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ODD-DIMENSIONAL MULTI-PULLBACK QUANTUM SPHERES

Piotr M. Hajac
(IMPAN / University of New Brunswick)

Joint work with D. Pask, A. Sims and B. Zieliński

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Motivation, goal and plan

1. Motivated by a problem in lattice theory, a new family of quantum complex projective spaces was introduced in 2012 via a combinatorial multi-pullback construction (Hajac, Kaygun, Zieliński).
2. The goal of this talk is to unravel the construction of tautological line bundles over these quantum complex projective spaces and to prove that they are *not* stably trivial.
3. Plan:
   1. Classical recall
   2. Construction of multi-pullback quantum spheres.
   3. Gauging the diagonal $U(1)$-action.
   4. Constructing a strong connection.
   5. Proving the main result.

Odd-dimensional spheres from solid tori

$$\boxed{S^{2N+1} := \left\{(z_0, \ldots, z_N) \in \mathbb{C}^{N+1} \mid |z_0|^2 + \cdots + |z_N|^2 = 1\right\}}$$

Let $V_i := \left\{(z_0, \ldots, z_N) \in S^{2N+1} \mid |z_i| = \max\{|z_0|, \ldots, |z_N|\}\right\}$.
Then

$$S^{2N+1} := \bigcup_{i=0}^{N} V_i \, .$$

Homeomorphism implementing $V_i \cong D^{\times^i} \times S^1 \times D^{\times^{N-i}}$

$$\phi_i : V_i \ni (z_0, \ldots, z_N) \longmapsto \left(\frac{z_0}{|z_i|}, \ldots, \frac{z_N}{|z_i|}\right) \in D^{\times^i} \times S^1 \times D^{\times^{N-i}},$$

$$\begin{aligned}\phi_i^{-1} : D^{\times^i} \times S^1 \times D^{\times^{N-i}} &\ni (d_0, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_N) \\ &\longmapsto \frac{1}{\sqrt{1 + \sum_{j \neq i} |d_j|^2}} (d_0, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_N) \in V_i \, .\end{aligned}$$

S^{2N+1} as a multi-pushout of solid tori

Assume that $i < j$ and suppress $\times$ for brevity.

$$
\begin{tikzcd}
 & & & S^{2N+1} & & & \\
D^iS^1D^{N-i} \arrow[urrr, dashed] & & V_i \arrow[ll, "\phi_i"'] \arrow[ur, hook] & & V_j \arrow[rr, "\phi_j"] \arrow[ul, hook'] & & D^jS^1D^{N-j} \arrow[ulll, dashed] \\
D^iS^1D^{j-i-1}S^1D^{N-j} \arrow[u, hook] & & & V_i \cap V_j \arrow[lll, "\phi_{ij}"'] \arrow[rrr, "\phi_{ji}"] \arrow[ul, hook'] \arrow[ur, hook] & & & D^iS^1D^{j-i-1}S^1D^{N-j} \arrow[u, hook']
\end{tikzcd}
$$

Note that $\phi_{ji} \circ \phi_{ij}^{-1} = \mathrm{id}_{D^iS^1D^{j-i-1}S^1D^{N-j}}$.

S^{2N+1} is homeomorphic to $\coprod_{0 \leq i \leq N} D^{\times i} \times S^1 \times D^{\times N-i}$ divided by the identifications prescribed by the diagrams $(0 \leq i < j \leq N)$:

$$D^iS^1D^{N-i} \hookleftarrow D^iS^1D^{j-i-1}S^1D^{N-j} \hookrightarrow D^jS^1D^{N-j}.$$

$C(S^{2N+1})$ as a multi-pullback C*-algebra

Definition

The multi-pullback algebra A^π of a finite family $\{\pi^i_j : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j\in J,\, i\neq j}$ of algebra morphisms is defined as

$$A^\pi := \left\{ (a_i)_{i\in J} \in \prod_{i\in J} A_i \;\middle|\; \pi^i_j(a_i) = \pi^j_i(a_j),\ \forall\, i,j \in J,\, i \neq j \right\}.$$

$C(S^{2N+1})$ is isomorphic as a C*-algebra to the subalgebra of

$$\prod_{0\leq i\leq N} C(D)^{\otimes i} \otimes C(S^1) \otimes C(D)^{\otimes N-i}$$

defined by the compatibility conditions ($0 \leq i < j \leq N$, $\otimes$ suppressed):

$$\begin{tikzcd}[column sep=small]
C(D)^iC(S^1)C(D)^{N-i} \arrow[dr, "\pi^i_j"] & & C(D)^jC(S^1)C(D)^{N-j} \arrow[dl, "\pi^j_i"'] \\
& C(D)^iC(S^1)C(D)^{j-i-1}C(S^1)C(D)^{N-j}. &
\end{tikzcd}$$

S^{2N+1} as a $U(1)$-principal bundle

The diagonal action of $U(1)$ on S^{2N+1}

$$S^{2N+1} \times U(1) \ni ((z_0, \dots, z_N), \lambda) \longmapsto (z_0\lambda, \dots, z_N\lambda) \in S^{2N+1}$$

carries componentwise to the multi-pushout presentation, e.g.

$$D^{\times i} \times S^1 \times D^{\times N-i} \times S^1 \ni ((d_0, \dots, d_{i-1}, s, d_{i+1}, \dots d_N), \lambda)$$
$$\longmapsto (d_0\lambda, \dots, d_{i-1}\lambda, s\lambda, d_{i+1}\lambda, \dots d_N\lambda) \in D^{\times i} \times S^1 \times D^{\times N-i}.$$

Hence we can determine a multi-pushout structure of $\mathbb{P}^N(\mathbb{C}) \cong S^{2N+1}/U(1)$ using the multi-pushout presentation of S^{2N+1}. However, to determine quotients by diagonal actions, we need to gauge them to actions on the rightmost components. This will yield an alternative multi-pushout presentation of S^{2N+1}.

From the diagonal to the rightmost action

Let G be a group and let X be a G-space. Then:

- $(X \times G)^R$ is $X \times G$ understood as a G-space with G-action $(X \times G) \times G \ni ((x,g),h) \longmapsto (x, gh) \in X \times G$.
- $(X \times G)^D$ is $X \times G$ understood as a G-space with G-action $(X \times G) \times G \ni ((x,g),h) \longmapsto (xh, gh) \in X \times G$.

G-space isomorphisms:

$$\kappa : (X \times G)^R \ni (x,g) \longmapsto (xg, g) \in (X \times G)^D,$$
$$\kappa^{-1} : (X \times G)^D \ni (x,g) \longmapsto (xg^{-1}, g) \in (X \times G)^R.$$

Gauged multi-pushout presentation of S^{2N+1}

S^{2N+1} is homeomorphic to $\coprod_{0\leq i\leq N} D^{\times N} \times S^1$ divided by the identifications prescribed by the diagrams, $0 \leq i < j \leq N$,

$$\begin{tikzcd}
& \overset{i}{D^{\times N} \times S^1} & & \overset{j}{D^{\times N} \times S^1} & \\
D^{\times j-1} \times S^1 \times D^{\times N-j} \times S^1 \arrow[ur, hook] \arrow[rr, "\chi_{ij}"] & & D^{\times i} \times S^1 \times D^{\times N-i-1} \times S^1 \arrow[ur, hook] &
\end{tikzcd},$$

where

$$\chi_{ij} : (d_1, \ldots, d_{j-1}, t, d_{j+1}, \ldots, d_N, s) \longmapsto$$
$$(t^{-1}d_1, \ldots, t^{-1}d_i, t^{-1}, t^{-1}d_{i+1}, \ldots, t^{-1}d_{j-1}, t^{-1}d_{j+1}, \ldots, t^{-1}d_N, st).$$

The diagrams are $U(1)$-equivariant with respect to actions on the rightmost components, whence they yield a multi-pushout presentation of $S^{2N+1}/U(1)$.

Definition

$\mathbb{P}^N(\mathbb{C}) := (\mathbb{C}^{N+1} \setminus \{0\})/\sim$, where

$$(x_i)_{0\le i\le N} \sim (y_i)_{0\le i\le N} \quad \Leftrightarrow \quad \exists\, \alpha \in \mathbb{C}\setminus\{0\}\ \forall\, i:\ x_i = \alpha y_i\,.$$

We denote by $[x_0 : \ldots : x_N]$ the class of $(x_0, \ldots, x_N)$ in $\mathbb{P}^N(\mathbb{C})$.

Affine open covering of $\mathbb{P}^N(\mathbb{C})$

Put $U_i := \{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \mid x_i \neq 0\}$.
Then $U_i \cong \mathbb{C}^N$, $[x_0 : \ldots : x_N] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}\right)$.

Affine closed covering of $\mathbb{P}^N(\mathbb{C})$

Put $V_i := \big\{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \mid |x_i| = \max\{|x_0|, \ldots, |x_N|\}\big\}$.
Then $V_i \cong D^{\times N}$ with a homeomorphism is given by:

$$\psi_i : V_i \ni [x_0 : \ldots : x_N] \longmapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}\right) \in D^{\times N},$$

$$\psi_i^{-1} : (d_1, \ldots, d_N) \longmapsto [d_1 : \ldots : d_i : 1 : d_{i+1} : \ldots : d_N].$$

Complex projective spaces as multi-pushouts

Let $0 \leq i < j \leq N$. We have the multi-pushout diagram:

$$
\begin{tikzcd}
 & & \mathbb{P}^N(\mathbb{C}) & & \\
D^{\times N} \arrow[urr, dashed] & V_i \arrow[l, "\psi_i"'] \arrow[ur, hook] & & V_j \arrow[ul, hook] \arrow[r, "\psi_j"] & D^{\times N} \arrow[ull, dashed] \\
D^{\times j-1} \times S^1 \times D^{\times N-j} \arrow[u, hook] & & V_i \cap V_j \arrow[ll, "\psi_{ij}"'] \arrow[ul, hook] \arrow[ur, hook] \arrow[rr, "\psi_{ji}"] & & D^{\times i} \times S^1 \times D^{\times N-i-1} \arrow[u, hook]
\end{tikzcd}.
$$

Here

$$\Upsilon_{ij} := \psi_{ji} \circ \psi_{ij}^{-1} : D^{\times j-1} \times S^1 \times D^{\times N-j} \longrightarrow D^{\times i} \times S^1 \times D^{\times N-i-1}$$

$$\Upsilon_{ij}(d_1, \ldots, d_{j-1}, s, d_{j+1}, \ldots, d_N) =$$

$$(s^{-1}d_1, \ldots, s^{-1}d_i, s^{-1}, s^{-1}d_{i+1}, \ldots, s^{-1}d_{j-1}, s^{-1}d_{j+1}, \ldots, s^{-1}d_N).$$

Since Υ_{ij} coincides with χ_{ij} with the rightmost component deleted, we infer that the quotient multi-pushout structure of $S^{2N+1}/U(1)$ agrees with the above multi-pushout presentation.

$C(\mathbb{P}^N(\mathbb{C}))$ as a multi-pullback C*-algebra

The C*-algebra $C(\mathbb{P}^N(\mathbb{C}))$ is isomorphic with the subalgebra of $\prod_{i=0}^{N} C(D)^{\otimes N}$ defined by the compatibility conditions:

$$
\begin{tikzcd}
i & & j \\
C(D)^{\otimes N} \arrow[d, two heads] & & C(D)^{\otimes N} \arrow[d, two heads] \\
C(D)^{\otimes j-1} \otimes C(S) \otimes C(D)^{\otimes N-j} & & C(D)^{\otimes i} \otimes C(S) \otimes C(D)^{\otimes N-i-1}, \arrow[ll, "\Upsilon_{ij}^*"']
\end{tikzcd}
$$

where $0 \leq i < j \leq N$.

The Toeplitz algebra

Definition

The Toeplitz algebra $\mathcal{T}$ is the universal C*-algebra generated by z and z^* satisfying $z^*z = 1$.

We have a short exact sequence of $U(1)$-equivariant C*-homomorphisms:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

Here u is the unitary generator of $C(S^1)$, $\mathcal{K}$ is the ideal of compact operators, and σ is the symbol map ($\sigma(z) := u$). The action of $U(1)$ on $\mathcal{T}$ is given by $z \mapsto \lambda z$.

We dualize this action to a coaction of $C(U(1))$ on $\mathcal{T}$. Explicitly, we have:

$$\rho : \mathcal{T} \longrightarrow \mathcal{T} \otimes C(U(1)) \cong C(U(1), \mathcal{T}),$$
$$\rho(z) := z \otimes u, \quad \rho(z)(\lambda) = \lambda z.$$

We use the Heyneman-Sweedler notation $\rho(t) =: t_{(0)} \otimes t_{(1)}$.

$C(S_H^{2N+1})$ is the C*-subalgebra of $\prod_{i=0}^{N} \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i}$ defined by the compatibility conditions prescribed by the following diagrams ($0 \leq i < j \leq N$, $\otimes$-supressed):

$$
\begin{tikzcd}
\mathcal{T}^i C(S^1) \mathcal{T}^{N-i} \arrow[dr, "\sigma_j"] & & \mathcal{T}^j C(S^1) \mathcal{T}^{N-j} \arrow[dl, "\sigma_i"'] \\
& \mathcal{T}^i C(S^1) \mathcal{T}^{j-i-1} C(S^1) \mathcal{T}^{N-j}. &
\end{tikzcd}
$$

Here $\sigma_j := \mathrm{id}^j \otimes \sigma \otimes \mathrm{id}^{n-j}$.

We equip all C*-algebras in the diagrams with the diagonal actions of $U(1)$. Since all morphisms in the diagrams are $U(1)$-equivariant, we obtain the diagonal $U(1)$-action on $C(S_H^{2N+1})$.

Gauging coactions

Let H be a *commutative* Hopf algebra, and P be an H-comodule algebra. Then:

- $(P \otimes H)^D$ is an H-comodule algebra $P \otimes H$ with the diagonal coaction $p \otimes h \longmapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)}h_{(2)}$.
- $(P \otimes H)^R$ is an H-comodule algebra $P \otimes H$ with the coaction on the rightmost factor $p \otimes h \longmapsto p \otimes h_{(1)} \otimes h_{(2)}$.

H-comodule algebra isomorphisms:

$$F : (P \otimes H)^D \longrightarrow (P \otimes H)^R, \quad p \otimes h \longmapsto p_{(0)} \otimes p_{(1)}h,$$
$$F^{-1} : (P \otimes H)^R \longrightarrow (P \otimes H)^D, \quad p \otimes h \longmapsto p_{(0)} \otimes S(p_{(1)})h.$$

$C(S_H^{2N+1})$ as a gauged multi-pullback

The following diagrams ($0 \leq i < j \leq N$, $\otimes$ suppressed) are $U(1)$-equivariant with respect to the $U(1)$-actions on the rightmost factors.

$$\begin{tikzcd} i & \mathcal{T}^N C(S^1) \arrow[d, "\sigma_{j-1}"'] & & \mathcal{T}^N C(S^1) \arrow[d, "\sigma_i"] & j \\ & \mathcal{T}^{j-1}C(S^1)\mathcal{T}^{N-j}C(S^1) & & \mathcal{T}^{i}C(S^1)\mathcal{T}^{N-i-1}C(S^1), \arrow[ll, "\tilde{\Psi}_{ij}"'] & \end{tikzcd}$$

$$\tilde{\Psi}_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes v \otimes \bigotimes_{\substack{l=i+1 \\ l\neq j}}^{N} t_l \otimes s$$

$$\longmapsto \bigotimes_{\substack{k=0 \\ k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\prod_{\substack{m=0 \\ m\neq i,j}}^{N} t_{m(1)} \right) S(v)s_{(1)} \otimes \bigotimes_{l=j+1}^{N} t_{l(0)} \otimes s_{(2)}.$$

$C(S_H^{2N+1})$ is isomorphic as a $U(1)$-C*-algebra to the multi-pullback $U(1)$-C*-algebra of the above diagrams.

Quantum complex projective spaces $\mathbb{P}^N(\mathcal{T})$

$C(\mathbb{P}^N(\mathcal{T}))$ is the C*-subalgebra of $\prod_{i=0}^N \mathcal{T}^{\otimes N}$ defined by the compatibility conditions prescribed by the diagrams $(0 \leq i < j \leq N)$:

$$\begin{tikzcd} i \quad \mathcal{T}^{\otimes N} \arrow[d, "\sigma_j"'] & \mathcal{T}^{\otimes N} \quad j \arrow[d, "\sigma_{i+1}"] \\ \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1}, \arrow[l, "\Psi_{ij}"'] \end{tikzcd}$$

$$\Psi_{ij} : \bigotimes_{k=0}^{i-1} t_k \otimes h \otimes \bigotimes_{l=i+1}^{N-1} t_l \mapsto \bigotimes_{\substack{k=0 \\ k\neq i}}^{j-1} t_{k(0)} \otimes S\left(\Big(\prod_{\substack{m=0 \\ m\neq i}}^{N-1} t_{m(1)}\Big) h \right) \otimes \bigotimes_{l=j}^{N-1} t_{l(0)}.$$

It follows from the gauged presentation of $C(S_H^{2N+1})$ that $C(\mathbb{P}^N(\mathcal{T})) \cong C(S_H^{2N+1})^{U(1)}$.

Universal presentation of $C(S_H^{2N+1})$

Let us define the following elements of $C(S_H^{2N+1})$:

$$a_i := \left(\bigotimes_{l=0}^{i-1} 1 \otimes \left(\begin{cases} z & \text{if } i \neq k \\ u & \text{if } i = k \end{cases} \right) \otimes \bigotimes_{m=i+1}^{N} 1 \right)_{k=0}^{N}.$$

It straightforward to check that $\forall\, i, j \in \{0, \ldots, N\},\ i \neq j$:

$$a_i a_j = a_j a_i, \quad a_i a_j^* = a_j^* a_i, \quad a_i^* a_i = 1, \quad \prod_{i=0}^{N} (1 - a_i a_i^*) = 0.$$

Lemma (Key Lemma)

$C(S_H^{2N+1})$ *is isomorphic as a* $U(1)$*-C*-algebra with the universal C*-algebra generated by* a_i*'s satisfying the above relations. The* $U(1)$*-action on the latter is given by rephasing the generators.*

Corollary

$C(S_H^{2N+1}) \cong \mathcal{T}^{\otimes N+1} / \mathcal{K}^{\otimes N+1}$, $K_0(C(S_H^{2N+1})) = \mathbb{Z}[C(S_H^{2N+1})] = \mathbb{Z}$, $K_1(C(S_H^{2N+1})) = \mathbb{Z}$.

Strong-connection formula

Let H be a Hopf algebra with bijective antipode and P a right H-comodule algebra. A strong connection is a unital bicolinear map $\ell : H \to P \otimes P$ such that *multiplication* $\circ\, \ell = \varepsilon$.

The dense subalgebra $\mathcal{P}_{U(1)}(C(S_H^{2N+1})) := \bigoplus_{m\in\mathbb{Z}} L_m^{2N+1}$, where

$$L_m^{2N+1} := \{p \in C(S_H^{2N+1}) \mid \rho(p) = p \otimes u^m\},$$

is an $\mathcal{O}(U(1))$-comodule algebra.

A strong connection on $\mathcal{P}_{U(1)}(C(S_H^{2N+1}))$

$$\ell : \mathcal{O}(U(1)) \to \mathcal{P}_{U(1)}(C(S_H^{2N+1})) \underset{\text{alg}}{\otimes} \mathcal{P}_{U(1)}(C(S_H^{2N+1})),$$

$$\ell(1) := 1 \otimes 1, \quad \ell(u^m) := (a_0^*)^m \otimes a_0^m,$$

$$\ell((u^*)^m) := \sum_{0\le k_1 \le \cdots \le k_m \le N} \left(\prod_{i=1}^{m} a_{k_i}\right) \otimes H_{k_1} \left(\prod_{j=1}^{m} a_{k_j}^*\right).$$

Here $m \in \mathbb{N} \setminus \{0\}$, $H_N = 1$, $H_i = \prod_{j=i+1}^{N}(1 - a_j a_j^*)$, $i \in \{0, \ldots, N-1\}$.

Main result

Theorem

$\forall\, m \in \mathbb{Z} \setminus \{0\},\ N \in \mathbb{N} \setminus \{0\}\ :$
$$[L_m^{2N+1}] \notin \mathbb{Z}[C(\mathbb{P}^N(\mathcal{T}))] \subseteq K_0(C(\mathbb{P}^N(\mathcal{T}))).$$

<u>Proof outline:</u>

1. By the preceding lemma, there exist $U(1)$-equivariant surjections $C(S_H^{2N+1}) \to C(S_H^{2N-1})$ given by
$$a_k \mapsto b_k \text{ when } k < N, \quad a_N \mapsto b_{N-1}.$$
Here $a_0, \ldots, a_N$ are isometries generating $C(S_H^{2N+1})$ and $b_0, \ldots, b_{N-1}$ are isometries generating $C(S_H^{2N-1})$.
2. Combining this fact with the stable triviality criterion [Hajac], we conclude that the stable freeness of L_m^{2N+1} implies the stable freeness of L_m^3.
3. Finally, as by an index pairing computation [Hajac, Matthes, Szymański] L_m^3 is not stably free, neither is L_m^{2N+1}.

Erdös nr 3

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3. **Hajac, P. M.**; Majid, S.: Projective module description of the q-monopole. Comm. Math. Phys. 206 (1999), no. 2, 247–264.

Obama nr 1

Obama nr 2