Quantum κ-Minkowski spacetime

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Warszawa 1 Oct. 2010

Aim:

Study 'noncommuting coordinates' of the κ-Minkowski spacetime by operator methods (as in Q.M.)

- irreducible representations
- κ-analogue of Weyl Operators, Heisenberg Group and Weyl calculus
- covariance
- uncertainty relations

CCR

In Q.M. momentum and position obey

$$[P, Q] = -iI.$$

A rep. can't be bounded; eg. Schrödinger rep on $L^2(\mathbb{R})$

$$P\psi(s) = -i\psi'(s), \quad Q\psi(s) = s\psi(s),$$

which is also (Von Neumann: unique) regular irrep., ie. (stronger) Weyl relations hold

$$e^{i\alpha P} e^{i\beta Q} = e^{i\alpha\beta} e^{i\beta Q} e^{i\alpha P}, \quad \alpha, \beta \in \mathbb{R}.$$

The Weyl operators:

$$W(\alpha, \beta) := e^{i\alpha P + i\beta Q} = e^{i\alpha\beta/2} e^{i\alpha P} e^{i\beta Q}$$

form a proj rep of abelian group $\mathbb{R}^2$

$$W(\alpha_1, \beta_1) \circ W(\alpha_2, \beta_2) = e^{i(\alpha_1\beta_2 - \alpha_2\beta_1)/2} W(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

or a unitary rep of the Heisenberg group $H = \mathbb{R}^3$, with

$$(\alpha_1, \beta_1, c_1)(\alpha_2, \beta_2, c_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, c_1 + c_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)/2).$$

Weyl quantisation

$W(\alpha,\beta)$ are the quantised "plane waves". Weyl quantisation of $f \in \widehat{L^1(\mathbb{R}^2)}$ is

$$f(P,Q) = (2\pi)^{-1} \int d\alpha\, d\beta\; \hat{f}(\alpha,\beta) W(\alpha,\beta),$$

where

$$\hat{f}(\alpha,\beta) = (2\pi)^{-1} \int dp\, dq\; f(p,q) e^{-i(\alpha p + \beta q)}.$$

Twisted star product [Groenewold, Moyal'40, Baker'60]

$$f(P,Q)\, g(P,Q) = (f \star g)(P,Q)$$

best studied as twisted convolution in Fourier space

$$\widehat{(f \star g)}(\alpha,\beta) = \int d\alpha' d\beta \hat{f}(\alpha',\beta') \hat{g}(\alpha-\alpha',\beta-\beta') e^{i(\alpha\beta'-\alpha'\beta)/2}$$

(by relating to the representation of the group algebra of the Heisenberg group).

If the operator $f(P,Q)$ is trace class (eg. for Schwarz f),

$$Tr f(P,Q) = \int dp\, dq\; f(p,q).$$

The κ-Minkowski spacetime $> 90's$ [Lukierski et.al., Majid-Ruegg, ..]

$$[q^0, q^j] = \frac{i}{\kappa} q^j, \quad [q^j, q^k] = 0, \qquad j, k = 1, \dots 4;\ (d),$$

studied as algebra. We fix $\kappa = 1$ in absolute units. First $d + 1 = 2$,

$$[T, X] = iX. \tag{1}$$

A *representation* of (1) is a pair of seladjoint operators (T, X) (on $\mathcal{H}$) satisfying (1).
It is *regular* if also (Weyl)

$$e^{i\alpha T} e^{i\beta X} = e^{i\beta e^{-\alpha} X} e^{i\alpha T}, \quad \alpha, \beta \in \mathbb{R}.$$

PROP. [Unterberger1984, Bertrand1997, Agostini2007, GG-BV2007] The regular irreps are

$$(T^{0,t}, X^{0,t}) = (t, 0),\ t \in \mathbb{R}; \quad \textit{trivial (onedim.)},$$

$$(T^+, X^+) = (P, e^{-Q}), \quad (T^-, X^-) = (P, -e^{-Q}); \quad \textit{nontrivial},$$

in terms of the Schröd. ops P, Q on $L^2(\mathbb{R})$ (but this is not QM!). $\diamond$
Notation for some reducible reps:

$$(T^0, X^0) = \int (T^{0,t}, X^{0,t}) dt = (Q, 0) \approx (P, 0),$$

$$(T^u, X^u) = (T^-, X^-) \oplus (T^0, X^0) \oplus (T^+, X^+); \quad \textit{universal.}$$

Proof. Trivial by Schur's lemma. Nontrivial (T, X) by reducing to CCR; from (2)

$$e^{i\alpha T} X e^{-i\alpha T} = e^{-\alpha} X. \tag{2}$$

Consequently, for f a (Borel) function of the spectrum of X,

$$f(e^{i\alpha T} X e^{-i\alpha T}) = e^{i\alpha T} f(X) e^{-i\alpha T} = f(e^{-\alpha} X R).$$

Now $0 \notin \text{spec}(X)$ and we may take $f(x) = e^{-i\beta \log |x|}$, obtaining

$$e^{i\alpha T} e^{i\beta(-\log |X|)} = e^{i\alpha\beta} e^{i\beta(-\log |X|)} e^{i\alpha T},$$

namely the Weyl relations for the CCR. By von Neumann uniqueness we may assume

$$T = P \qquad Q = -\log |X|.$$

Let $C = \text{sign}(X)$, which commutes strongly with Q. We rewrite again (2) in terms of $T = P$, $X = Ce^{-Q}$:

$$e^{i\alpha P} C e^{-Q} e^{-i\alpha P} = e^{-\alpha} C e^{-Q}$$

and, using $e^{-Q} e^{-i\alpha P} = e^{-\alpha} e^{-i\alpha P} e^{-Q}$ and strict positivity of e^{-Q},

$$e^{i\alpha P} C = C e^{i\alpha P},$$

namely C strongly commutes with P, too. By the generalised Schur's lemma, C is a multiple of the identity, $C = \pm I$, and $X = \pm e^{-Q}$. $\square$

As a matter of fact they could be written 60 years ago, by linking to certain group...

Given a rep. (T, X), the Weyl operators

$$W(\alpha, \beta) = e^{i(\alpha T + \beta X)}$$

must satisfy

$$W(\alpha, 0) = e^{i\alpha T}, \quad W(0, \beta) = e^{i\beta X},$$

$$W(\alpha, \beta)^{-1} = W(\alpha, \beta)^*,$$

$$W(\lambda\alpha, \lambda\beta)W(\lambda'\alpha, \lambda'\beta) = W((\lambda + \lambda')\alpha, (\lambda + \lambda')\beta).$$

Solution:

$$e^{i\alpha T + i\beta X} = e^{i\alpha T} e^{i\beta \frac{e^{\alpha} - 1}{\alpha} X}.$$

Explicitly, with $T_{\pm} = P, X_{\pm} = \pm e^{-Q}$,

$$e^{i\alpha T_{\pm} + i\beta X_{\pm}} \xi(s) = e^{i\alpha P \pm i\beta e^{-Q}} \xi(s) = e^{\pm i\beta \frac{1 - e^{-\alpha}}{\alpha} e^{-s}} \xi(s + \alpha), \quad \xi \in L^2(\mathbb{R}).$$

With $T_0 = Q, X_0 = 0$,

$$e^{i\alpha T_0 + \beta X_0} \xi(s) = e^{i\alpha s} \xi(s).$$

The Weyl operators are closed under composition (not up to a constant as for CCR) and form a (strongly continuous) unitary rep π of $H = \mathbb{R}^2$, with the group law:

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, w(\alpha_1 + \alpha_2, \alpha_1)e^{\alpha_2}\beta_1 + w(\alpha_1 + \alpha_2, \alpha_2)\beta_2),$$

where

$$w(\alpha, \alpha') = \frac{\alpha(e^{\alpha'} - 1)}{\alpha'(e^{\alpha} - 1)}.$$

Rem: $w(0,0) = 1, \quad w(\alpha, \alpha') > 0, \quad w(\alpha_1, \alpha_2)w(\alpha_2, \alpha_3) = w(\alpha_1, \alpha_3).$

Conversly, $\pi \rightsquigarrow W(\alpha, \beta)$ and both link to regular reps of the κ-Minkowski relations. The "κ-Heisenberg" group H is isomorphic to $A_o(1)$, the connected affine group of $\mathbb{R}$, known also as "$ax + b$" group; its irreps were classified by Gelfand-Naimark in 1947...

Rem: our $W(\alpha, \beta)$ do not depend on ordering of operators (e.g. "time-first", [Agostini]) and on BCH formula (cf. [Agostini et al, Gracia-Bondia et al, Kosiński et al]).

Quantisation à la Weyl

$W(\alpha, \beta)$ are the quantised “plane waves”. Following Weyl we define the quantisation

$$f(T, X) = \int d\alpha\, d\beta\, \hat{f}(\alpha, \beta) e^{i(\alpha T + \beta X)},$$

where

$$\hat{f}(\alpha, \beta) = \frac{1}{(2\pi)^2} \int dt\, dx\, f(t, x) e^{-i(\alpha t + \beta x)},$$

and (T, X) is the universal representation of the commutation relations.
Notation for the components:

$$f(T, X) = f(T^-, X^-) \oplus f(T^0, X^0) \oplus f(T^+, X^+).$$

Two “good” (and indispensable) features:
- $\bar{f}(T, X) = f(T, X)^\dagger$, so real f goes to selfadjoint operator,
- if f depends only on t, $f(T, X) = f(T)$ (the ordinary functional calculus),
- similarly for x.

Twisted Products

$$f(T,X)\,g(T,X) = (f\star g)(T,X)$$

again provides $\star$-product, given by some explicit integral; hard to study directly.
In Fourier space looks like a deformed convolution

$$\widehat{(f\star g)}(\alpha,\beta) =$$

$$\int d\alpha' d\beta'\; w(\alpha-\alpha',\alpha)\hat{f}(\alpha',\beta')\hat{g}(\alpha-\alpha', w(\alpha-\alpha',\alpha)\beta - w(\alpha-\alpha',\alpha')e^{\alpha-\alpha'}\beta').$$

but as for CCR, the best is to relate it to reps of the group algebra of H.

However our H is not unimodular; need to use the left Haar measure and modular function

$$d\mu(\alpha,\beta) = \frac{e^{\alpha}-1}{\alpha} d\alpha\, d\beta, \quad \Delta(\alpha,\beta) = e^{\alpha}.$$

Fortunately, we can cure this:

$$u : L^1(\mathbb{R}^2) \to L^1(H), \quad (u\varphi)(\alpha,\beta) = \frac{\alpha}{e^\alpha - 1}\varphi(\alpha,\beta),$$

is an isometric isomorphism to the group algebra $L^1(H)$, with the convolution product and involution (involving $d\mu$ and Δ).

Given a unitary representation W of the group H,

$$\pi(\varphi) := \int d\mu(\alpha,\beta)\varphi(\alpha,\beta)W(\alpha,\beta), \quad \varphi \in L^1(H),$$

defines a *-representation of the group algebra $L^1(H)$, which fulfills

$$f(T,X) = \pi(u\widehat{f}) \quad \text{but} \quad \neq \pi(\widehat{f}).$$

This makes $\star$ O.K.:

$$(f \star g)(T,X) = \pi(u\widehat{f})\pi(u\widehat{g}).$$

REM. If $f(\cdot,0)=g(\cdot,0)$, then $f(T_0,0)=g(T_0,0)$ (functional calculus).

Now fix (T^+,X^+). $R:=X^+$ looks like the quantisation of $x|_{\mathbb{R}_+}$.
If $f(\cdot,x)=g(\cdot,x), \quad x\in(0,\infty)$, then $f(T,R)=g(T,R)$.

Moreover, $f(T,R)$ 'appears' as a function of P,Q.
PROP.

$$(\gamma f)(t,x)=\int d\alpha\ e^{i\alpha t}\,\mathscr{F}_1 f\left(\alpha, e^{-x}(e^{\alpha/2}-e^{-\alpha/2})/\alpha\right)$$

fulfils

$$(\gamma f)(P,Q)=f(T,R),$$

where lhs. is the CCR Weyl quantisation.
Pf. Check that for $g=\gamma f$

$$K_f(s,u):=\mathscr{F}_1 f\left(u-s,\frac{e^{-s}-e^{-u}}{u-s}\right)=H_g(s,u):=\mathscr{F}_1 g\left(u-s,\frac{u+s}{2}\right).$$

CORR. If $f\in L^1(\mathbb{R}^2)\cap\widehat{L^1(\mathbb{R}^2)}$ and the operator $f(T,R)$ is trace class

$$Tr f(T,R)=\int_{\{r>0\}} dt\ dr\,\frac{1}{r}f(t,r).$$

We expect the classical limit (large κ) of each $\pm$ component is same as the small $\hbar$ limit of CCR up to restrictions to open halflines, which are separated from each other and the origin.
This is consistent with the spectrum of X being continuous $\mathbb{R}\backslash\{0\}$ and pure point $\{0\}$ for all κ, which suggests the classical limit of $d = 2$ κ-Mikowski is $\mathbb{R} \times \tilde{\mathbb{R}}$, where

$$\tilde{\mathbb{R}} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty).$$

What is the C*-algebra $\mathcal{A}$ of the (regular) κ-Minkowski relations, ie. $C^*(A^+(1))$?
(Recall C^*(CCR)$= \mathcal{K}$ (compacts)).

PROP. [D.P.2010, preprint]

$$\mathcal{A} = \mathcal{K} \oplus \mathcal{C}_\infty(\mathbb{R}) \oplus \mathcal{K}.$$

But there is a flaw in the prove that norm closure of image of $\pi^\pm$ is $\mathcal{K}$, and actually $\pi^\pm(f) \in \mathcal{K}$ for $f \in L^1(H)$ iff $\int f(a,b)\,db = 0$ [Khalil1974].
(In fact we can easily see that $\pi^\pm(f) \in HS$). So, now I try again:

PROP. (Almost as above) TFIES:

$$0 \to \mathcal{K} \oplus \mathcal{K} \to \mathcal{A} \to \mathcal{C}_\infty(\mathbb{R}) \to 0.$$

"Pf". In [Ziep1983] we find

$$0 \to \mathcal{K} \to \mathcal{A}^+ \to \mathcal{C}(S^1) \to 0.$$

The presence of $C(S^1) = C_\infty(\mathbb{R})^+$ is clear due to the unitalization $\mathcal{A}^+$ of $\mathcal{A}$ but $\mathcal{K}$ arises from a curious "$\mathcal{K} \oplus \mathcal{K} = \mathcal{K}$" in his proof (which also invalidates the proof based on [BrownDouglasFillmore] that the extension is nontrivial).

[Zep1975, Zep1983] studies $C^*(A(1))$ (with $a \in \mathbb{R} \setminus 0$), which has two $\mathbb{R}$-lines of trivial regular irreps $\pi^{t,\pm}$ and just one nontrivial π, and gets

$$\mathcal{K} \to \mathcal{A}^+ \to \mathcal{C}(\infty) = \mathcal{C}(''8'') = \mathcal{C}_\infty(\mathbb{R} \sqcup \mathbb{R})^+ .$$

For that he shows that π is faithful and that TFAE:

i) Khalil condition on f,

ii) $f \in \cap_{t,\pm} ker(\pi^{t,\pm}) = \bigoplus_{t,\pm} ker(\pi^{t,\pm}) = ker(\pi^0)$ and

iii) $\pi(f) \in \mathcal{K}$.

Now $d+1$ dimension: T, X_j selfadjoint &

$$[T, X_j] = iX_j, \quad [X_j, X_k] = 0,$$

regular form:

$$e^{i\alpha T} e^{i\boldsymbol{\beta X}} = e^{ie^{-\alpha}\boldsymbol{\beta X}} e^{i\alpha T}, \qquad \alpha \in \mathbb{R}, \boldsymbol{\beta} \in \mathbb{R}^d,$$
$$[e^{i\boldsymbol{\beta X}}, e^{i\boldsymbol{\beta' X}}] = 0, \qquad \boldsymbol{\beta}, \boldsymbol{\beta'} \in \mathbb{R}^d,$$

where $\boldsymbol{\beta} = (\beta_j), \quad \boldsymbol{X} = (X_j), \quad \boldsymbol{\beta X} = \left(\sum_j \beta_j X_j\right)$.

PROP. Irreps up to unitary equivalence:
trivial: $\forall\ t \in \mathbb{R},\ T^{(0,t)} = t,\ \boldsymbol{X}^{(0,t)} = \boldsymbol{0}$,
nontrivial: $\forall\ \boldsymbol{c} = (c_j) \in S^{d-1} \subset \mathbb{R}^d, \quad T^{(\boldsymbol{c})} = P,\ X_j^{(\boldsymbol{c})} = c_j e^{-Q}$.
Note $S^0 = \{\pm 1\}$.

Notation: We need as before the direct integral $T^{(0)} = P,\ \boldsymbol{X}^{(0)} = \boldsymbol{0}$ and the universal rep $T^u = I \otimes P,\ X_j^u = C_j^u R^u = (c_j \cdot) \otimes e^{-Q}$, on the Hilbert space

$$\mathfrak{H}^u = L^2(S^{d-1} \sqcup \{\boldsymbol{0}\}, d\mu(\boldsymbol{c})) \otimes L^2(\mathbb{R}),$$

where $d\mu(\boldsymbol{c})$ is the ususal measure on $S^{(d-1)} + \delta_{\boldsymbol{0}}$.

Proof. With $R^2 = (X_1^2 + \ldots + X_d^2)^{**}$, let $E = \chi_{(0,\infty)}(R)$. By (2)

$$e^{i\alpha T} E e^{-i\alpha T} = \chi_{(0,\infty)}(e^{-\alpha} R) = E,$$

so that E commutes strongly both with R and T. Hence, by the generalised Schur's lemma, either $E = I$ or $E = 0$. If $E = 0$, the representation is trivial: T is a real number and $\boldsymbol{X} = \boldsymbol{C} = 0$. Otherwise R is invertible, and the bounded operators

$$C_k = X_k R^{-1}, \quad k = 1, \ldots, d,$$

strongly commute pairwise and with R. By (2) and the properties of functional calculus,

$$e^{i\alpha T} C_k e^{-i\alpha T} = e^{i\alpha T} X_k e^{-i\alpha T} e^{i\alpha T} R^{-1} e^{-i\alpha T} = (e^{-\alpha} X_k)(e^{-\alpha} R)^{-1} = C_k,$$

so that C_k strongly commutes with T, too. Hence by Schur's lemma $C_k = c_k I$ for some c_k, and $X_k = c_k R$.

Since the representation is not trivial, there is some $c_j \neq 0$: thus strong relation written for $\boldsymbol{\beta} = \beta \boldsymbol{e}_j$ is like in $d + 1 = 2$. It follows that $T = P$, $R = e^{-Q}$, by positivity of R and proposition in $d + 1 = 2$.

With the unitary operator $U = e^{iP \log |\boldsymbol{c}|}$ can rescale $Ue^{-Q}U^* = (1/|\boldsymbol{c}|)e^{-Q}$, so can assume $|\boldsymbol{c}| = 1$, i.e. $\boldsymbol{c} \in S^{d-1}$. $\square$

Given rep. $(T, \boldsymbol{X})$ the Weyl op. are

$$e^{i(\alpha T+\boldsymbol{\beta X})} = e^{i\alpha T} e^{i\frac{e^{\alpha}-1}{\alpha}\boldsymbol{\beta X}}, \quad (\alpha, \boldsymbol{\beta}) \in \mathbb{R} \times \mathbb{R}^d.$$

In particular,

$$e^{i(\alpha T^u+\boldsymbol{\beta X}^u)} = e^{i\alpha P+(\boldsymbol{\beta c}\cdot)\, e^{-Q}} = e^{i\alpha P} e^{i\frac{e^{\alpha}-1}{\alpha}(\boldsymbol{\beta c}\cdot)\, e^{-Q}}.$$

The quantisation of the function $f \in L^1(\mathbb{R}^{d+1}) \cap \widehat{L^1(\mathbb{R}^{d+1})}$ is the operator

$$f(T^u, \boldsymbol{X}^u) = \frac{1}{\sqrt{(2\pi)^{d+1}}} \int d\alpha\, d\boldsymbol{\beta}\, \hat{f}(\alpha, \boldsymbol{\beta}) e^{i(\alpha T^u+\boldsymbol{\beta X^u})}.$$

We have also (unbounded) trace functional

$$\tau_c(f) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dt\, d\boldsymbol{x} |\boldsymbol{x}|^{-d} f(t, \boldsymbol{x}),$$

whenever the rhs. exists.

Covariance I (automorphisms):

time translations, $T \mapsto T + a$, are implemented in $(T^{(\boldsymbol{c})}, X^{(\boldsymbol{c})})$ and $(T^{(0)}, X^{(0)})$ by e^{iaQ} (but not in $(T^{(0,t)}, X^{(0,t)})$),

space rotations, $r \in O(\mathbb{R}^d)$, are trivial in $(T^{(0,t)} X^{(0,t)})$ and are implemented in (T^u, X^u) by pull back of r^{-1} (but not in $(T^{(\boldsymbol{c})}, X^{(\boldsymbol{c})})$),

space dilations, $e^s \in \mathbb{R}_+$, are trivial in $(T^{(0,t)} X^{(0,t)})$ and are implemented in $(T^{(\boldsymbol{c})}, X^{(\boldsymbol{c})})$ by e^{ihP}. But in $(T^{(\boldsymbol{c})}, X^{(\boldsymbol{c})})$ the commuting (!) time translations $\times$ space dilations are implemented only *projectively*, and an amplification $T = P \otimes I$, $X_j = c_j e^{-Q} \otimes e^{-Q}$ on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$, is needed to implement them not projectively - by $e^{iaQ} \otimes e^{ihP}$.

Accordingly in (T^u, X^u).

Covariance II. Of course (due to its origins) there is a deformed Poincare covariance under the deformed (twisted) quantum Poincare algebra .
It is interesting to implemented it (in the reps above ?) upcoming [A. Sitarz, B. Durhuus]...

Covariance III. Cf. [P] for a relation between two other well known models

$$[Moyal] \underset{reduction}{\overset{\text{extension}}{\Longleftrightarrow}} [DFR]$$

and for the result that the deformed covariance (deformed Lorentz action, same product) is just usual (form) covariance (usual Lorentz action, Lorentz transformed product) [P2009] (the former is also equivalent to the extended model, with the covariance broken by dismissing lot of admissible localisation states.)

This works also here. We introduced in [DGP2010] the Lorentz covariant (extension of the) κ-Minkowski spacetime ($d + 1 = 4$) where κ is vieved as time-component of a 4-vector of (central) generators.

Recently we constructed a (minimal) Poincare covariantization of of kappa Minkowski: generators $X^\mu, V^\mu, A^\mu, \mu = 0, 1, \ldots d$ & (at most) quadratic relations

$$[X^\mu, X^\nu] = i\Big(V^\mu (X - A)^\nu - V^\nu (X - A)^\mu\Big), \tag{4a}$$

$$[X^\mu, V^\nu] = [X^\mu, A^\nu] = 0, \tag{4b}$$

$$[A^\mu, V^\nu] = [A^\mu, A^\nu] = [V^\mu, V^\nu] = 0, \tag{4c}$$

$$V_\mu V^\mu = I. \tag{4d}$$

More precisely, we construct selfadjoint operators fulfilling (1a-d) strongly (later) together with a unitary representation U of the full Poincaré group $\mathscr{P}$ such that

$$U(\Lambda, a)^{-1} X^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu X^\nu + a^\mu I, \tag{5a}$$

$$U(\Lambda, a)^{-1} V^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu V^\nu, \tag{5b}$$

$$U(\Lambda, a)^{-1} A^\mu U(\Lambda, a) = \Lambda^\mu{}_\nu A^\nu + a^\mu I \tag{5c}$$

How?

The Covariant Coordinates

Now, for simplicity, specify $d+1=4$ and use relativistic notation and conventions.
Consider first an irreducible representation X^μ, V^μ, A^μ of (4).
By the Schur's lemma, $V^\mu = v^\mu I, A^\mu = a^\mu I$ for some real 4-vectors v, a, so that

$$[X^\mu, X^\nu] = i(v^\mu(X^\nu - a^\nu I) - v^\nu(X^\mu - a^\mu I)) \tag{6}$$

(kind of a combination of a"canonical" a "Lie type" contribution) and

$$v_\mu v^\mu = 1.$$

The special case $a = 0, v = v_{(0)}$, where

$$v_{(0)} = (1, 0, 0, 0)$$

is just the usual κ-Minkowski relations:

$$[X^0_{(0)}, X^j_{(0)}] = iX^j_{(0)}, \quad [X^j_{(0)}, X^k_{(0)}] = 0. \tag{7}$$

For every $(\Lambda, a) \in \mathscr{P}$, the operators

$$X^\mu = \Lambda^\mu{}_\nu X^\nu_{(0)} + a^\mu I$$

fulfill (6) with $v = \Lambda v_{(0)}$.
We can so obtain an irrep for any pair $(v, a) \in H \times \mathbb{R}^4$, where $H = \mathscr{L} v_{(0)}$ is the orbit of $v_{(0)}$ under the full Lorentz group $\mathscr{L} = O(1, 3)$; it is a two sheeted hyperboloid.

Taking a direct integral over the Haar measure $d(\Lambda, a)$ of $\mathscr{P}$ of all the irreps above, it is easy (see e.g. [DFR], [DGP]) to construct selfadjoint operators X^μ, V^μ, A^μ and a unitary representation U of $\mathscr{P}$, fulfilling (4-5).
The result of the construction is equivalent to the following covariant representation.
Consider the Hilbert space of $\mathfrak{H}_{(0)}$-valued functions $\psi(\Lambda, a)$, with scalar product

$$(\psi, \psi') = \int d(\Lambda, a)(\psi(\Lambda, a), \psi'(\Lambda, a))_{(0)}.$$

Then set

$$(X^\mu \psi)(\Lambda, a) = (\Lambda X_{(0)} + aI)^\mu \, \psi(\Lambda, a), \tag{8a}$$

$$(V^\mu \psi)(\Lambda, a) = (\Lambda v_{(0)})^\mu \, \psi(\Lambda, a), \tag{8b}$$

$$A^\mu \psi(\Lambda, a) = a^\mu \, \psi(\Lambda, a), \tag{8c}$$

$$U(M, b)\psi(\Lambda, a) = \psi((M, b)^{-1}(\Lambda, a)). \tag{8d}$$

Note that U is a strongly continuous representation of $\mathscr{P}$ and we may define momentum operators by setting $e^{ia^\mu P_\mu} = U(\mathbb{I}, a)$; they fulfil the commutation relations

$$[P^\mu, P^\nu] = 0, \quad [P^\mu, V^\nu] = 0, \tag{9a}$$

$$[P^\mu, A^\nu] = [P^\mu, X^\nu] = ig^{\mu\nu} I. \tag{9b}$$

Above, the first of (9a) means abelian translations; and others express the effect of infinitesimal translations. Analogously one can define the generators of infinitesimal Lorentz transformations, and write down all the remaining commutation relations.

Weyl Symbols

will be functions $f = f(v, a; x)$, where $x \in \mathbb{R}^4$, $\quad (v, a) \in H \times \mathbb{R}^4$ (surviving the classical limit as extra dimensions).
Again the Weyl ops e^{ikX} govern the quantisation

$$f(V, A; X) = \frac{1}{(2\pi)^2} \int dk \; \hat{f}(V, A; k) e^{ikX},$$

where

$$\hat{f}(V, A; k) = \frac{1}{(2\pi)^2} \int dx \; f(V, A; x) e^{-ikx}.$$

and the replacement of v^μ, a^μ by the operators V^μ, A^μ respectively is in the usual sense of functions of pairwise commuting operators. This intertwines operator adjunction and pointwise conjugation:

$$\bar{f}(V, A; X) = f(V, A; X)^*.$$

One may obtain the symbolic calculus with the star product

$$(f \star g)(V, A; X) = f(V, A; X) g(V, A; X).$$

To find the Weyl ops and their composition explicitly (the latter suffices) consider the irreducible case when $V = vI, A = aI$, ie. $X = \Lambda X_{(0)} + aI$, where the Lorentz matrix s.t. $\Lambda v^{(0)} = v$ is

$$\Lambda = \left(\begin{array}{c|c} v^0 & \vec{v} \\ \hline \vec{v}^t & \mathbb{I} + \frac{\vec{v}^t\vec{v}}{1+v^0} \end{array} \right).$$

(Here regard $\vec{v}, \vec{h}, k, \vec{0}$ as row 3-vectors; and $\vec{v}^t, \vec{h}^t, \vec{k}^t, \vec{0}^t$ as column vectors).
Write now the composition of Weyl operators (p.13) for the original κ Mink (7)

$$e^{ih_\mu X^\mu_{(0)}} e^{ik_\mu X^\mu_{(0)}} = e^{i\phi_\mu(h,k) X^\mu_{(0)}} \tag{10}$$

for any $h, k \in \mathbb{R}^4$, where (cf. $d + 1 = 2$)

$$\phi^0(h,k) = h^0 + k^0, \tag{11a}$$

$$\vec{\phi}(h,k) = -w(h^0 + k^0, h^0) e^{ik_0} \vec{h} - w(h^0 + k^0, k^0) \vec{k}. \tag{11b}$$

By relativistic invariance of scalar product,

$$e^{ikX} = e^{ika} e^{i(\Lambda^{-1}k)X_{(0)}},$$

hence by (7,10) we have

$$e^{ihX} e^{ikX} = e^{i(h+k)a}\, e^{i\phi(\Lambda^{-1}h,\Lambda^{-1}k)X_{(0)}}$$

which can be rearranged again in terms of X as

$$e^{ihX} e^{ikX} = e^{i(h+k-\varphi(h,k;v))a}\, e^{i\varphi(h,k;v)X},$$

where

$$\varphi(h,k;v) = \Lambda\phi(\Lambda^{-1}h, \Lambda^{-1}k)).$$

Substituting the expression for ϕ, we compute

$$\varphi^0(h,k;v) = w((h{+}k)v, hv)e^{ikv}(v^0\vec{h}{-}h^0\vec{v})\vec{v}^T + w((h{+}k)v, kv)(v^0\vec{k}{-}k^0\vec{v})\vec{v}^T$$
$$+(h{+}k)v\, v^0,$$

$$\vec{\varphi}(h,k;v) = w((h{+}k)v, hv)e^{ikv}(hv\,\vec{v}{-}\vec{h}) + w((h{+}k)v, kv)(kv\,\vec{v}{-}\vec{k}v) - (h{+}k)v\,\vec{v}.$$

(as expected, φ does not depend on the choice of Λ, provided that $\Lambda v_{(0)} = v$).

Finally we obtain the regular (Weyl) form of our full model relations

$$e^{ihX}e^{ikX} = e^{i(h_\mu + k_\mu - \varphi_\mu(h,k;V))A^\mu}\, e^{i\varphi_\mu(h,k;V)X^\mu}, \qquad (12)$$

where v_μ is replaced by the (commuting) operators V_μ in φ.

Uncertainty Relations. First $d+1=2$.

$$\Delta_\omega(T)\Delta_\omega(X) \geqslant \frac{1}{2}\omega(|[T,X]|) = \frac{1}{2}\omega(|X|)$$

for a state ω with $\omega(X)$ small do not exclude small product of uncertainties.
E.g. for any ω pure and supported at 0, T, X have null uncertainty (cheap).
Other states localised close to 0? Let $\varepsilon, \eta > 0$.
There always is $\xi^\varepsilon \in \mathscr{D}(P)$ derivable and with compact support such that $\Delta_{\xi^\varepsilon}(P) < \varepsilon$.
Same for the state shifted by λ, $\xi^\varepsilon_\lambda(s) = (e^{-i\lambda P}\xi^\varepsilon)(s) = \xi^\varepsilon(s-\lambda)$.
But $\xi^\varepsilon_\lambda \in \mathscr{D}(e^{-Q})$ and due to the compactness of the support,

$$\lim_{\lambda\to\infty} \Delta_{\xi^\varepsilon_\lambda}(e^{-Q}) = 0.$$

In particular, there is a λ_η such that $\Delta_{\xi^\varepsilon_{\lambda_\eta}}(e^{-Q}) < \eta$. We found a state not belonging to the trivial component, and such that

$$\Delta(T) < \varepsilon, \quad \Delta(R) < \eta.$$

Message: no limit on the localisation precision of all the spacetime coordinates, at least in the region close to the space origin (asymptotically classical).
This contrasts the standard motivations for spacetime quantisation, namely to prevent the formation of closed horizons as an effect of localisation *alone* (see [DFR]).

Noncommutativity at large

For a state localised at distance $\omega(R) = L$ from the origin, rewrite the uncertainty relations (with the Planck mass $\kappa \sim 10^{35}/m$)

$$L \lesssim 2\kappa\, c\Delta T \Delta R.$$

The usual scale of strong interaction physics is $\Delta R \sim c\Delta T \ll 10^{-19}m$, so

$$L \ll 10^{-3}m,$$

which is the peak nominal beam size at LHC...

If LHC was placed somewhere else on the Earth (diameter $L \sim 10^{7}m$), it would be insensitive to this, provided κ should be $> 10^{45}/m$.

The usual atomic physics scales $\ell = 1$ Å $= 10^{-10}m$ (diameter of the classical electron orbit in the Hydrogen atom, and $\tau = 137\pi\ell$ (the period of the orbit) need

$$L \ll \kappa c\tau\ell \sim 10^{17}m \sim 10 \text{ light–years}.$$

But α-Centauri is already five light–years far.

Conclusions

On the mathematical side, we found an explicit quantisation prescription of κ-Minkowski, which realises precisely the underlying relations, and more...

Instead regarding the physical interpretation, we find some problematic features:

- The main motivation for spacetime quantisation, namely to prevent arbitrarily precise localisation (which could lead to horizon formation) is lost for this model.

- On the contrary, the noncommutativity grows dangerously at large distances.

- No Lorentz and translation covariance; hovewer it is restored in extended model, in which two dimensionful parameters κ, c coexist;
ie. a universal length may well exist in a Poincaré covariant setting (and so in DSR).

- Despite these problems, no obstacles to start a QFT, i.e. the 2nd (or '3rd') quantisation...

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