_{(1)} \otimes \ell(h)^{\langle 1 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}.$$

STRONG CONNECTION FORMULA

The dense subalgebra $\mathcal{P}_{U(1)}(C(S_H^{2N+1})) := \bigoplus_{m\in\mathbb{Z}} L_m^{2N+1}$, where

$$L_m^{2N+1} := \{p \in C(S_H^{2N+1}) \mid \rho(p) = p \otimes u^m\},$$

is an $\mathcal{O}(U(1))$-comodule algebra.

A STRONG CONNECTION ON $C(S_H^{2N+1})$

$$\ell : \mathcal{O}(U(1)) \to \mathcal{P}_{U(1)}(C(S_H^{2N+1})) \underset{\mathrm{alg}}{\otimes} \mathcal{P}_{U(1)}(C(S_H^{2N+1}))$$

is defined by the following formulae, where $m > 0$:

$$\ell(1) := 1 \otimes 1, \quad \ell(u^m) := (a_0^*)^m \otimes a_0^m,$$

$$\ell((u^*)^m) := \sum_{0\leqslant k_1\leqslant\cdots\leqslant k_m\leqslant n} \left(\prod_{i=1}^m a_{k_i}\right) \otimes H_{k_1} \left(\prod_{j=1}^m a_{k_j}^*\right).$$

Here $H_N = 1$, $H_i = \prod_{j=i+1}^n (1 - a_j a_j^*)$, $i \in \{0, \ldots, N-1\}$.

Universal Presentation of $C(S_H^{2N+1})$

Lemma

$C(S_H^{2N+1})$ is isomorphic as a C*-algebra with the universal C*-algebra generated by a_i's satisfying the following identities:

$$a_i a_j = a_j a_i, \quad a_i a_j^* = a_j^* a_i, \quad \text{when } i \neq j,$$
$$a_i^* a_i = 1, \quad \prod_{i=0}^{n} (1 - a_i a_i^*) = 0.$$

Corollary 1. $C(S_H^{2N+1}) \cong \mathcal{T}^{\otimes N+1} / \mathcal{K}^{\otimes N+1}$.

Corollary 2. $K_0(C(S_H^{2N+1})) = \mathbb{Z} = K_1(C(S_H^{2N+1}))$ and $[C(S_H^{2N+1})]$ is a generator of $K_0(C(S_H^{2N+1}))$.

Main Result

For any at least three dimensional odd quantum sphere, all non-zero winding number noncommutative line bundles are not stably trivial:

Theorem

$$\forall m \in \mathbb{Z}\backslash\{0\}, N \in \mathbb{N}\backslash\{0\} \;:\; [L_m^{2N+1}] \notin \mathbb{Z}[C(\mathbb{P}^N(\mathcal{T}))] \subseteq K_0(C(\mathbb{P}^N(\mathcal{T})))$$

Proof Outline

1. By the preceding lemma, there exist $U(1)$-equivariant surjections $C(S_H^{2N+1}) \to C(S_H^{2N-1})$ given by

$$a_k \mapsto b_k \text{ when } k < N, \quad a_N \mapsto b_{N-1}.$$

Here $a_0, \ldots, a_N$ are isometries generating $C(S_H^{2N+1})$ and $b_0, \ldots, b_{N-1}$ are isometries generating $C(S_H^{2N-1})$. Hence there exists a $U(1)$-equivariant surjection

$$C(S_H^{2N+1}) \to C(S_H^3).$$

2. Combining this fact with the stable triviality criterion [Hajac], we conclude that the stable freeness of L_m^{2N+1} implies the stable freeness of L_m^3.
3. Finally, as by an index pairing computation [Hajac, Matthes, Szymański] L_m^3 is not stably free, neither is L_m^{2N+1}.