One-one correspondence between cocycles and generating functionals on quantum groups in presence of symmetry

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All inner products will be left conjugate linear and right linear.

Let Γ be a discrete group.

Definition 0.1. *A function $f : \Gamma \longrightarrow \mathbb{C}$ is called conditionally positive definite if for all $n \in I\!N$, $\{g_1, g_2, ...g_n\} \subset \Gamma$ and $\{\alpha_1, \alpha_2, ...\alpha_n\} \subset \mathbb{C}$ we have*

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j f(g_i^{-1}g_j) \geq 0 \quad (\textit{whenever } \sum_{i=1}^{n} \alpha_i = 0).$$

It is called normalized if $f(e) = 0$, where e is the identity of Γ.

Definition 0.2. *Let $U : \Gamma \longrightarrow B(H)$ be a unitary representation of Γ. A map $c : \Gamma \longrightarrow H$ is called a cocycle with respect to U if*

$$c(gh) = c(g) + U_g(c(h)) \quad (g, h \in \Gamma).$$

Let $\mathbb{C}\Gamma$ be the group algebra, which is thought of as the coefficient algebra of the compact quantum group $C^*(\Gamma)$ (the full group C*-algebra of Γ). For $g \in \Gamma$, we will denote the corresponding element of $\mathbb{C}\Gamma$ by λ_g. By definition we see that the set $\{\lambda_g : g \in \Gamma\}$ is a linear basis for $\mathbb{C}\Gamma$. Moreover $(\mathbb{C}\Gamma, \Delta)$ is a CQG-algebra, where Δ is the standard coproduct on the compact quantum group $C^*(\Gamma)$ given by $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in \Gamma$. Let $\epsilon : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ be the counit so that $\epsilon(\lambda_g) = 1$ for all $g \in \Gamma$.

Let us see how f and c look like at the level of group algebra.

(a) How f looks like on $\mathbb{C}\Gamma$:

Define a functional $\gamma : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ as follows:

$$\gamma(\lambda_g) := f(g) \quad \text{(and extend linearly).}$$

This is well-defined as $\{\lambda_g : g \in \Gamma\}$ is a basis for $\mathbb{C}\Gamma$. Take any element $x \in \mathbb{C}\Gamma$ and let $x := \sum_{k=1}^{m} \alpha_k \lambda_{g_k}$. We have $\epsilon(x) = \sum_{k=1}^{m} \alpha_k$ since $\epsilon(\lambda_g) = 1$ for all $g \in \Gamma$. Moreover we see that $\gamma(x^*x) = \sum_{i,j=1}^{m} \overline{\alpha_i}\alpha_j f(g_i^{-1}g_j)$, using the fact that $(\lambda_g)^* = \lambda_{g^{-1}}$.

Thus the fact that f is conditionally positive definite implies that $\gamma(x^*x) \geq 0$ whenever $\epsilon(x) = 0$. The fact that f is normalized implies that $\gamma(1) = 0$.

(b) How c looks like on $\mathbb{C}\Gamma$:

Define a linear map $\eta : \mathbb{C}\Gamma \longrightarrow H$ as follows:

$$\eta(\lambda_g) := c(g) \quad \text{(and extend linearly).}$$

As before η is a well-defined map. Let π be the unique unital $*$-homomorphism $\pi : \mathbb{C}\Gamma \longrightarrow B(H)$ such that $\pi(\lambda_g) = U_g$. The fact that c is a cocyle translates now to the following:

$$\eta(xy) = \eta(x)\epsilon(y) + \pi(x)(\eta(y)) \quad (x, y \in \mathbb{C}\Gamma).$$

The above considerations motivate the following two definitions:

Definition 0.3. *Let (A, Δ) be a CQG-algebra. A functional $\gamma : A \longrightarrow \mathbb{C}$ is called conditionally positive definite and normalized if*

$$\gamma(x^*x) \geq 0 \quad (\textit{whenever } \epsilon(x) = 0)$$

and $\gamma(1) = 0$.

Definition 0.4. *Let (A, Δ) be a CQG-algebra and $\pi : A \longrightarrow B(H)$ be a unital *-homomorphism. A map $\eta : A \longrightarrow H$ is called a cocycle with respect to π if*

$$\eta(xy) = \eta(x)\epsilon(y) + \pi(x)(\eta(y)) \quad (x, y \in A).$$

These objects are important and they arise in various branches of mathematics, e.g.

(a) A discrete countable group Γ has Haagerup approximation property if there exists a proper conditionally positive definite function on Γ, whereas it has Kazhdan property (T) (which is a strong negation of Haagerup approximation property) if for any unitary representation $U : \Gamma \longrightarrow B(H)$, all cocycles on Γ with respect to U are inner.

(b) Let G be a compact group. Let $(\mu_t)_{t\geq 0} \subset M(G)$ be a convolution semigroup of probability measures satisfying : $\mu_0 = \delta_e$, the point mass at the identity e and $\lim_{t\to 0^+} \mu_t(f) = f(e)$ for all $f \in C(G)$. Define

$$\gamma(f) := \lim_{t\to 0^+} \frac{f(e) - \mu_t(f)}{t} \quad \text{(whenever it exists)}.$$

It can be shown (e.g. using fundamental theorem of coalgebra) that $\mathrm{Pol}(G) \subset \mathrm{Dom}\gamma$, where $\mathrm{Pol}(G)$ denotes the trigonometric polynomial algebra of G.

Then one can show that $\gamma|_{\mathrm{Pol}(G)} : \mathrm{Pol}(G) \longrightarrow \mathbb{C}$ is a conditionally positive definite functional, with $\gamma(1) = 0$.

Let (A, Δ) be a CQG-algebra and $\gamma : A \longrightarrow \mathbb{C}$ be a conditionally positive definite functional such that $\gamma(1) = 0$. Such a functional will be called a generating functional.

Let $a \in A$ and $x := a - \epsilon(a)1$. Then $\epsilon(x) = 0$. Thus $\gamma(x^*x) \geq 0$. Thus we have $\gamma(a^*a) - \epsilon(a^*)\gamma(a) - \gamma(a^*)\epsilon(a) \geq 0$ for all $a \in A$. This immediately allows us to define an inner-product on A:

For $a, b \in A$ define $\langle a, b\rangle := \gamma(a^*b) - \epsilon(a^*)\gamma(b) - \gamma(a^*)\epsilon(b)$. Let $\mathcal{I} := \{a \in A : \langle a, a\rangle = 0\}$. Note that $\mathcal{I} \neq \phi$ as $1 \in \mathcal{I}$. Let $D := A/\mathcal{I}$. Thus D becomes a pre-Hilbert space. Let $H := \overline{D}$. Let $\eta : A \longrightarrow H$ denote the quotient map. Let $\mathcal{L}^\dagger(D)$ denotes the set of adjointable linear maps on D i.e. if $T \in \mathcal{L}^\dagger(D)$, then there exists a linear map $T^* : D \longrightarrow D$ such that $\langle \xi, T(\eta)\rangle = \langle T^*(\xi), \eta\rangle$

for all $\xi, \eta \in D$. Define $\pi : A \longrightarrow \mathcal{L}^{\dagger}(D)$ by $\pi(a)(\eta(b)) := \eta(ab) - \eta(a)\epsilon(b)$. Can check that π is well-defined, $\pi(1) = 1$, $\langle \xi, \pi(ab)(\eta)\rangle = \langle \pi(a)^*(\xi), \pi(b)(\eta)\rangle$ for all $\xi, \eta \in D$. (Lindsay-Skalski): Since (A, Δ) is a CQG-algebra, it is spanned by the matrix coeficients of all finite dimensional unitary representations. Let $u := ((u_{ij})) \in M_n(A)$ be a unitary corepresentation of A of dimension n. For $\xi \in D$ we have that

$$\|\pi(u_{ij})(\xi)\|^2 = \langle \xi, \pi(u_{ij}^* u_{ij})(\xi)\rangle \leq \langle \xi, \sum_{k=1}^{n} \pi(u_{kj}^* u_{kj})(\xi)\rangle = \|\xi\|^2,$$

since $\sum_{k=1}^{n} u_{kj}^* u_{kj} = 1$ and $\pi(1) = 1$. This shows that $\pi(A) \subset B(H)$. Thus $\pi : A \longrightarrow B(H)$ becomes a unital *-homomorphism. Obviously by the definition of η we see that η is a cocycle with respect to π and by the definition of the inner-product we see that

$$\langle \eta(a), \eta(b)\rangle = \gamma(a^*b) - \epsilon(a^*)\gamma(b) - \gamma(a^*)\epsilon(b) \quad (\forall\ a, b \in A). \tag{1}$$

Thus given a CQG-algebra (A, Δ) and a generating functional $\gamma : A \longrightarrow \mathbb{C}$, we can associate a unital *-representation $\pi : A \longrightarrow B(H)$, a cocyle $\eta : A \longrightarrow H$ with respect to π such that they are related by equation (1).

Question: Let (A, Δ) be a CQG-algebra, $\pi : A \longrightarrow B(H)$ be a unital *-representation and $\eta : A \longrightarrow H$ be a cocycle with respect to π. Can you find a generating functional $\gamma : A \longrightarrow \mathbb{C}$ such that it is related to η by equation (1)?

NO! e.g.:

Consider the 2-dimensional torus $\mathbb{T}^2$ and the CQG-algebra $(\mathrm{Pol}(\mathbb{T}^2), \Delta)$, where Δ is the standard coproduct on $C(\mathbb{T}^2)$. We have that

$$C(\mathbb{T}^2) = C^*\{U, V :\ UU^* = U^*U = VV^* = V^*V = 1,\ UV = VU\}.$$

The set $\{U^m V^n \ :\ \ m, n \in \mathbb{Z}\}$ is a basis for $\mathrm{Pol}(\mathbb{T}^2)$. Take $\pi := \epsilon$, the counit of $C(\mathbb{T}^2)$ and $\eta : \mathrm{Pol}(\mathbb{T}^2) \longrightarrow \mathbb{C}$ be defined by extending the rule $\eta(U) := r_u$, $\eta(U^*) := -r_u$, $\eta(V) := r_v$, $\eta(V^*) := -r_v$ on elements like $U^m V^n$, using the identity $\eta(xy) = \eta(x)\epsilon(y) + \epsilon(x)\eta(y)$. Moreover assume that $\overline{r_u} r_v \in \mathbb{C}/\mathbb{R}$.

Now suppose there is a generating functional $\gamma : \mathrm{Pol}(\mathbb{T}^2) \longrightarrow \mathbb{C}$ such that η is related to γ via equation (1). We have $\gamma(UV) = \gamma(VU)$. Now

$$\gamma(UV) = \gamma(U) + \gamma(V) + \langle \eta(U^*), \eta(V)\rangle$$

and

$$\gamma(VU) = \gamma(V) + \gamma(U) = \langle \eta(V^*), \eta(U)\rangle.$$

Thus equating both sides we have

$$\langle \eta(U^*), \eta(V)\rangle = \langle \eta(V^*), \eta(U)\rangle$$

or in other words

$$\langle \eta(U), \eta(V)\rangle = \langle \eta(V), \eta(U)\rangle$$

which implies that $\langle \eta(U), \eta(V)\rangle = \overline{r_u} r_v \in \mathbb{R}$ which is a contradiction.

Proposition 0.5 (Kyed & Vergnioux, 2011)**.** *Let (A, Δ) be a CQG-algebra, $\pi : A \longrightarrow B(H)$ be a unital *-representation and $\eta : A \longrightarrow H$ be a cocycle with respect to π. Let furthermore η satisfies*

$$\langle \eta(S(a)^*), \eta(S(b^*)) \rangle = \langle \eta(b), \eta(a) \rangle \quad (a, b \in A) \tag{2}$$

where S is the antipode. Then there exists a generating functional $\gamma : A \longrightarrow \mathbb{C}$ satisfying

- $\gamma \circ S = \gamma$.
- *Equation* (1) *holds.*

Conversely given a generating functional $\gamma : A \longrightarrow \mathbb{C}$ satisfying $\gamma \circ S = \gamma$, the cocycle η obtained from it satisfies condition (2)*. Moreover the functional γ is given by*

$$\gamma(a) = -\frac{1}{2} \langle \eta(S(a_{(1)})^*), \eta(a_{(2)}) \rangle \quad (a \in A).$$

We will generalize this result.

Definition 0.6. *Let (A, Δ) be a CQG-algebra. A linear map $\alpha : A \longrightarrow A$ is called an admissible bijection if we have*

1. *α is a homomorphism.*
2. *$\alpha((\alpha(x^*))^*) = x$ for all $x \in A$.*
3. *$(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha$.*
4. *$\mathrm{id} + \alpha$ is a bijection on A.*
5. *$\mathrm{id} \otimes \mathrm{id} + \alpha \otimes \alpha$ is a bijection on $A \otimes A$.*

Given such an admissible bijection, define $S_\alpha := S \circ \alpha$, which we call α-twisted antipode. Note that for $\alpha = \mathrm{id}$ $S_\alpha = S$ and for $\alpha = \tau_{\frac{i}{2}}$ where $(\tau_t)_{t \geq 0}$ is the scaling automorphism group of (A, Δ), we have that $S_\alpha = R$, the unitary antipode of (A, Δ).

Note that S_α satisfies the identities:

$$S_\alpha(a_{(1)})\alpha(a_{(2)}) = \epsilon(a)1 = \alpha(a_{(1)})S_\alpha(a_{(2)}) \quad (a \in A).$$

Compare this with the usual antipode identities:

$$S(a_{(1)})a_{(2)} = \epsilon(a)1 = a_{(1)}S(a_{(2)}) \quad (a \in A).$$

Theorem 0.7 (BD, Franz, Kula & Skalski)**.** *Let (A, Δ) be a CQG-algebra, $\pi : A \longrightarrow B(H)$ be a unital *-representation, $\eta : A \longrightarrow H$ be a cocycle with respect to π and α be an admissible bijection. Let furthermore η satisfies*

$$\langle \eta(S_\alpha(a)^*), \eta(S_\alpha(b^*)) \rangle = \langle \eta(b), \eta(a) \rangle \quad (a, b \in A) \tag{3}$$

Then there exists a generating functional $\gamma : A \longrightarrow \mathbb{C}$ satisfying

- $\gamma \circ S_\alpha = \gamma$.
- *Equation* (1) *holds.*

Conversely given a generating functional $\gamma : A \longrightarrow \mathbb{C}$ *satisfying* $\gamma \circ S_\alpha = \gamma$, *the cocycle* η *obtained from it satisfies condition* (3).

Taking $\alpha = \mathrm{id}$ gives the theorem of Kyed & Vergnioux. Taking $\alpha = \tau_{\frac{i}{2}}$ gives a new interesting result. It produces a KMS-symmetric generating functional, which complements Kyed & Vergnioux's result which produces a GNS-symmetric functional.

A few comments about our theorem:

The formula for γ coming out of our theorem is slightly complicated. It is as follows. Let $a \in A$. Then by property 4. of admissible bijection, there exists $c \in A$ such that $(\mathrm{id} + \alpha)(c) = a$. Then

$$\gamma(a) = -\langle \eta(S_\alpha(c_{(1)})^*), \eta(\alpha(c_{(2)}))\rangle.$$

How we guess such a formula:

Suppose there is such a γ for this η related by equation (1). Then applying γ to the identity $S_\alpha(a_{(1)})\alpha(a_{(2)}) = \epsilon(a)1$, using the fact that $\gamma(1) = 0$ and equation (1) we arrive exactly at this formula.

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