Invariant means on the compactifications of locally compact quantum groups

November 3, 2014.

Our motivation behind this work is the following result in the classical Harmonic analysis on groups:

Proposition 0.1. *[2, Chapter 2] Let G be a locally compact group and $WAP(G)$ denotes the C^* algebra of weakly almost periodic functions on G. Then $WAP(G)$ admits a unique invariant mean. Morever, letting $\mathcal{I}$ denoting the kernel (which is a two sided ideal) and $AP(G)$ denoting the C^* algebra of almost periodic functions on G, we have $WAP(G) = AP(G) \oplus \mathcal{I}$ as Banach spaces.*

We would like to prove such a statement for quantum groups. However, at the moment it is unclear what would be the right generalization in the quantum set-up.

In this talk, we will consider the first step towards this, namely to examine the question of existence of invariant means on certain C* algebraic objects associated with quantum groups.

Definition 0.2. *We call a pair (A, Δ) a comact semitopological quantum semigroup where :*

(i) A is a unital C^* -algebra, considered as a norm closed subalgebra of A^{**} (the universal envelopping von Neumann algebra of A).

*(ii) $\Delta : A \longrightarrow A^{**}\overline{\otimes}A^{**}$ is a unital *-homomorphism satisfying $(\widetilde{\Delta} \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \widetilde{\Delta}) \circ \Delta$, where $\widetilde{\Delta}$ is the normal lift of Δ to A^{**}.*

(iii) For $f \in A^*$, $(f \otimes \mathrm{id})(\Delta(a)) \in A$, $(\mathrm{id} \otimes f)(\Delta(a)) \in A$ for all $a \in A$. Note here we have used (without explicitly using a different notation, to avoid complication) the fact that any element in A^* admits a unique extension to a normal functional on A^{**}.

Note that since $A \subset A^{**}$, any element in A^* can be extended to A^{**} by Hahn-Banach theorem. However not all such extensions will be normal functionals on A^{**}. In fact it can be shown that *there exists only one Hahn-Banach extension, which yields a normal functional on A^{**}. We will consider only this extension.*

The set-up in Definition 0.2 includes the following cases:

- All compact quantum groups.
- All compact semitopological semigroups. In fact it can be shown that if A in Definition 0.2 is abelian, then $A = C(S)$ for a compact semitopological semigroup, and conversely given a compact semitopological semigroup S, $C(S)$ can be given a structure which makes it look like the object considered in Definition 0.2. Thus in particular, this includes the weakly periodic compactification of a locally compact group.
- All C* Eberlein algebras defined in [3].

Two immediate consequences of (iii) in Definition 0.2 are the following:

- A becomes an $A^* - -A^*$ bimodule:

$$f \cdot a := (\mathrm{id} \otimes f)(\Delta(a)) \ ; \ a \cdot f := (f \otimes \mathrm{id})(\Delta(a)) \quad (f \in A^*, a \in A).$$

- Defining $\langle , \rangle : A^* \times A \longrightarrow \mathbb{C}$ by $A^* \times A \ni (f, a) \mapsto f(a) \in \mathbb{C}$, A^* becomes a dual Banach algebra with the product $\star$:

$$(f \star g)(a) := \langle f, g \cdot a \rangle \quad (a \in A, f, g \in A^*).$$

Definition 0.3. *Let (A, Δ) be a compact semitopological quantum semigroup. A state $M \in A^*$ is called left invariant if*

$$M \cdot a = M(a).1 \quad (\forall\ a \in A).$$

Similarly we define a right invariant mean. A mean which is both right and left invariant is called invariant mean.

Our aim is to show that under some conditions on (A, Δ), there always exists an invariant mean.

Theorem 0.4. *Let (A, Δ) be a compact semitopological quantum semigroup. For $a \in A$, let $K(a) := \{\mu \cdot a : \mu$ is a state on $A\}$. Then the following are equivalent:*

1. *A has a left invariant mean;*
2. *$K(a) \cap \mathbb{C}1 \neq \emptyset$ for all $a \in A$;*
3. *$0 \in K(a - a \cdot \mu)$ for all $a \in A$ and all states $\mu \in A^*$.*

Proof. By definition, $M \in A^*$ is left invariant when $\mu \star M = \langle \mu, 1 \rangle M$ for all $\mu \in A^*, a \in A$, or equivalently, if $M \cdot a = \langle M, a \rangle 1$ for all $a \in A$. So (1)⇒(2). If (2) holds then for $a \in A$ there is a state $\mu \in A^*$ and $t \in \mathbb{C}$ with $\mu \cdot a = t1$. Then for a state $\lambda \in A^*$ we have that $\mu \cdot (a - a \cdot \lambda) = t1 - (\mu \cdot a) \cdot \lambda = t1 - t1 \cdot \lambda = 0$ as $1 \cdot \lambda = (\lambda \otimes \mathrm{id})\Delta(1) = 1$. So (2)⇒(3).

Now suppose that (3) holds. For each $a \in A$ and state μ let

$$M(a, \mu) = \{\lambda \in A^* \text{ a state } : \lambda \cdot (a - a \cdot \mu) = 0\},$$

which is non-empty by assumption. If (λ_α) is a net in $M(a, \mu)$ converging weak* to λ, then λ is a state, and for any $\phi \in A^*$,

$$\langle \phi, \lambda \cdot (a - a \cdot \mu) \rangle = \langle \lambda, (a - a \cdot \mu) \cdot \phi \rangle = \lim_\alpha \langle \lambda_\alpha, (a - a \cdot \mu) \cdot \phi \rangle = \lim_\alpha \langle \phi, \lambda_\alpha \cdot (a - a \cdot \mu) \rangle = 0.$$

So $\lambda \in M(a, \mu)$ and we conclude that $M(a, \mu)$ is weak*-closed.

We claim that the family $\{M(a, \mu) : a \in A, \mu \text{ a state}\}$ has the finite intersection property. If so, then as the unit ball of A^* is weak*-compact, there is $\lambda \in M(a, \mu)$ for all a, μ. Set $M = \lambda \star \lambda$ so

$$\langle M, a \cdot \mu \rangle = \langle \lambda, \lambda \cdot (a \cdot \mu) \rangle = \langle \lambda, \lambda \cdot a \rangle = \langle M, a \rangle,$$

that is, M is left invariant. As Δ is a unital $*$-homomorphism, and λ is a state, also M is a state as required to show (1). Note in deriving this we have crucially used the fact that A is unital.

To show the finite intersection property, we use induction. Let $a_1, \cdots, a_n \in A$ and $\mu_1, \cdots, \mu_n$ be states on A, and suppose that $\lambda \in \bigcap_{j=1}^{n-1} M(a_j, \mu_j)$. As (3) holds, we can find $\phi \in M(\lambda \cdot a_n, \mu_n)$, so

$$0 = \phi \cdot (\lambda \cdot a_n - \lambda \cdot a_n \cdot \mu_n) = (\phi \star \lambda) \cdot (a_n - a_n \cdot \mu_n).$$

However, for $1 \leq j < n$,

$$(\phi \star \lambda) \cdot (a_j - a_j \cdot \mu_j) = \phi \cdot \big(\lambda \cdot (a_j - a_j \cdot \mu_j)\big) = \phi \cdot 0 = 0.$$

Thus $\phi \star \lambda \in \bigcap_{j=1}^{n} M(a_j, \mu_j)$, and so the result follows by induction. □

Note that here we obtained the invariant mean as a square of a certain functional.

How we apply the above theorem in practice:

We will state without proof the following situation:

Theorem 0.5. *Let (A, Δ) be a compact semitopological quantum semigroup such that for some Hilbert space H, there exists a unitary operator $V \in A^{**}\overline{\otimes}B(H)$ satisfying*

- $(\widetilde{\Delta} \otimes \mathrm{id})(V) = V_{13}V_{23}$.
- $A_v := \{(\mathrm{id} \otimes \omega)(V) : \ \omega \in B(H)_*\}$ *is norm dense in* A.

Then (A, Δ) *has a unique invariant mean.*

If we define a compact quantum group to be a quantum group in the sense of [5] with the underlying C* algebra being unital algebra, then it can be shown that the set-up of Theorem 0.5 accommodates this. Thus in particular, we have the existence of Haar state on such objects.

All the C*-Eberlein algebras as described in [3] are also like this. So in particular Theorem 0.5 can be taken as a quantum version of the classical result that Eberlein compactification of a locally compact group admits a unique invariant mean.

An interesting observation in this context is a reducing procedure, the proof being along the same line of Theorem 2.1 in [1]:

Theorem 0.6. *Let* (A, Δ) *be a compact semitopological quantum semigroup with an invariant mean. If the left kernel of the invariant mean say* $\mathcal{I}$*, is a two sided* C^* *ideal in* A*, then* $A/\mathcal{I}$ *can be given the structure of a compact semitopological quantum semigroup. Moreover,* $A/\mathcal{I}$ *will admit a faithful invariant mean.*

The hypothesis of Theorem 0.6 will always be satisfied if (A, Δ) is a compact quantum group. Moreover, Theorem 7.5 in [3] gives another instance when this is true. In fact an interesting question in context of this reduction is whether the reduced object becomes a compact quantum group. This is related to the decomposition result we stated in the beginning of this talk. It can be shown [3] that for a Kac algebra $\mathbb{G}$, the Eberlein compactification of $\mathbb{G}$ (which in particular is a compact semitopological quantum semigroup) has this property.

In an upcoming paper [4] we will discuss other situations when this is true.

This talk is based on the following joint works:

- B. Das and M. Daws, Quantum Eberlein compactifications and invariant means, to appear in *Indiana Univ. Maths. J.* arXiv: 1406.1109v1 [math F.A.]
- B. Das and C. Mrozinski, From compact semitopological quantum semigroup to compact quantum groups, in preparation.

References

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[2] R.B. Burckel, Weakly almost periodic functions on semigroups, *Gordon and Breach Science Publishers, New York*, (1970).

[3] B. Das and M. Daws, Quantum Eberlein compactifications and invariant means, to appear in *Indiana Univ. Maths. J.* arXiv: 1406.1109v1 [math F.A.]

[4] B. Das and C. Mrozinski, From compact semitopological quantum semigroup to compact quantum groups, in preparation.

[5] S.L. Woronowicz, From multiplicative unitaries to quantum groups, *Int. J. Math.* **7** no. 1 (1996) 127–149.