Deformation quantization of Hamiltonian actions in Poisson geometry

Chiara Esposito

Universitat Autonoma de Barcelona

February 18, 2013

Outline

Canonical action

Definition

Let G be a Lie group acting on a Poisson manifold (M, π). The action $\Phi : G \times M \to M$ is said canonical if

$$\Phi_g^*\{f, h\} = \{\Phi_g^* f, \Phi_g^* h\} \qquad \forall f, h \in C^\infty(M)$$

What is Hamiltonian action?

Momentum map

A momentum map is a tool associated with a canonical action of a Lie group on a Poisson manifold, used to construct conserved quantities for the action.

Momentum map

A momentum map is a tool associated with a canonical action of a Lie group on a Poisson manifold, used to construct conserved quantities for the action.

Supposing that exist a linear map $H : \mathfrak{g} \to C^\infty(M)$ such that

$$\xi_M = \{H_\xi, \cdot\} = X_{H_\xi}$$

the map $\boldsymbol{\mu} : M \to \mathfrak{g}^*$ defined by

Momentum map

A momentum map is a tool associated with a canonical action of a Lie group on a Poisson manifold, used to construct conserved quantities for the action.

Supposing that exist a linear map $H : \mathfrak{g} \to C^\infty(M)$ such that

$$\xi_M = \{H_\xi, \cdot\} = X_{H_\xi}$$

the map $\boldsymbol{\mu} : M \to \mathfrak{g}^*$ defined by

$$H_\xi(m) = \langle \boldsymbol{\mu}(m), \xi \rangle$$

is the momentum map for the canonical action.

Hamiltonian action

A momentum map $\boldsymbol{\mu} : M \to \mathfrak{g}^*$ is equivariant if the correspondent $H : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism.

Theorem

A canonical action is Hamiltonian if and only if there is a Lie algebra homomorphism $\psi : \mathfrak{g} \to C^\infty(M)$ such that $X_{\psi(\xi)} = \xi_M$ for all $\xi \in \mathfrak{g}$. If ψ exists, an equivariant momentum map $\boldsymbol{\mu}$ is determined by $H = \psi$. Conversely, if $\boldsymbol{\mu}$ is equivariant, we can take $\psi = H$.

Symplectic Reduction

Marsden and Weinstein idea: we can reduce the size of the phase space by taking advantage of the momentum map and the invariance of the system under the given symmetry group.

Symplectic Reduction

Marsden and Weinstein idea: we can reduce the size of the phase space by taking advantage of the momentum map and the invariance of the system under the given symmetry group.

Theorem (Marsden-Weinstein Reduction)

Let $\Phi : G \times M \to M$ be a Hamiltonian action of the Lie group G on the Poisson manifold (M, π) with momentum map $\boldsymbol{\mu} : M \to \mathfrak{g}^*$. Let $u \in \mathfrak{g}^*$ be a regular value of $\boldsymbol{\mu}$ and suppose that G_u acts freely and properly on the manifold $\boldsymbol{\mu}^{-1}(u)$. Then there is a Poisson structure π_u on the reduced space $M//G := \boldsymbol{\mu}^{-1}(u)/G_u$.

Ingredients

What is a Hamiltonian action in this context?

Ingredients:

- Poisson Lie groups
- Lie bialgebras

Poisson Lie groups

Definition

A Poisson Lie group (G, π) is a Lie group equipped with a Poisson structure π which preserves multiplication and inverse.

Example

$\pi = 0$ is obviously multiplicative, hence any Lie group G with the trivial Poisson structure is a Poisson Lie group.

Lie bialgebra

Definition

A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ such that

- ${}^t\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$, and
- δ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$

$$ad_\xi(\delta(\eta)) - ad_\eta(\delta(\xi)) - \delta([\xi, \eta]) = 0$$

Dual Poisson Lie group

Theorem

If (G, π_G) is a Poisson Lie group, then the linearization of π_G at e defines a Lie algebra structure on $\mathfrak{g}^*$. Conversely, if G is connected and simply connected, then every Lie bialgebra $(\mathfrak{g}, \delta)$ defines a unique multiplicative Poisson structure π_G on G.

This implies that G^* is also a Poisson Lie group, called dual.

Poisson action

Definition

The action of (G, π_G) on (M, π) is called Poisson action if the map $\Phi : G \times M \to M$ is Poisson, where $G \times M$ is a Poisson manifold with structure $\pi_G \oplus \pi$.

Generalization of canonical action! If $\pi_G = 0$, the action is Poisson if and only if it preserves π.

Momentum map

Definition (Lu)

A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\xi_M = \pi^\sharp(\boldsymbol{\mu}^*(\theta_\xi))$$

where θ_ξ is the left invariant 1-form on G^* defined by the element $\xi \in \mathfrak{g} = (T_e G^*)^*$ and $\boldsymbol{\mu}^*$ is the cotangent lift $T^*G^* \to T^*M$.

Momentum map

Definition (Lu)

A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\boldsymbol{\mu} : M \to G^*$ such that

$$\xi_M = \pi^\sharp(\boldsymbol{\mu}^*(\theta_\xi))$$

where θ_ξ is the left invariant 1-form on G^* defined by the element $\xi \in \mathfrak{g} = (T_e G^*)^*$ and $\boldsymbol{\mu}^*$ is the cotangent lift $T^*G^* \to T^*M$.

A Hamiltonian action is a Poisson action induced by an equivariant momentum map.

Infinitesimal momentum map

Let's focus on the map $\alpha : \mathfrak{g} \to \Omega^1(M)$

Definition

Let M be a Poisson manifold and G a Poisson Lie group. An infinitesimal momentum map is a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\ ,\ ]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [\ ,\ ]_\pi).$$

Poisson Reduction

Theorem

Let $\Phi : G \times M \to M$ *be a Hamiltonian action with momentum map* $\boldsymbol{\mu} : M \to G^*$ *and* $u \in G^*$ *a regular value of* $\boldsymbol{\mu}$*. The Poisson reduction of* (M, G) *is the quotient*

$$M//G := \boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$$

$M//G$ *inherits a Poisson structure from* M*.*

Poisson Reduction

Theorem

Let $\Phi : G \times M \to M$ be a Hamiltonian action with momentum map $\boldsymbol{\mu} : M \to G^*$ and $u \in G^*$ a regular value of $\boldsymbol{\mu}$. The Poisson reduction of (M, G) is the quotient

$$M//G := \boldsymbol{\mu}^{-1}(\mathcal{O}_u)/G$$

$M//G$ inherits a Poisson structure from M.

Example

Suppose that $\pi_G = 0$. Then

$$C^\infty(M//G) \simeq (C^\infty(M)/\mathcal{I})^G$$

Deformation quantization approach

Goal: quantize Hamiltonian actions and Poisson reduction.

Steps:

1. Quantize a Poisson action
2. Quantize Momentum map
3. Quantize Poisson reduction

Quantization of Poisson manifold

Given (M, π) we define

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} \hbar^n P_n(f, g)$$

where

$$P_1(f, g) - P_1(g, f) = \{f, g\}$$

Quantization of Lie bialgebra

Given $(\mathfrak{g}, \delta)$ we associate the Hopf algebra $(\mathcal{U}(\mathfrak{g}), \Delta)$, where

$$\Delta X = X \otimes 1 + 1 \otimes X$$

Quantum group $(\mathcal{U}_\hbar(\mathfrak{g}),\ \Delta_\hbar,\ [\cdot,\cdot]_\hbar)$

$$\Delta_\hbar = \Delta + \sum_{n=1}^{\infty} \hbar^n \Delta_n$$

Quantum action

How can we define a quantum action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{A}_\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action

Quantum action

How can we define a quantum action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{A}_\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action

Definition

The quantum action is a linear map

$$\Phi_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to End\ \mathcal{A}_\hbar : \xi \mapsto \Phi_\hbar(\xi)(f)$$

such that

1. Hopf algebra action
2. Algebra homomorphism

Quantum Hamiltonian action

1. Quantum momentum map which, as in the classical case, factorizes the quantum action
2. $\hbar \to 0$ classical momentum map

Non commutative forms

The non-commutative analogue of the de Rham complex is $(\Omega(\mathcal{A}_\hbar), d)$ with the universal derivation

$$d : \mathcal{A}_\hbar \to \Omega(\mathcal{A}_\hbar)$$

Quantum momentum map

The map

$$adb \longmapsto a[b, \cdot\,]_*$$

induces a non commutative product on $\Omega(\mathcal{A}_\hbar)$ and natural morphism of differential graded algebras

$$\Omega^1(\mathcal{A}_\hbar) \longrightarrow C^1(\mathcal{A}_\hbar, \mathcal{A}_\hbar)$$

This induces a first definition of the momentum map.

Quantum momentum map

Definition

A quantum momentum map is defined to be a linear map

$$\boldsymbol{\mu}_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \to \Omega^1(\mathcal{A}_\hbar) : \xi \mapsto \sum_i a^i_\xi db^i_\xi .$$

Such $\boldsymbol{\mu}_\hbar$ defines an action of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{A}_\hbar$ via the map

$$\Omega^1(\mathcal{A}_\hbar) \to C^1(\mathcal{A}_\hbar, \mathcal{A}_\hbar).$$

Extension

Definition

A quantum momentum map is defined to be a linear map

$$\boldsymbol{\mu}_\hbar : T(\mathcal{U}_\hbar(\mathfrak{g})[1]) \to \Omega^\bullet(\mathcal{A}_\hbar) : \xi_1 \otimes \cdots \otimes \xi_n \mapsto a_1 db_1 \otimes \cdots \otimes a_n db_n$$

such that

$$\Phi_\hbar(\xi_1 \otimes \cdots \otimes \xi_n)(f_1, \ldots, f_n) = \frac{1}{\hbar^n} a_1[b_1, f_1] \ldots a_n[b_n, f_n]$$

Quantum Reduction

Definition

Le $\mathcal{I}_\hbar$ be the left ideal of $\mathcal{A}_\hbar$ generated by $\boldsymbol{\mu}_\hbar$. The action of $\mathcal{U}_\hbar(\mathfrak{g})$ descends to an action on $\mathcal{A}_\hbar/\mathcal{I}_\hbar$ and we define the reduced algebra by

$$\mathcal{A}_\hbar^{red} = (\mathcal{A}_\hbar/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathfrak{g})}$$

Hopf algebra action condition

Assume that ξ acts by

$$\Phi_\hbar(\xi) = \frac{1}{\hbar} a[b, \cdot\,]$$

for some $a, b \in C^\infty_\hbar(M)$. Note that $a \neq 0$ as soon as ξ is not killed by the cocycle δ.

Hopf algebra action $\Longrightarrow \Phi_\hbar(\eta) = \frac{1}{\hbar} a[a^{-1}, \cdot]$

$$\Delta_\hbar(\xi) = \xi \otimes 1 - \hbar\, \eta \otimes \xi + 1 \otimes \xi$$
$$\Delta_\hbar(\eta) = \eta \otimes 1 - \hbar\, \eta \otimes \eta + 1 \otimes \eta$$

Algebra homomorphism condition

We calculate the bracket of generators to get the deformed algebra structure of $\mathfrak{g}$:

$$[\Phi_\hbar(\xi), \Phi_\hbar(\eta)]\, f = \frac{1}{\hbar^2}(a[b, a[a^{-1}, f]] - a[a^{-1}, a[b, f]]) \\ = a[b, a][a^{-1}, f] + a^2[[b, a^{-1}], f].$$

Imposing that $\Phi_\hbar$ is a Lie algebra homomorphism we obtain different algebra structures that we discuss case by case.

Two dimensions: $[a, b] = 0$

Consider the Lie bialgebra $\mathfrak{g} = \mathbb{R}^2$ with generators ξ, η and a deformation quantization $C^\infty_\hbar(M)$ of a Poisson manifold M.

Algebra homomorphism $\Longrightarrow \mathcal{U}_\hbar(\mathbb{R}^2)$ generated by $[\xi, \eta] = 0$

deformation quantization of

Abelian Lie bialgebra $\mathfrak{g} = \mathbb{R}^2$ with cobracket

$$\delta(\xi) = -\frac{1}{2}\eta \wedge \xi$$
$$\delta(\eta) = 0$$

Two dimensions: $[a, b] = 0$

Classical action

$$\Phi(\xi) = a_0\{b_0, \cdot\}$$
$$\Phi(\eta) = a_0\{a_0^{-1}, \cdot\}.$$

Quantum reduction

$$(C_\hbar^\infty(M)/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathbb{R}^2)} = \{a = \lambda, b = \mu\}^{\mathcal{U}_\hbar(\mathbb{R}^2)}$$

Quantization of the Poisson reduced algebra

$$(C^\infty(M)/\mathcal{I})^{\mathbb{R}^2} = \{a_0 = \lambda, b_0 = \mu\}^{\mathbb{R}^2}$$

Three dimensions: $\mathfrak{su}(2)$

Consider $a, b, c \in C^\infty_\hbar(M)$ satisfying

$$aba^{-1} = e^{2\hbar} b$$
$$aca^{-1} = e^{-2\hbar} c$$
$$[b, c] = \frac{\hbar^2}{e^{-\hbar} - e^{\hbar}} a^{-2} - (1 - e^{2\hbar}) cb$$

and the generators ξ, η, ζ acting respectively by

$$\Phi_\hbar(\xi) f = \frac{1}{\hbar} a[b, f]$$
$$\Phi_\hbar(\eta) f = \frac{1}{\hbar} [c, f] a$$
$$\Phi_\hbar(\zeta) f = afa^{-1}.$$

Three dimensions: $\mathfrak{su}(2)$

1. Lie algebra homomorphism

$$\zeta\xi\zeta^{-1} = e^{2\hbar}\xi$$
$$\zeta\eta\zeta^{-1} = e^{-2\hbar}\eta$$
$$[\xi, \eta] = \frac{\zeta^{-1} - \zeta}{e^{-\hbar} - e^{\hbar}}$$

2. Hopf algebra action

$$\Delta_\hbar(\zeta) = \zeta \otimes \zeta$$
$$\Delta_\hbar(\xi) = \xi \otimes 1 + \zeta \otimes \xi$$
$$\Delta_\hbar(\eta) = 1 \otimes \eta + \eta \otimes \zeta^{-1}.$$

Three dimensions: $\mathfrak{su}(2)$

Let

$$\Lambda = a^{-2} - e^{\hbar} \frac{(1 - e^{2\hbar})^2}{\hbar^2} cb$$

The ideal $\mathcal{I}_\hbar$ generated by Λ in $\mathcal{A}_\hbar$ is $\mathcal{U}_\hbar(\mathfrak{su}(2))$-invariant, and

$$(C^\infty_\hbar(M)/\mathcal{I}_\hbar)^{\mathcal{U}_\hbar(\mathfrak{su}(2))}$$

is the deformation quantization of the Poisson reduction

$$M//SU(2)$$

corresponding to the symplectic leaf $a_0^{-2} - 4b_0c_0 = 0$ in $SU(2)^* = SB(2, \mathbb{C})$.

Idea

Given a Poisson manifold (M, π) we can associate $\Sigma(M) \rightrightarrows M$.

Question: given a Poisson action of (G, π_G) on (M, π), can we associate an action of (G, π_G) on $\Sigma(M)$?

Theorem

Given a Poisson action $G \times M \to M$ there exists a lifted Poisson action of G on $\Sigma(M)$ which is Hamiltonian with momentum map $J : \Sigma(M) \to G^*$.

Groupoid

Groupoid Γ over M is defined by

$$\Gamma \rightrightarrows M$$

Groupoid

Groupoid Γ over M is defined by

$$\Gamma \rightrightarrows M$$

Composition map $m : \Gamma_2 \to \Gamma$ where

$$\Gamma_2 = \{(g, h) \in \Gamma \times \Gamma | s(g) = t(h)\}$$

Unit map $u : M \to \Gamma : x \mapsto 1_x$

Inverse map $i : \Gamma \to \Gamma : g \mapsto g^{-1}$

Lie groupoid

Definition

A Lie groupoid is a groupoid $(\Gamma \rightrightarrows M, m, i, u)$ where

- Γ and M are manifolds
- s, t, m, i and u are smooth maps
- s and t are submersions

Symplectic groupoid

Definition

A Poisson groupoid is a Lie groupoid $\Gamma \rightrightarrows M$ with a multiplicative structure π on Γ.

When π is non degenerate, $\Omega = \pi^{-1}$ is a symplectic form. Thus, $(\Gamma \rightrightarrows M, \Omega)$ symplectic groupoid.

Poisson action on symplectic groupoid

A momentum map for the action of a Poisson Lie group (G, π_G) on symplectic groupoid $\Gamma \rightrightarrows M$ is a map $J : \Gamma \to G^*$ such that

$$\xi_M = \pi^\sharp(J^*(\theta_\xi))$$

Theorem

If $G \times \Gamma \to \Gamma$ is Hamiltonian action with momentum map $J : \Gamma \to G^*$ such that $J(M) = e$ then the following are equivalent:

1. *$J : \Gamma \to G^*$ is a groupoid morphism*
2. *twisted multiplicativity*

Quantization of J

Hamiltonian action $G \times \Gamma \to \Gamma$ with momentum map $J : \Gamma \to G^*$

1. Quantize symplectic groupoid given by quantum groupoid

$$C^\infty_\hbar(M) \rightrightarrows C^\infty_\hbar(\Gamma)$$

2. Quantize action $\mathfrak{g} \to \mathrm{End}\, C^\infty(\Gamma)$
3. Quantize groupoid homomorphism $\alpha : \mathfrak{g} \to \Omega^1(\Gamma)$

Lifted momentum map

Hamiltonian action $G \times M \to M$ with momentum map

$$\boldsymbol{\mu} : M \to G^*$$

If (M, π) is integrable, we associate $\Sigma(M) \rightrightarrows M$ and we get a lifted Hamiltonian action $G \times \Sigma(M) \to \Sigma(M)$ with momentum map

$$J : \Sigma(M) \to M : J(x) = \boldsymbol{\mu}(t(x))\boldsymbol{\mu}(s(x))^{-1}$$

Thanks!

Now we have our "towards" :)